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Finsler Metrics with $\mathbf{K} = 0$ and $\mathbf{S} = 0$

Zhongmin Shen

Abstract. In the paper, we study the shortest time problem on a Riemannian space with an external force. We show that such problem can be converted to a shortest path problem on a Randers space. By choosing an appropriate external force on the Euclidean space, we obtain a non-trivial Randers metric of zero flag curvature. We also show that any positively complete Randers metric with zero flag curvature must be locally Minkowskian.

1 Introduction

One of the fundamental problems in Riemann-Finsler geometry is to study and characterize Finsler metrics of constant flag curvature. For a Finsler manifold (M, F), the flag curvature $\mathbf{K} = \mathbf{K}(P, \mathbf{y})$ is a function of tangent planes $P \subset T_xM$ and non-zero vectors $\mathbf{y} \in P$. When F is Riemannian, $\mathbf{K}(P, \mathbf{y}) = \mathbf{K}(P)$ is independent of $\mathbf{y} \in P$ and $\mathbf{K}(P)$ is just the sectional curvature of tangent planes $P \subset T_xM$. Thus the flag curvature in Finsler geometry is an analogue of the sectional curvature in Riemannian geometry. Riemannian metrics of constant sectional curvature were classified by E. Cartan a long time ago. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However, the local metric structure of a Finsler metric with constant flag curvature is much more complicated. Mathematicians have discovered several important Finsler metrics of constant flag curvature (*e.g.*, the Funk metrics ($\mathbf{K} = -1/4$), the Hilbert-Klein metrics ($\mathbf{K} = -1$) [BaChSh], [Sh1] and the Bryant metrics ($\mathbf{K} = 1$) [Br1], [Br2], [Br3]). All of them are locally projectively flat.

In [Be], L. Berwald constructs a locally projectively flat Finsler metric with zero flag curvature $\mathbf{K} = 0$. It is defined on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ by

$$F(\mathbf{y}) := \frac{\Big(\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2\,|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)} + \langle \mathbf{x}, \mathbf{y}\rangle\Big)^2}{(1 - |\mathbf{x}|^2)^2\sqrt{|\mathbf{y}|^2 - (|\mathbf{x}|^2\,|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2)}}, \quad \mathbf{y} \in T_{\mathbf{x}}\mathbb{B}^n = \mathrm{R}^n, \tag{1}$$

where $|\cdot|$ and $\langle\, ,\, \rangle$ denote the standard Euclidean norm and inner product. See also [Sh2]. This Finsler metric is positively complete in the sense that every geodesic on an interval (a, b) can be extended to a geodesic on $(a, +\infty)$. According to [AZ], any positively complete Finsler metric with zero flag curvature $\mathbf{K} = 0$ must be locally Minkowskian if the first and second Cartan torsions are bounded (see also [Sh1]). Thus the Finsler metric in (1) does not have bounded Cartan torsions.

For a Finsler manifold (M, F), there is an interesting quantity $\tau = \tau(\mathbf{y})$ defined on each tangent space (T_xM, F_x),

$$\tau(\mathbf{y}) := \ln\left[\sqrt{\det(g_{ij})}\,\mathrm{Vol}\left\{(y^i) \in \mathrm{R}^n \;\Big|\; F\Big(y^i \frac{\partial}{\partial x^i}\Big|_x\Big) < 1\right\} \Big/ \mathrm{Vol}\big(\mathbb{B}^n(1)\big)\right],$$

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where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ and $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}(\mathbf{y})$. It is known that at any point $x \in M$, the Minkowski norm F_x is Euclidean if and only if $\tau(\mathbf{y}) = 0, \forall\, \mathbf{y} \in T_xM$ (see Section 7.3 in [Sh7]). For a vector $\mathbf{y} \in T_xM$, let $c(t)$, $-\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y}$. Define

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt}\Big[\tau\big(\dot{c}(t)\big)\Big]\Big|_{t=0}.$$

We call $\mathbf{S} = \mathbf{S}(\mathbf{y})$ the S-curvature. The S-curvature measures the rate of change of τ along geodesics. This quantity was first introduced in [Sh4] for a volume comparison theorem (see also [Sh1], [Sh7]). It is easily seen that Minkowski spaces are Finsler spaces with $\mathbf{K} = 0$ and $\mathbf{S} = 0$. Recently, the author and D. Bao discovered a family of non-projectively flat Finsler metrics of constant flag curvature $\mathbf{K} = 1$ on S^3 using the Hopf-fibration structure [BaSh]. This family of Finsler metrics also have vanishing S-curvature $\mathbf{S} = 0$. The Bonnet-Myers theorem tells us that the diameter is always less than or equal to π. Applying the volume comparison theorem, we can show that the injectivity radius is equal to π. Thus for any point $p \in \mathrm{S}^3$, there is another point $p^* \in \mathrm{S}^3$ such that any unit speed geodesic $\sigma(t)$ issuing from a point $p \in \mathrm{S}^3$ ($\sigma(0) = p$) is minimizing on $[0, \pi]$ and passes through p^* ($\sigma(\pi) = p^*$). See Sections 9 and 18 in [Sh7].

In this paper, motivated by the shortest time problem , we are going to construct a non-projectively flat Finsler metrics with $\mathbf{K} = 0$ and $\mathbf{S} = 0$ in each dimension. Further, these Finsler metrics have bounded first and second Cartan torsions. Thus, they are not positively complete.

Theorem 1.1 *Let* $n \geq 2$ *and*

$$\Omega := \{p = (x, y, \bar{p}) \in \mathrm{R}^2 \times \mathrm{R}^{n-2} \mid x^2 + y^2 < 1\}.$$

Define

$$F(\mathbf{y}) := \frac{\sqrt{(-yu + xv)^2 + |\mathbf{y}|^2(1 - x^2 - y^2)} - (-yu + xv)}{1 - x^2 - y^2}, \tag{2}$$

where $\mathbf{y} = (u, v, \bar{\mathbf{y}}) \in T_p\Omega = \mathrm{R}^n$ *and* $p = (x, y, \bar{p}) \in \Omega$*. F is a Finsler metric on* Ω *with vanishing flag curvature* $\mathbf{K} = 0$ *and vanishing S-curvature* $\mathbf{S} = 0$.

It is known that every Berwald metric satisfies $\mathbf{S} = 0$ [Sh1], [Sh4]. Since every Berwald metric with $\mathbf{K} = 0$ must be locally Minkowskian (see [AIM], [BaChSh]), the Finsler metric in (2) is not Berwaldian.

The Finsler metric in (2) is in the following form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha(x) := \sup_{y \in T_xM} \beta(y)/\alpha(y) < 1$ for all $x \in M$. This type of Finsler metrics was first studied by G. Randers in 1941 [Ra] from the standard point of general relativity (see also [AIM]). Therefore they are called *Randers metrics*.

The Randers metric in (2) is not locally projectively flat, hence not locally Minkowskian. Theorem 1.1 is inconsistent with the main result in [SSAY], where they claim that every Randers metric with $\mathbf{K} = 0$ must be locally Minkowskian. See also [YaSh], [Ma] for related discussion. Nevertheless, if a Randers metric with $\mathbf{K} = 0$ is positively complete, *i.e.*, every geodesic on an interval (a, b) can be extended to a geodesic on $(a, +\infty)$, then it must be locally Minkowskian.

Theorem 1.2 *Let $F = \alpha + \beta$ be a positively complete Randers metric on a manifold M. Then the flag curvature vanishes $\mathbf{K} = 0$ if and only if it is locally Minkowskian. In this case, α is a flat Riemannian metric and β is parallel with respect to α.*

Example (2) shows that the positive completeness can not be dropped. In [Sh6], we show that every locally projectively flat Randers metric with $\mathbf{K} = 0$ must be locally Minkowskian. In this case, the positive completeness is not required.

2 Preliminaries

In this section, we recall some basic definitions in Riemann-Finsler geometry [BaChSh], [Sh1].

A Finsler metric on a manifold M is a function $F\colon TM \to [0, \infty)$ with the following properties:

(a) $F(\lambda \mathbf{y}) = \lambda F(\mathbf{y})$, $\lambda > 0$;
(b) For any non-zero vector $\mathbf{y} \in T_xM$, the induced bilinear form $g_{\mathbf{y}}$ on T_xM is an inner product, where

$$g_{\mathbf{y}}(\mathbf{u}, \mathbf{v}) := \frac{1}{2}\frac{\partial^2}{\partial s\,\partial t}[F^2(\mathbf{y} + s\mathbf{u} + t\mathbf{v})]|_{s=t=0}, \quad \mathbf{u}, \mathbf{v} \in T_xM.$$

Riemannian metrics are special Finsler metrics. Traditionally, a Riemannian metric is denoted by $a_{ij}(x)dx^i \otimes dx^j$. It is a family of inner products on tangent spaces. Let $\alpha(\mathbf{y}) := \sqrt{a_{ij}(x)y^iy^j}$, $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. α is a family of Euclidean norms on tangent spaces. Throughout this paper, we also denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)y^iy^j}$.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ a 1-form on a manifold M. Define

$$F = \alpha + \beta.$$

Then F satisfies (a). If we assume that

$$\|\beta\|_\alpha(x) := \sup_{y \in T_xM} \frac{\beta(\mathbf{y})}{\alpha(\mathbf{y})} = \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1, \quad x \in M,$$

then F satisfies (b). By definition, F is a Finsler metric. We call F a Randers metric.

Let F be a Finsler metric. For a non-zero vector $\mathbf{y} \in T_pM \setminus \{0\}$, define

$$\mathbf{C}_{\mathbf{y}}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{4}\frac{\partial^3}{\partial s\,\partial t\,\partial r}[F^2(\mathbf{y} + s\mathbf{u} + t\mathbf{v} + r\mathbf{w})]|_{s=t=r=0}$$

$$\tilde{\mathbf{C}}_{\mathbf{y}}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}) := \frac{1}{4}\frac{\partial^4}{\partial s\,\partial t\,\partial r\,\partial h}[F^2(\mathbf{y} + s\mathbf{u} + t\mathbf{v} + r\mathbf{w} + h\mathbf{z})]|_{s=t=r=h=0}.$$

By the homogeneity of F, we have

$$\mathbf{C}_{\mathbf{y}}(\mathbf{u}, \mathbf{v}, \mathbf{y}) = 0, \quad \tilde{\mathbf{C}}_{\mathbf{y}}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{y}) = -\mathbf{C}_{\mathbf{y}}(\mathbf{u}, \mathbf{v}, \mathbf{w}).$$

We call $\mathbf{C}$ and $\tilde{\mathbf{C}}$ the first and second Cartan torsion respectively. In literatures, $\mathbf{C}$ is simply called the Cartan torsion. It is obvious that F is Riemannian if and only if $\mathbf{C} = 0$.

The essential bounds of $\mathbf{C}$ and $\tilde{\mathbf{C}}$ at $x \in M$ are defined by

$$\|\mathbf{C}\|_x := \sup_{\mathbf{u} \in T_x M} F(\mathbf{y}) \frac{|\mathbf{C}_{\mathbf{y}}(\mathbf{u}, \mathbf{u}, \mathbf{u})|}{[g_{\mathbf{y}}(\mathbf{u}, \mathbf{u})]^{3/2}}$$

$$\|\tilde{\mathbf{C}}\|_x := \sup_{\mathbf{u} \in T_x M} F^2(\mathbf{y}) \frac{|\tilde{\mathbf{C}}_{\mathbf{y}}(\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u})|}{[g_{\mathbf{y}}(\mathbf{u}, \mathbf{u})]^2},$$

where the supremum is taken over all non-zero vectors $\mathbf{y}, \mathbf{u} \in T_x M$ with $g_{\mathbf{y}}(\mathbf{y}, \mathbf{u}) = 0$.

For a Finsler metric F on an n-dimensional manifold M, the (Busemann-Hausdorff) volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}\big(\mathbb{B}^n(1)\big)}{\mathrm{Vol}\{(y^i) \in \mathrm{R}^n \mid F(y^i \frac{\partial}{\partial x^i}|_x) < 1\}}. \tag{3}$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary functions, even F is locally expressed by elementary functions. However, for Randers metrics, the volume form is expressed by a very simple formula, since each indicatrix $S_x M := \{\mathbf{y} \in T_x M \mid F(\mathbf{y}) = 1\}$ is a shifted Euclidean sphere in $(T_x M, \alpha_x)$. More precisely, for a Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta(y) = b_i(x) y^i$, its volume form dV_F is given by

$$dV_F = \big(1 - \|\beta\|^2_\alpha(x)\big)^{\frac{n+1}{2}} dV_\alpha. \tag{4}$$

See [Sh1] for more details.

Let F be a Finsler metric on a manifold M. Let

$$g_{ij}(x, y) := g_{\mathbf{y}}\left(\frac{\partial}{\partial x^i}\Big|_x, \frac{\partial}{\partial x^j}\Big|_x \right) = \frac{1}{2}[F^2]_{y^i y^j}(\mathbf{y}), \quad \mathbf{y} = y^i \frac{\partial}{\partial x^i}\Big|_x.$$

Express the volume form dV_F by

$$dV_F = \sigma_F(x) dx^1 \cdots dx^n,$$

where $\sigma_F(x)$ is defined in (3). Set

$$\tau(\mathbf{y}) := \ln \left[\frac{\sqrt{\det\big(g_{ij}(x, y)\big)}}{\sigma_F(x)} \right].$$

The quantity τ is a scalar function on $TM \setminus \{0\}$. We call it the distortion. At each point $x \in M$, τ depends only on F_x on T_xM. When F_x is Euclidean, *i.e.*, $F_x(\mathbf{y}) = \sqrt{g_{ij}(x)y^j y^k}$, $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x$,

$$\sigma_F(x) = \sqrt{\det\big(g_{ij}(x)\big)}.$$

Thus $\tau(\mathbf{y}) = 0, \forall\, \mathbf{y} \in T_xM$. In fact, the converse is true too. See Section 7.3 in [Sh7].

Locally minimizing constant speed curves (geodesics) are characterized by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i(x, y)$ are given by

$$G^i := \frac{1}{4} g^{il}\left\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right\} y^j y^k. \tag{5}$$

G^i are called the geodesic coefficients in a local coordinate system. If F is Riemannian, then $G^i(x, y) = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in (y^i) at every point $x \in M$. A Finsler metric is called a *Berwald metric* if the geodesic coefficients have this property. There are many non-Riemannian Berwald metrics. The classification of Berwald metrics is done by Z. I. Szabo [Sz].

Let $G^i(x, y)$ denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is expressed by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}[\ln \sigma_F(x)], \tag{6}$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric [Sh4]. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$.

Now, we recall the definition of Riemann curvature. Let F be a Finsler metric on an n-manifold and G^i denote the geodesic coefficients of F. For a vector $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, define $\mathbf{R}_\mathbf{y} = R^i_k(x, y)dx^k \otimes \frac{\partial}{\partial x^i}|_x \colon T_xM \to T_xM$ by

$$R^i_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{7}$$

The Ricci curvature is defined by

$$\mathrm{Ric}(\mathbf{y}) := R^i_i(x, y).$$

In dimension two, let $x := x^1$, $y := x^2$, $u := y^1$, $v := y^2$. We can express the Ricci curvature by

$$\begin{aligned}\mathrm{Ric}(\mathbf{y}) = 2&\left\{\frac{\partial G^1}{\partial x} + \frac{\partial G^2}{\partial y} + \frac{\partial G^1}{\partial u}\frac{\partial G^2}{\partial v} - \frac{\partial G^1}{\partial v}\frac{\partial G^2}{\partial u}\right\}\\ &- S^2 - \left(u\frac{\partial}{x} + v\frac{\partial}{\partial y} - 2G^1\frac{\partial}{\partial u} - 2G^2\frac{\partial}{\partial v}\right)(S),\end{aligned} \tag{8}$$

where $S = \frac{\partial G^1}{\partial u} + \frac{\partial G^2}{\partial v}$.

For a two-dimensional tangent plane $P \subset T_xM$ and a non-zero vector $\mathbf{y} \in P$, define

$$\mathbf{K}(P, \mathbf{y}) := \frac{g_{\mathbf{y}}\big(\mathbf{R}_{\mathbf{y}}(\mathbf{u}), \mathbf{u}\big)}{g_{\mathbf{y}}(\mathbf{y}, \mathbf{y}) g_{\mathbf{y}}(\mathbf{u}, \mathbf{u}) - g_{\mathbf{y}}(\mathbf{y}, \mathbf{u}) g_{\mathbf{y}}(\mathbf{y}, \mathbf{u})}, \tag{9}$$

where $P = \operatorname{span}\{\mathbf{y}, \mathbf{u}\}$. $\mathbf{K}$ is called the flag curvature. Usually, $\mathbf{K}(P, \mathbf{y})$ depends on the direction $\mathbf{y} \in P$. If F is Riemannian, then $\mathbf{K}(P, \mathbf{y})$ is independent of $\mathbf{y} \in P$. The flag curvature is an analogue of the sectional curvature in Riemannian geometry. It is obvious that $\mathbf{K} = 0$ if and only if $\mathbf{R} = 0$.

The following theorem is due to Akbar-Zadeh [AZ].

Theorem 2.1 ([AZ]) *Let* (M, F) *be a positively complete Finsler manifold with* $\mathbf{K} = 0$. *Assume that*

$$\sup_{x \in M} \|\mathbf{C}\|_x < \infty, \qquad \sup_{x \in M} \|\tilde{\mathbf{C}}\|_x < \infty.$$

Then F is locally Minkowskian.

See also Theorem 10.3.7 in [Sh1] for a proof.

Let F be a Finsler metric on a manifold M and $G^i(x, y)$ denote the geodesic coefficients of F. Let $\tilde{F}$ be another metric and $\tilde{G}^i(x, y)$ denote the geodesic coefficients of $\tilde{F}$. To find the relationship between the Riemann curvature $R^i_k(x, y) dx^k \otimes \frac{\partial}{\partial x^i}$ of F and the Riemann curvature $\tilde{R}^i_k(x, y) dx^k \otimes \frac{\partial}{\partial x^i}$ of $\tilde{F}$, we introduce

$$H^i(x, y) := G^i(x, y) - \tilde{G}^i(x, y).$$

Define

$$H^i_{|k} := \frac{\partial H^i}{\partial x^k} + H^j \frac{\partial^2 \tilde{G}^i}{\partial y^j \partial y^k} - \frac{\partial H^i}{\partial y^j} \frac{\partial \tilde{G}^j}{\partial y^k}.$$

We have the following useful formula

$$R^i_k = \tilde{R}^i_k + 2H^i_{|k} - y^j (H^i_{|j})_{y^k} + 2H^j (H^i)_{y^j y^k} - (H^i)_{y^j} (H^j)_{y^k}. \tag{10}$$

The proof is straightforward, so is omitted. See [Sh1].

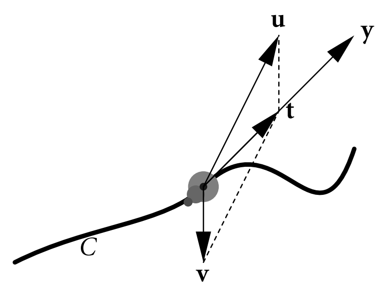
3 Shortest Time Problem

Let (M, F) be a Finsler space. Suppose that an object on (M, F) is pushed by an internal force $\mathbf{u}$ with constant length, $F(\mathbf{u}) = 1$. Due to the friction, the object moves on M at a constant speed, but it can change direction freely. Without external force acting on the object, any path of shortest time is a shortest path of F.

Now given an external force $\mathbf{v}$ pushing the object. We assume that $F(-\mathbf{v}) < 1$, otherwise, the object can not move forward in the direction $-\mathbf{v}$. Due to friction, the speed of the object is proportional to the length of the combined force

$$\mathbf{t} := \mathbf{v} + \mathbf{u}.$$

For simplicity, we may assume that the speed of the object is equal to $F(\mathbf{t})$. That is, $\mathbf{t} := \mathbf{v} + \mathbf{u}$ is the velocity vector once the direction of the internal force is chosen.



Express the velocity vector in the following form

$$\mathbf{t} = F(\mathbf{t})\mathbf{y},$$

where $\mathbf{y}$ is a unit vector with respect to F. Since $F(\mathbf{u}) = 1$, $F(\mathbf{t})$ is determined by

$$F\big(F(\mathbf{t})\mathbf{y} - \mathbf{v}\big) = F(\mathbf{u}) = 1. \tag{11}$$

Now we are going to find the Finsler metric $\tilde{F}$ such that the $\tilde{F}$-length of any curve is equal to the time for which the object travels along it.

Lemma 3.1 *Let (M, F) and $\mathbf{v}$ be a vector field on M with $F(-\mathbf{v}) < 1$. Define $\tilde{F}$: $TM \to [0, \infty)$ by*

$$F\left(\frac{\mathbf{y}}{\tilde{F}(\mathbf{y})} - \mathbf{v}\right) = 1, \quad \mathbf{y} \in T_xM \setminus \{0\}. \tag{12}$$

For any curve C in M, the $\tilde{F}$-length of C is equal to the time for which the object travels along it.

Proof Take an arbitrary curve C from p to q, and a coordinate map $c\colon [0, T] \to C$ such that $c(0) = p$, $c(T) = q$ and the velocity vector $\dot{c}(t)$ is equal to the combined force at $c(t)$. Express $\dot{c}(t) = F\big(\dot{c}(t)\big)\mathbf{y}(t)$, where $\mathbf{y}(t)$ is tangent to C at $c(t)$ with $F\big(\mathbf{y}(t)\big) = 1$. By (11),

$$F\Big(F\big(\dot{c}(t)\big)\mathbf{y}(t) - \mathbf{v}(t)\Big) = 1, \tag{13}$$

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where $\mathbf{v}(t) := \mathbf{v}_{c(t)}$. Consider the following equation

$$F\big(\lambda \mathbf{y}(t) - \mathbf{v}(t)\big) = 1, \quad \lambda > 0.$$

Since $F\big(-\mathbf{v}(t)\big) < 1$, the above equation has a unique solution, that is $F\big(\dot{c}(t)\big)$. By the definition of $\tilde{F}$, we obtain

$$\tilde{F}\big(\mathbf{y}(t)\big) = \frac{1}{F\big(\dot{c}(t)\big)}. \tag{14}$$

From (13) and (14), we obtain

$$\tilde{F}\big(\dot{c}(t)\big) = F\big(\dot{c}(t)\big)\,\tilde{F}\big(\mathbf{y}(t)\big) = 1.$$

This implies

$$T = \int_0^T \tilde{F}\big(\dot{c}(t)\big)\,dt.$$

Thus $\tilde{F}$ is the desired Finsler metric. ■

The relation between F and $\tilde{F}$ is actually very simple. For a point $x \in M$, the indicatrix of F at x is related to that of $\tilde{F}$ at x,

$$\{\mathbf{y} \in T_xM \mid \tilde{F}(\mathbf{y}) = 1\} = \{\mathbf{y} \in T_xM \mid F(\mathbf{y}) = 1\} + \mathbf{v}_x.$$

This leads to the following:

Proposition 3.2 *Let (M, F) be a Finsler manifold and $\mathbf{v}$ be a vector field on M with $F(-\mathbf{v}) < 1$. Let $\tilde{F}\colon TM \to [0, \infty)$ denote the Finsler metric defined by (12). The volume form of F equals that of $\tilde{F}$,*

$$dV_F = dV_{\tilde{F}}.$$

Proof Let (x^i) be a local coordinate system at $x \in M$ and

$$\mathcal{U}_F := \left\{ (y^i) \in \mathrm{R}^n \;\middle|\; F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1 \right\},$$
$$\mathcal{U}_{\tilde{F}} := \left\{ (y^i) \in \mathrm{R}^n \;\middle|\; \tilde{F}\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1 \right\}.$$

From (12), we have

$$\mathcal{U}_{\tilde{F}} = \mathcal{U}_F + (v^i),$$

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where $\mathbf{v} = v^i \frac{\partial}{\partial x^i}|_x$. This gives that

$$dV_F = \frac{\mathrm{Vol}\big(\mathbb{B}^n(1)\big)}{\mathrm{Vol}(\mathcal{U}_F)} dx^1 \cdots dx^n = \frac{\mathrm{Vol}\big(\mathbb{B}^n(1)\big)}{\mathrm{Vol}(\mathcal{U}_{\tilde{F}})} dx^1 \cdots dx^n = dV_{\tilde{F}}. \qquad \blacksquare$$

For a Finsler metric F and a vector $\mathbf{v}$ with $F(-\mathbf{v}) < 1$, let $\tilde{F}$ be the Finsler metric defined in (12). The relation between $\tilde{F}$ and F can also be described in the following way. Let $F^* \colon T^*M \to [0, \infty)$ denote the Finsler metric dual to F, *i.e.*,

$$F^*_x(\xi) := \sup_{\mathbf{y} \in T_xM} \frac{\xi(\mathbf{y})}{F(\mathbf{y})}, \quad \xi \in T^*_xM.$$

Let $\beta^* \colon T^*_xM \to R$ be defined by $\beta^*(\xi) := \xi(\mathbf{v})$, $\xi \in T^*M$. Then $\tilde{F} \colon TM \to [0, \infty)$ is the Finsler metric dual to $\tilde{F}^* := F^* + \beta^*$.

Consider the shortest time problem on a Riemannian manifold (M, α) with an external force field $\mathbf{v}$. Denote by $\langle \, , \, \rangle_\alpha$ the family of inner products on tangent spaces, which are determined by α,

$$\alpha(\mathbf{y}) = \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle_\alpha}, \quad \mathbf{y} \in T_pM.$$

Solving the following equation for $\tilde{F}(\mathbf{y})$,

$$\alpha\left(\frac{\mathbf{y}}{\tilde{F}(\mathbf{y})} - \mathbf{v}\right) = 1,$$

we obtain

$$\tilde{F}(\mathbf{y}) = \tilde{\alpha}(\mathbf{y}) + \tilde{\beta}(\mathbf{y}), \quad \mathbf{y} \in T_pM \tag{15}$$

where

$$\tilde{\alpha}(\mathbf{y}) := \frac{\sqrt{\langle \mathbf{v}, \mathbf{y} \rangle_\alpha^2 + \alpha(\mathbf{y})^2 \big(1 - \alpha(\mathbf{v})^2\big)}}{1 - \alpha(\mathbf{v})^2}, \quad \tilde{\beta}(\mathbf{y}) := -\frac{\langle \mathbf{v}, \mathbf{y} \rangle_\alpha}{1 - \alpha(\mathbf{v})^2}.$$

By Proposition 3.2, we have

$$dV_{\tilde{F}} = dV_\alpha.$$

The Finsler metric $\tilde{F}$ in (15) is called a *Randers metric* in Finsler geometry.

Example 3.1 Let α denote the standard Euclidean metric on the unit ball $\mathbb{B}^n$ and $\mathbf{v}$ denote the radial vector field on $\mathbb{B}^n$, which is given by

$$\mathbf{v}_p = -(x^i), \quad p = (x^i) \in \mathbb{B}^n.$$

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The Randers metric associated with $(\alpha, \mathbf{v})$ as defined in (15) is given by

$$\tilde{F}(\mathbf{y}) = \frac{\sqrt{\langle \mathbf{v}, \mathbf{y}\rangle^2 + |\mathbf{y}|^2(1 - |\mathbf{v}|^2)} + \langle \mathbf{v}, \mathbf{y}\rangle}{1 - |\mathbf{v}|^2}, \quad \mathbf{y} \in T_p\mathbb{B}^n,$$

where $|\cdot|$ and $\langle\ ,\ \rangle$ denote the standard Euclidean norm and inner product. This is just the Funk metric on $\mathbb{B}^n$. Geodesics of $\tilde{F}$ are straight lines. Moreover, the flag curvature is a negative constant, $\mathbf{K} = -1/4$. If an object moves away from the center, it takes infinite time to reach the boundary. However, it takes finite time to reach the center along any shortest path. Thus $\tilde{F}$ is positively complete, but not negatively complete.

Example 3.2 (**[BaSh]**) Let S^3 be the standard unit sphere in $\mathbb{R}^4$. Let α denote the standard Riemannian metric on S^3 and $\mathbf{v} := \epsilon\mathbf{w}$, where $|\epsilon| < 1$ and $\mathbf{w}$ is a left-invariant unit vector field on S^3. The Randers metric associated with $(\alpha, \mathbf{v})$ as defined in (15) is given by

$$\tilde{F}(\mathbf{y}) = \frac{\sqrt{\epsilon^2\langle \mathbf{w}, \mathbf{y}\rangle_\alpha^2 + (1 - \epsilon^2)\alpha(\mathbf{y})^2} - \epsilon\langle \mathbf{w}, \mathbf{y}\rangle_\alpha}{1 - \epsilon^2}, \quad \mathbf{y} \in T_p\mathrm{S}^3. \tag{16}$$

We have shown that $\tilde{F}$ has constant flag curvature $\mathbf{K} = 1$ for any ϵ with $|\epsilon| < 1$. In [Sh7], we have verified that $\tilde{F}$ has vanishing S-curvature for any ϵ with $|\epsilon| < 1$.

Other examples will be discussed in Section 7 below and [Sh5].

4 Proof of Theorem 1.2

According to Theorem 2.1, to prove Theorem 1.2, it suffices to prove that the first and second Cartan torsions have uniform upper bounds.

Consider a Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$ with $\|\beta\|_\alpha = \sqrt{a^{ij}b_ib_j} < 1$. It is proved that the first Cartan torsion of F satisfies the bound

$$\|\mathbf{C}\|_x \le \frac{3}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \|\beta\|^2(x)}} < \frac{3}{\sqrt{2}}, \quad x \in M. \tag{17}$$

This is verified by B. Lackey in dimension two [BaChSh], and can be extended to higher dimensions with a simple argument [Sh3].

Now we are going to find an upper bound on the second Cartan torsion for Randers metrics. First, we consider a special two-dimensional case.

Let $p \in M$ and $\kappa := \|\beta\|_\alpha(p) < 1$. There is an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for T_pM such that

$$F(\mathbf{y}) = \sqrt{u^2 + v^2} + \kappa u, \quad \mathbf{y} = u\mathbf{e}_1 + v\mathbf{e}_2 \in T_pM.$$

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Lemma 4.1 *Let κ be a constant with $0 \leq \kappa < 1$ and*

$$F := \sqrt{u^2 + v^2} + \kappa u. \tag{18}$$

The second Cartan torsion of F satisfies the following bound

$$\|\tilde{\mathbf{C}}\| \leq \frac{27}{2}\kappa < \frac{27}{2}.$$

Proof Let $\mathbf{y} = r(\cos\theta, \sin\theta)$, where $r > 0$, and $\mathbf{y}^\perp$ denote the vector perpendicular to $\mathbf{y}$ with respect to $g_\mathbf{y}$,

$$g_\mathbf{y}(\mathbf{y}, \mathbf{y}^\perp) = 0, \quad g_\mathbf{y}(\mathbf{y}^\perp, \mathbf{y}^\perp) = F^2(\mathbf{y}).$$

We have

$$\mathbf{y}^\perp = \frac{r}{\sqrt{1 + \kappa\cos\theta}}(-\sin\theta, \kappa + \cos\theta).$$

By a direct computation, we obtain

$$\tilde{\mathbf{C}}_\mathbf{y}(\mathbf{y}^\perp, \mathbf{y}^\perp, \mathbf{y}^\perp, \mathbf{y}^\perp) = F^2(\mathbf{y}) \left\{ 6\kappa \frac{\kappa + \cos\theta}{1 + \kappa\cos\theta} - \frac{15}{2}\kappa\cos\theta \right\}.$$

This gives

$$\begin{aligned}
\|\tilde{\mathbf{C}}\| &= \max_{0\leq\theta\leq 2\pi} \left| 6\kappa \frac{\kappa + \cos\theta}{1 + \kappa\cos\theta} - \frac{15}{2}\kappa\cos\theta \right| \\
&\leq \max_{0\leq\theta\leq 2\pi} \left\{ 6\kappa \left| \frac{\kappa + \cos\theta}{1 + \kappa\cos\theta} \right| + \frac{15}{2}\kappa|\cos\theta| \right\} \\
&\leq 6\kappa + \frac{15}{2}\kappa = \frac{27}{2}\kappa.
\end{aligned}$$

This proves the lemma. ■

We can extend Lemma 4.1 to higher dimensions.

Lemma 4.2 *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold M. The second Cartan torsion of F satisfies*

$$\|\tilde{\mathbf{C}}\| \leq \frac{27}{2}\|\beta\|_\alpha < \frac{27}{2}.$$

Proof At a point $p \in M$, there are two vectors $\mathbf{y}, \mathbf{u} \in T_pM$ with

$$F(\mathbf{y}) = 1, \quad g_\mathbf{y}(\mathbf{y}, \mathbf{u}) = 0, \quad g_\mathbf{y}(\mathbf{u}, \mathbf{u}) = 1$$

such that

$$\|\tilde{\mathbf{C}}\|_p = |\tilde{\mathbf{C}}_{\mathbf{y}}(\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u})|.$$

Let $V := \operatorname{span}\{\mathbf{y}, \mathbf{u}\}$ and

$$\kappa := \sup_{\mathbf{v} \in V} \frac{\beta(\mathbf{v})}{\alpha(\mathbf{v})}.$$

By Lemma 4.1,

$$|\tilde{\mathbf{C}}_{\mathbf{y}}(\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u})| \le \frac{27}{2}\kappa.$$

Note that

$$\kappa = \sup_{\mathbf{v} \in V} \frac{\beta(\mathbf{v})}{\alpha(\mathbf{v})} \le \sup_{\mathbf{v} \in T_pM} \frac{\beta(\mathbf{v})}{\alpha(\mathbf{v})} =: \|\beta\|_\alpha(p).$$

We obtain

$$\|\tilde{\mathbf{C}}\|_p \le \frac{27}{2}\|\beta\|_\alpha(p). \qquad \blacksquare$$

Proof of Theorem 1.2 By (17), we know that $\|\mathbf{C}\| < 3/\sqrt{2}$. By (4.2), we know that $\|\tilde{\mathbf{C}}\| < 13.5$. Then Theorem 1.2 follows from Theorem 2.1. $\blacksquare$

5 Randers Metrics with $\mathbf{S} = 0$

In this section, we are going to find a sufficient and necessary condition on α and β for $\mathbf{S} = 0$. In particular, we will show that if β is a Killing form of constant length, then $\mathbf{S} = 0$.

Let $F = \alpha + \beta$ be a Randers metric on a manifold M, where

$$\alpha(y) = \sqrt{a_{ij}(x)y^iy^j}, \quad \beta(y) = b_i(x)y^i$$

with $\|\beta\|_x := \sup_{y \in T_xM} \beta(y)/\alpha(y) < 1$.

In a standard local coordinate system (x^i, y^i) in TM, define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \tilde{\Gamma}_{ik}^j dx^k$ denote the Levi-Civita connection forms of α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$s^i_j = a^{ip}s_{pj} \quad s_j := b_i s^i_j.$$

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The geodesic coefficients G^i of F are related to the geodesic coefficients $\tilde{G}^i$ of α by

$$G^i = \tilde{G}^i + Py^i + Q^i, \tag{19}$$

where

$$P := \frac{1}{2F}\{r_{ij}y^i y^j - 2\alpha s_i y^i\}$$
$$Q^i := \alpha s^i_j y^j.$$

See [AIM]. Observe that

$$\frac{\partial Q^i}{\partial y^i} = \alpha^{-1} y_i s^i_j y^j + \alpha s^i_i = \alpha^{-1} s_{ij} y^i y^j + \alpha a^{ij} s_{ij} = 0,$$

where $y_i := a_{ij}y^j$. Thus

$$\frac{\partial G^i}{\partial y^i} = \frac{\partial \tilde{G}^i}{\partial y^i} + (n+1)P.$$

Put

$$dV_F := \sigma_F(x) dx^1 \cdots dx^n, \quad dV_\alpha = \sigma_\alpha(x) dx^1 \cdots dx^n.$$

According to (19), we have

$$\sigma_F = (1 - \|\beta\|^2_\alpha)^{\frac{n+1}{2}} \sigma_\alpha.$$

Note that

$$d[\ln \sigma_\alpha] = \frac{\partial \tilde{G}^i}{\partial y^i}.$$

By (6), we obtain a formula for $\mathbf{S}$,

$$\mathbf{S} = (n+1)\{P - d[\ln \sqrt{1 - \|\beta\|^2_\alpha}]\}. \tag{20}$$

We have the following:

Proposition 5.1 *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Then*

$$\mathbf{S} = 0 \tag{21}$$

if and only if

$$r_{ij} + b_i s_j + b_j s_i = 0. \tag{22}$$

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Proof For the sake of simplicity, we choose an orthonormal basis for T_xM such that $a_{ij} = \delta_{ij}$. Let

$$\rho := \ln \sqrt{1 - \|\beta\|_\alpha^2}$$

and $d\rho = \rho_i dx^i$, *i.e.*,

$$\rho_i = -\frac{b_j b_{j|i}}{1 - \|\beta\|_\alpha^2}. \tag{23}$$

By (20), $\mathbf{S} = 0$ if and only if

$$r_{ij}y^i y^j - 2\alpha s_i y^i = 2(\alpha + \beta)\rho_i y^i. \tag{24}$$

(24) is equivalent to the following equations

$$r_{ij} = b_j\rho_i + b_i\rho_j \tag{25}$$

$$-s_i = \rho_i. \tag{26}$$

First we assume that $\mathbf{S} = 0$. Then (25) and (26) hold. Plugging (26) into (25) gives (22).

Now we assume that (22) holds. Note that $s_j b_j = b_i s_{ij} b_j = 0$. Contracting (22) with b_j yields

$$b_j r_{ij} = -\|\beta\|_\alpha^2 s_i, \tag{27}$$

that is,

$$b_j b_{i|j} + b_j b_{j|i} = -\|\beta\|^2 (b_j b_{j|i} - b_j b_{i|j}).$$

We obtain

$$b_j b_{i|j} = -\frac{1 + \|\beta\|_\alpha^2}{1 - \|\beta\|_\alpha^2} b_j b_{j|i}. \tag{28}$$

It follows from (23) and (28) that

$$s_i = \frac{1}{2}\left(b_j b_{j|i} + \frac{1 + \|\beta\|_\alpha^2}{1 - \|\beta\|_\alpha^2} b_j b_{j|i} \right) = -\rho_i. \tag{29}$$

Thus

$$\begin{aligned} r_{ij}y^i y^j - 2\alpha s_i y^i &= -2\beta s_i y^i - 2\alpha s_i y^i \\ &= -2(\alpha + \beta) s_i y^i \\ &= 2F\rho_i y^i. \end{aligned}$$

We obtain

$$\mathbf{S} = (n + 1)\{\rho_i y^i - \rho_i y^i\} = 0.$$

This gives (21). ∎

6 Randers Metrics with $\mathbf{K} = 0$ and $\mathbf{S} = 0$

In this section, we are going to compute the Riemann curvature of a Randers metric satisfying $\mathbf{S} = 0$.

Let $F = \alpha + \beta$ be a Randers metric on a manifold M. Denote by $b_{i|j}$, $b_{i|j|k}$, *etc.*, the coefficients of the covariant derivatives of β with respect to α. Set $r_{ij} := (b_{i|j} + b_{j|i})/2$, $s_{ij} := (b_{i|j} - b_{j|i})/2$, $s^i_j = a^{ik}s_{kj}$ and $s_j := b_i s^i_j$. Further, we set $s_0 := s_p y^p$, $s^i_0 := s^i_p y^p$, $s_{0|j} = s_{p|j} y^p$ and $s_{0|0} = s_{p|q} y^p y^q$, *etc.* Denote the Riemann curvature of α by $\bar{R}^i_k dx^k \otimes \frac{\partial}{\partial x^i}$. We have the following:

Theorem 6.1 *Let $F = \alpha + \beta$ be a Randers metric satisfying $\mathbf{S} = 0$. Then $\mathbf{K} = 0$ if and only if the following two equations hold,*

$$
\begin{aligned}
\bar{R}^i_k = &-(s_{0|0}\delta^i_k - s_{0|k}y^i) - (s_{k|0} - s_{0|k})y^i \\
&- s_0(s_0\delta^i_k - s_k y^i) + (\alpha^2 s^i_j s^j_k - s^i_j s^j_0 y_k) - 3s_{k0}s^i_0,
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
0 = &\ s_j s^j_0(\alpha^2\delta^i_k - y_k y^i) + \alpha^2(s_j s^j_0 \delta^i_k - s_j s^j_k y^i) \\
&+ \alpha^2(s^i_{k|0} - s^i_{0|k}) - (\alpha^2 s^i_{0|k} - s^i_{0|0} y_k).
\end{aligned} \tag{31}
$$

Proof By assumption $\mathbf{S} = 0$, we obtain from (20) and (26), we obtain

$$P = -s_0.$$

Thus $G^i = \tilde{G}^i + H^i$, where

$$H^i = -s_0 y^i + \alpha s^i_0.$$

By a direct computation, we obtain

$$
\begin{aligned}
H^i_{|k} &= -s_{0|k}y^i + \alpha s^i_{0|k} \\
(H^i_{|j})_{y^k} &= -s_{k|j}y^i - s_{0|j}\delta^i_k + \alpha^{-1}y_k s^i_{0|j} + \alpha s^i_{k|j} \\
y^j(H^i_{|j})_{y^k} &= -s_{k|0}y^i - s_{0|0}\delta^i_k + \alpha^{-1}y_k s^i_{0|0} + \alpha s^i_{k|0} \\
(H^i)_{y^j} &= -s_j y^i - s_0\delta^i_j + \alpha^{-1}y_j s^i_p y^p + \alpha s^i_j \\
(H^i)_{y^j y^k} &= -s_j\delta^i_k - s_k\delta^i_j + \alpha^{-3}(a_{jk}\alpha^2 - y_j y_k)s^i_0 + \alpha^{-1}(y_j s^i_k + y_k s^i_j),
\end{aligned}
$$

$y_k := a_{jk}y^j$. Plugging them into (10), we obtain

$$
\begin{aligned}
R^i_k = &\ \bar{R}^i_k + (s_{0|0}\delta^i_k - s_{0|k}y^i) + (s_{k|0} - s_{0|k})y^i \\
&+ s_0(s_0\delta^i_k - s_k y^i) - (\alpha^2 s^i_j s^j_k - s^i_j s^j_0 y_k) + 3s_{k0}s^i_0 \\
&- \{s_j s^j_0(\alpha^2\delta^i_k - y_k y^i) + \alpha^2(s_j s^j_0\delta^i_k - s_j s^j_k y^i) \\
&\qquad + \alpha^2(s^i_{k|0} - s^i_{0|k}) - (\alpha^2 s^i_{0|k} - s^i_{0|0}y_k)\}\alpha^{-1}
\end{aligned} \tag{32}
$$

According to (32), the coefficients of the Riemann curvature of F are in the following form

$$R^i_k = A + B\alpha^{-1},$$

where A and B are polynomials of y^i at each point $x \in M$. Thus $R^i_k = 0$ if and only if $A = 0$ and $B = 0$. This proves Theorem 6.1. ■

Taking the trace of R^i_k in (32), we obtain

$$\text{(33)} \qquad \begin{aligned} \mathrm{Ric} = \overline{\mathrm{Ric}} + (n-1)\{s_{0|0} + s_0 s_0\} + 2s_{k0}s^k_0 - \alpha^2 s^k_j s^j_k \\ + 2\{s^k_{0|k} - (n-1)s_j s^j_0\}\alpha. \end{aligned}$$

By (33), we immediately obtain the following:

Proposition 6.2 *Let* $F = \alpha + \beta$ *be a Randers metric with* $\mathbf{S} = 0$. *Then* F *is of zero Ricci curvature,* $\mathrm{Ric} = 0$, *if and only if*

$$\text{(34)} \qquad s^k_{0|k} = (n-1)s_j s^j_0 \quad \textit{and}$$

$$\text{(35)} \qquad \overline{\mathrm{Ric}} = -(n-1)(s_{0|0} + s_0 s_0) + \alpha^2 s^k_j s^j_k - 2s_{k0}s^k_0.$$

We do not know if there is a three-dimensional Randers metric satisfying $\mathrm{Ric} = 0$, $\mathbf{S} = 0$ and $\mathbf{K} \neq 0$. Such examples, if they exist, would be of interest.

7 Examples

In this section, we will construct two interesting examples, one in dimension two and the other in dimension three. Both metrics satisfy that $\mathbf{S} = 0$ and $\mathbf{K} = 0$. The examples are Randers metrics in the form $F = \tilde{\alpha} + \tilde{\beta}$, where $\tilde{\alpha}$ is a Riemannian metric and $\tilde{\beta}$ is a 1-form. It is known that if $F = \tilde{\alpha} + \tilde{\beta}$ is of constant flag curvature, then F is locally projectively flat if and only if $\tilde{\beta}$ is closed [BaMa], [Sh1]. The 1-forms in our examples are not closed. Thus F is not projectively flat.

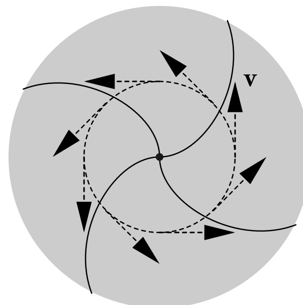
Examples in Dimension Two Let $\alpha = \sqrt{u^2 + v^2}$ denote the standard Euclidean metric on $\mathbb{R}^2$. Take a vector field $\mathbf{v}$ on the unit disk $\mathbb{D}^2$ given by

$$\mathbf{v} = (-y, x), \quad p = (x, y) \in \mathbb{D}^2.$$

The Finsler metric associated with $(\alpha, \mathbf{v})$ is a Randers metric $F = \tilde{\alpha} + \tilde{\beta}$, where

$$\tilde{\alpha} := \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2},$$

$$\tilde{\beta} := -\frac{-yu + xv}{1 - x^2 - y^2}.$$



$$a_{11} = \frac{1-x^2}{(1-x^2-y^2)^2}, \quad a_{12} = -\frac{xy}{(1-x^2-y^2)^2} = a_{21}, \quad a_{22} = \frac{1-y^2}{(1-x^2-y^2)^2}$$
$$b_1 = \frac{y}{1-x^2-y^2}, \quad b_2 = -\frac{x}{1-x^2-y^2}.$$

The geodesic coefficients $\tilde{G}^1$ and $\tilde{G}^2$ of $\tilde{\alpha}$ are given by

$$\tilde{G}^1 = -\frac{x(u^2+v^2)}{2(1-x^2-y^2)} - \frac{y(xu+yv)-v}{1-x^2-y^2}\tilde{\beta} + \frac{xu+yv}{1-x^2-y^2}u$$
$$\tilde{G}^2 = -\frac{y(u^2+v^2)}{2(1-x^2-y^2)} + \frac{x(xu+yv)-u}{1-x^2-y^2}\tilde{\beta} + \frac{xu+yv}{1-x^2-y^2}v.$$

The Gauss curvature $\tilde{\mathbf{K}}$ of $\tilde{\alpha}$ is given by

$$\tilde{\mathbf{K}} = -\frac{5+x^2+y^2}{1-x^2-y^2}.$$

By a direct computation, we obtain

$$r_{11} = -\frac{2xy}{(1-x^2-y^2)^2}$$
$$r_{12} = \frac{x^2-y^2}{(1-x^2-y^2)^2} = r_{21}$$
$$r_{22} = \frac{2xy}{(1-x^2-y^2)^2}$$
$$s_{11} = 0$$
$$s_{12} = \frac{1}{(1-x^2-y^2)^2} = -s_{21}$$
$$s_{22} = 0.$$

From the above equations, we obtain the following formulas for $s_i := b_r a^{rh} s_{hi}$.

$$s_1 = \frac{x}{1 - x^2 - y^2}$$

$$s_2 = \frac{y}{1 - x^2 - y^2}.$$

We immediately see that the following identity holds.

$$r_{ij} = -b_i s_j - b_j s_i.$$

By Proposition 5.1, we conclude that $\mathbf{S} = 0$.

By the above equations, we obtain the following formulas for $H^i := -s_j y^j y^i + \tilde{\alpha} a^{ir} s_{rl} y^l$:

$$H^1 = -\frac{xu + yv}{1 - x^2 - y^2} u - \frac{y(xu + yv) - v}{1 - x^2 - y^2} \tilde{\alpha}$$

$$H^2 = -\frac{xu + yv}{1 - x^2 - y^2} v + \frac{x(xu + yv) - u}{1 - x^2 - y^2} \tilde{\alpha}.$$

Then we obtain the following formulas for the geodesic coefficients $G^1 = \tilde{G}^1 + H^1$ and $G^2 = \tilde{G}^2 + H^2$,

$$G^1 = -\frac{x(u^2 + v^2)}{2(1 - x^2 - y^2)} - \frac{y(xu + yv) - v}{1 - x^2 - y^2} F \tag{36}$$

$$G^2 = -\frac{y(u^2 + v^2)}{2(1 - x^2 - y^2)} + \frac{x(xu + yv) - u}{1 - x^2 - y^2} F. \tag{37}$$

One can easily verify that G^1 and G^2 satisfy

$$S := \frac{\partial G^1}{\partial u} + \frac{\partial G^2}{\partial v} = 0. \tag{38}$$

By Proposition 3.2, we know that the area form of F is equal to the Euclidean area form

$$dA_F = dA_\alpha = dxdy.$$

By (6), we conclude that $\mathbf{S} = 0$ again.

A direct computation yields

$$\frac{\partial G^1}{\partial x} + \frac{\partial G^2}{\partial y} + \frac{\partial G^1}{\partial u}\frac{\partial G^2}{\partial v} - \frac{\partial G^1}{\partial v}\frac{\partial G^2}{\partial u} = 0. \tag{39}$$

Plugging (38) and (39) into (8), we obtain that $\mathrm{Ric} = 0$, hence $\mathbf{K} = 0$.

One can also plug the formulas of G^1 and G^2 in (36) and (37) directly into (7) to verify that $R^1_1 = 0$, $R^1_2 = 0$, $R^2_1 = 0$ and $R^2_2 = 0$.

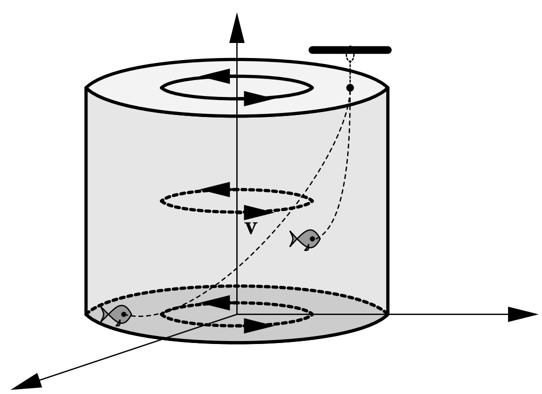
Examples in Dimension Three or Higher We shall only construct a Randers metric in dimension three. One can easily extend it to higher dimensions with a slight modification. Let $\alpha := \sqrt{u^2 + v^2 + w^2}$ denote the canonical Euclidean metric on $\mathbb{R}^3$. Take a vector $\mathbf{v}$ in the cylinder $\Omega := \{(x, y, z) \mid x^2 + y^2 < 1\}$ given by

$$\mathbf{v}_p := (-y, x, 0), \quad p = (x, y, z) \in \Omega.$$

We consider the shortest time problem under the influence of $\mathbf{v}$. Image that the water in a cylindrical fish tank is rotating around the central axis, and the fishes in the tank see the food hanging near the water surface. Each fish has to figure out the path of the shortest time to reach the food. The path of shortest time is the shortest path of the Randers metric, $F = \tilde{\alpha} + \tilde{\beta}$, where

$$\tilde{\alpha} := \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2},$$

$$\tilde{\beta} := -\frac{-yu + xv}{1 - x^2 - y^2}.$$



By a similar argument, we obtain the following formulas for the geodesic coefficients G^i of F,

$$G^1 = -\frac{x(u^2 + v^2 + w^2)}{2(1 - x^2 - y^2)} - \frac{y(xu + yv) - v}{1 - x^2 - y^2}F$$

$$G^2 = -\frac{y(u^2 + v^2 + w^2)}{2(1 - x^2 - y^2)} + \frac{x(xu + yv) - u}{1 - x^2 - y^2}F$$

$$G^3 = 0.$$

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First, one can easily verify that

$$\frac{\partial G^1}{\partial u} + \frac{\partial G^2}{\partial v} + \frac{\partial G^3}{\partial w} = 0.$$

By Proposition 3.2, the volume form of dV_F is given by

$$dV_F = dV_\alpha = dxdydz.$$

By (6), we conclude that $\mathbf{S} = 0$.

Plugging the above formulas into (7), we obtain $R^i_k = 0$. Thus $\mathbf{K} = 0$.

Certainly, one can extend the above Randers metric to higher dimensions with a slight modification. The formulas for geodesic coefficients and the volume form remain same. Thus $\mathbf{S} = 0$.

Since $G^3 = 0$, from (7), we immediately see that $R^3_1 = 0$, $R^3_2 = 0$ and $R^3_3 = 0$. Plugging the formulas of G^1, G^2, G^3 into (7), one can verify that $R^1_1 = 0$, $R^1_2 = 0$, $R^1_3 = 0$, $R^2_1 = 0$, $R^2_2 = 0$ and $R^2_3 = 0$.

References

[AIM] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The theory of sprays and Finsler spaces with applications in physics and biology*. Fund. Theories Phys. **58**, Kluwer Academic Publishers, 1993.

[AZ] H. Akbar-Zadeh, *Sur les espaces de Finsler à courbures sectionnelles constantes.* Acad. Roy. Belg. Bull. Cl. Sci. (5) **74**(1988), 281–322.

[BaChSh] D. Bao, S. S. Chern and Z. Shen, *An introduction to Riemann-Finsler geometry*. Springer, 2000.

[BaMa] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type. A generalization of the notion of Berwald space*. Publ. Math. Debrecen **51**(1997), 385–406.

[BaRo] D. Bao and C. Robles, *On Randers metrics of constant curvature*. Rep. Math. Phys., to appear.

[BaSh] D. Bao and Z. Shen, *Finsler metrics of constant curvature on the Lie group* S^3. J. London Math. Soc., to appear.

[Be] L. Berwald, *Über eine characteristic Eigenschaft der allgemeinen Räume konstanter Krümmung mit gradlinigen Extremalen.* Monatsh. Math. Phys. **36**(1929), 315–330.

[Br1] R. Bryant, *Finsler structures on the 2-sphere satisfying* $K = 1$. In: Finsler Geometry, Contemp. Math. **196**, Amer. Math. Soc., Providence, RI, 1996, 27–42.

[Br2] ———, *Projectively flat Finsler 2-spheres of constant curvature*. Selecta Math. (N.S.) **3**(1997), 161–203.

[Br3] ———, *Finsler manifolds with constant curvature*. Talk at the 1998 Geometry Festival in Stony Brook.

[Ma] M. Matsumoto, *Randers spaces of constant curvature*. Rep. Math. Phys. **28**(1989), 249–261.

[Ra] G. Randers, *On an asymmetric metric in the four-space of general relativity*. Phys. Rev. **59**(1941), 195–199.

[Sh1] Z. Shen, *Differential Geometry of Spray and Finsler Spaces.* Kluwer Academic Publishers, Dordrecht, 2001.

[Sh2] ———, *Funk metrics and R-flat sprays*. Preprint, unpublished.

[Sh3] ———, *On R-quadratic Finsler spaces*. Publ. Math. Debrecen **58**(2001), 263–274.

[Sh4] ———, *Volume comparison and its applications in Riemann-Finsler geometry*. Adv. Math. **128**(1997), 306–328.

[Sh5] ———, *Two-dimensional Finsler metrics with constant curvature*. Manuscripta Math., to appear.

[Sh6] ———, *Projectively flat Randers metrics with constant curvature*. Math. Ann., to appear.

[Sh7] ———, *Lectures on Finsler Geometry*. World Scientific, Singapore, 2001.

[SSAY] C. Shibata, H. Shimada, M. Azuma and H. Yasuda, *On Finsler spaces with Randers' metric.* Tensor (N.S.) **31**(1977), 219–226.

[Sz] Z. I. Szabó, *Positive definite Berwald spaces (structure theorems on Berwald spaces)*. Tensor (N.S.) **35**(1981), 25–39.
[YaSh] H. Yasuda and H. Shimada, *On Randers spaces of scalar curvature*. Rep. Math. Phys. **11**(1977), 347–360.

Math. Dept., IUPUI
402 N. Blackford Street
Indianapolis, IN 46202-3216
USA
email: zshen@math.iupui.edu