UNIVERSALLY MARKETABLE INSURANCE UNDER MULTIVARIATE MIXTURES

BY

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ABSTRACT

The study of desirable structural properties that define a marketable insurance contract has been a recurring theme in insurance economic theory and practice. In this article, we develop probabilistic and structural characterizations for insurance indemnities that are universally marketable in the sense that they appeal to all policyholders whose risk preferences respect the convex order. We begin with the univariate case where a given policyholder faces a single risk, then extend our results to the case where multiple risks possessing a certain dependence structure coexist. The non-decreasing and 1-Lipschitz condition, in various forms, is shown to be intimately related to the notion of universal marketability. As the highlight of this article, we propose a multivariate mixture model which not only accommodates a host of dependence structures commonly encountered in practice but is also flexible enough to house a rich class of marketable indemnity schedules.

KEYWORDS

1-Lipschitz, dependence, indemnity, marketability, mixtures

JEL codes: C02, G22

1. INTRODUCTION

Designing a menu of marketable insurance policies that appeal to policyholders with diverse risk preferences and risk profiles is a strategically challenging but practically significant business decision central to the financial viability of an insurance company. The quest for such policies in turn is closely related to the question of what constitutes widely accepted desirable characteristics of

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indemnities. Common examples of such characteristics include nullity at zero (no payment if there is no loss) and increasing monotonicity with respect to the ground-up loss (the heavier the loss, the more the indemnity). Given these characteristics, optimal (in a certain sense) insurance contracts can be formulated via optimizing an objective functional capturing the interests (e.g., expected utility of terminal wealth, risk measures of loss exposure) of relevant parties (e.g., policyholders and/or insurers) over a set of admissible indemnities possessing the desirable characteristics identified in the first place. The structure of the optimal contracts thus found hinges upon what fundamental properties are imposed on the set of admissible indemnities, with vastly different results obtained for different properties imposed *a priori*. The fact that the structure of the optimal indemnity can differ drastically speaks to the strategic importance of what insurance companies construe as favorable properties of an insurance indemnity.

Among the wide spectrum of feasible insurance indemnity schedules, those with the *non-decreasing and 1-Lipschitz condition* (i.e., those I such that $0 \leq I(x) - I(y) \leq x - y$ for all $y \leq x$), also informally known as the slowly growing condition, have unquestionably gained in popularity in the recent insurance literature. Ensuring that the indemnified loss never increases faster than the policyholder's ground-up loss, the non-decreasing and 1-Lipschitz condition has been used in an abundance of papers as a starting point for formulating optimal (re)insurance policies, which are typically in the form of insurance layers, see, for example, Chi and Tan (2011), Cai *et al.* (2017), Cheung and Lo (2017), and Lo (2017). For all the mathematical benefits it brings, theoretical or empirical justifications of the economic suitability and practical relevance of the non-decreasing and 1-Lipschitz condition to the insurance business have long been lacking, undermining the conceptual foundation of many existing results it underlies. The most common argument in favor of the condition is concerned with the issue of *ex post* moral hazard. By making the policyholder and the insurer both worse off when the ground-up loss becomes heavier, the condition ensures that both parties have a stake in the contract and eliminates their incentives to tamper with losses. In the expected utility paradigm, Young (1999) showed that the indemnity for an optimal insurance contract is necessarily a non-decreasing and 1-Lipschitz function.

As far as the authors are aware, the intimate connections between the non-decreasing and 1-Lipschitz condition and the marketability of indemnities were first formally brought out only recently in Cheung *et al.* (2014) as a natural application of the concept of risk reducers (see also a further study of risk reducers in He *et al.* (2016)). In a standard expected utility setting, Cheung *et al.* (2014) introduced a class of indemnities which are acceptable to all policyholders in the sense that it is always possible to price the indemnity in such a way to raise the expected utility of any given policyholder as well as to cover the expected cost of the insurer. Such indemnities are termed *universally marketable* to recognize the fact that the acceptability by policyholders is universal, which is of particular importance for the insurance business, where it is more

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common for an insurer to provide a fixed menu of indemnity schedules for policyholders of different risk profiles and risk preferences to choose from than to design tailor-made policies for individual policyholders. It is shown in Cheung *et al*. (2014) that an indemnity that is assumed to be non-decreasing *a priori* is universally marketable if and only if it is a 1-Lipschitz function.

Building upon the preliminary study of universal marketability initiated in Cheung *et al*. (2014), in this article we provide probabilistic and structural characterizations for universal marketability in a more general setting and, more importantly, generalize this concept from a single risk to a multivariate framework. A recurring theme is the role played by the non-decreasing and 1-Lipschitz condition and its multivariate counterparts in making an indemnity marketable to as many policyholders as possible. In the first part of the article, we extend the definition of universal marketability from Cheung *et al*. (2014)'s standard expected utility framework to a more general one, where policyholders are allowed to assess their risk level using any desired risk functional that preserves the convex order. The resulting decision-making framework encompasses preferences dictated by expected utilities as well as many other common risk measures in practice. Armed with this generalized definition of universal marketability, we formulate necessary and sufficient conditions for an indemnity to be universally marketable. It is established that a universally marketable indemnity is *universally risk-reducing* in the sense that it reduces a policyholder's risk with respect to convex order irrespective of his/her risk profile. Structure-wise, such an indemnity is a non-decreasing and 1-Lipschitz function of the ground-up loss. These probabilistic and algebraic properties cast light on the practical attractiveness of a universally marketable indemnity, lend theoretical support to the many works that hinge on the non-decreasing and 1-Lipschitz condition, and pave the way for the study of multivariate universally marketable indemnities in the later part of the article.

On the basis of our univariate results, we extend the notion of universal marketability to a multivariate framework, where policyholders are exposed to multiple losses (such as in multiple-peril insurance) and are allowed to purchase a multivariate indemnity from a given menu of policies, and characterize multivariate universally marketable indemnities. Such a multivariate extension is not only practically significant because of its intrinsic connections to portfolios of losses, but also mathematically challenging because of the intricate dependence between the different losses at work. We are particularly interested in how the non-decreasing and 1-Lipschitz condition should be appropriately modified for multivariate functions. As the first step, we pursue the extension in the universal losses setting which accommodates multivariate losses of arbitrary marginal distributions and dependence structures. It turns out that for a multivariate indemnity to be universally marketable, it is necessary and sufficient that it is a non-decreasing and 1-Lipschitz function of the sum of its arguments. That the indemnity is essentially univariate strips the problem of its inherent multivariate character, excludes many feasible multivariate indemnities, and arises from the presence of strongly negatively dependent losses

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which rarely occur in the typical insurance business. To eliminate such pathological cases with minimal relevance to the insurance business, we introduce, as the highlight of this article, a mixture structure which not only provides a unifying treatment of a wide variety of commonly used dependence structures but also effectively captures policyholders' exposure to systematic and systemic risks. It is shown within the mixture structure that a multivariate indemnity is universally marketable if and only if it is universally risk-reducing (as in the univariate setting), and if and only if it is a non-decreasing and *componentwise* 1-Lipschitz multivariate function. These results suggest that the mixture structure strikes a reasonable balance between accommodating portfolio losses of different kinds of dependence structures and allowing for a diversity of multivariate indemnities and have useful implications for the design of optimal multivariate insurance contracts, the study of which is still in a nascent stage.

The rest of this article is organized as follows. In Section 2, we put forward a general definition of universal marketability and present the accompanying probabilistic and structural characterizations, for both the univariate (Section 2.1) and multivariate (Section 2.2) settings. Section 3, the highlight of this article, formulates the multivariate mixture model, within which universally marketable indemnities are investigated and characterized as non-decreasing and componentwise 1-Lipschitz functions. Finally, Section 4 concludes the paper. To maximize the readability of this article, all proofs are deferred to the appendix.

2. UNIVERSALLY MARKETABLE INDEMNITIES

Throughout this article, all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite expectations and are interpreted as loss-profit variables, with positive values corresponding to losses and negative values corresponding to profits. An indemnity is identified with a real-valued function $I : \mathbb{R}^n \to \mathbb{R}$ for some positive integer n so that $I(\mathbf{x})$ is the compensation that the policyholder will receive from the insurer if $\mathbf{x}$ is the realization of the ground-up loss, which is a random vector with n components (when $n = 1$, the ground-up loss is simply a random variable). For the purpose of comparing the variability of different losses, the convex order is utilized. For any two random variables X and Y, we say that X is smaller than Y in *convex order*, denoted by $X \leq_{\text{cx}} Y$, if $\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)]$ holds for any convex function v such that the expectations exist. We also recall that $X \leq_{\text{cx}} Y$ is equivalent to any (and thus both) of the following statements:

- $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)]$ for any non-decreasing convex function v such that the expectations exist;
- $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{E}\left[(X-d)_+\right] \leq \mathbb{E}\left[(Y-d)_+\right]$ for any $d \in \mathbb{R}$, in which $(\cdot)_+$ is the positive part function defined by $x_+ = \max\{x, 0\}$.

For more information about convex order, we refer the reader to Müller and Stoyan (2002), Denuit *et al.* (2005), and Shaked and Shanthikumar (2007).

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2.1. The univariate case

We begin by formalizing and generalizing the notion of universal marketability first proposed in Cheung *et al.* (2014) in the case of a single loss. This notion plays a central role throughout this article.

Definition 2.1. *An insurance indemnity function* $I:\mathbb{R}\to\mathbb{R}$ *is said to be* universally marketable *if for any integrable random variable* X[1] *and any risk functional* π *that is defined on the space of integrable random variables and preserves the convex order (i.e.,* $Y_1 \leq_{\text{cx}} Y_2$ *implies* $\pi(Y_1)\leq\pi(Y_2)$*), we can find a premium* $P\geq\mathbb{E}[I(X)]$ *(dependent on* X *and* π*) such that*

$$\pi(X-I(X)+P)\leq\pi(X). \tag{2.1}$$

If we take $\pi(\cdot)=-\mathbb{E}[u(-\cdot)]$, where u is an arbitrary non-decreasing and concave utility function, then we retrieve Cheung *et al.* (2014)'s definition confined to the expected utility paradigm.

It follows by definition that under a universally marketable indemnity, a premium can always be found such that every risk-averse risk bearer, regardless of his/her risk preference (quantified by his/her convex-order-preserving risk functional π) and risk profile (represented by the loss X he/she faces), will be better off in the sense that he/she enjoys a lower level of risk[2] as a result of the purchase of insurance. In the insurance literature, this is commonly known as the participation constraint (the policyholder has the incentive to "participate" in the insurance contract). The qualifier "universal" is affixed to acknowledge the fact that the insurance is marketable to all risk-averse policyholders in the economy. Meanwhile, the insurer, who relies on the law of large numbers to diversify risks and is, for all intents and purposes, risk-neutral (see, e.g., page 45 of Eeckhoudt *et al.* (2005)), also finds the indemnity acceptable[3] with a premium that is sufficient for covering the expected indemnified loss.

It is often difficult to verify whether or not a given indemnity is universally marketable directly by definition. In this regard, Theorem 2.2 below presents two characterizations of universal marketability that not only point out the connections of universally marketable indemnities to risk management, but also provide them with an easily verifiable structural description. These characterizations provide a convenient framework for developing the results in the remainder of this article.

Theorem 2.2. *Let* $I:\mathbb{R}\to\mathbb{R}$ *be an insurance indemnity function. The following statements are equivalent:*

(a) *I is universally marketable;*
(b) *I is* universally risk-reducing *with respect to convex order, that is, for any integrable loss* X,[4]

$$X-I(X)\leq_{\text{cx}} X-\mathbb{E}[I(X)];$$

(c) *I is non-decreasing and 1-Lipschitz, that is,* $0\leq I(x)-I(y)\leq x-y$ *for all* $y\leq x$.

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Theorem 2.2 above integrates and generalizes several key results of Cheung *et al*. (2014). Not only is it liberated from the expected utility paradigm to a more general setting, but it also does not require that non-decreasing monotonicity be imposed *a priori* on the indemnity function in question, but shows that this mild property of the indemnity is a by-product of its universal marketability. Furthermore, our proof is self-contained and elementary without recourse to the notion of risk reducers studied in Cheung *et al*. (2014).

The two characterizations given in Theorem 2.2 are of independent interest and shed different light on what it takes for an indemnity to be universally marketable. Referred to as the *universally risk-reducing property*, Statement (b) compares the variability of the policyholder's loss before the purchase of insurance with that after the purchase of insurance with respect to convex order and asserts that a universally marketable indemnity always lowers the risk a policyholder faces irrespective of the type of risk he/she bears. This is a mathematical manifestation of the universal appeal of such an indemnity. Technically, the universal risk-reducing property is equally useful in that it reveals the links between universal marketability and convex order and allows us to analyze universally marketable indemnities using established tools in the theory of stochastic ordering.

While Statement (b) offers a risk management interpretation of universal marketability, Statement (c) translates the notion equivalently into the non-decreasing monotonicity and 1-Lipschitzity of an indemnity. Such a structural description is of practical importance because it showcases the concrete form of universally marketable indemnities and significantly eases their identification. For example, stop-loss contracts $I_1(x) = (x - d)_+,\ d \in \mathbb{R}$, quota-share contracts $I_2(x) = ax,\ a \in [0, 1]$, and insurance layer contracts $I_3(x) = (x \wedge u - d)_+,\ d \leq u$ are all commonly used examples of universally marketable indemnities because the corresponding indemnity functions are easily seen to be non-decreasing and 1-Lipschitz. Intuitively, the 1-Lipschitz condition can be viewed as a kind of size constraint that avoids the possibility of over-insurance, that is, the situation that the purchase of a disproportionate amount of insurance backfires and increases rather than decreases risk.

2.2. The multivariate case

In this subsection, we turn to multivariate insurance indemnities, which are written on a portfolio of multiple losses, and strive to extend Theorem 2.2 by characterizing universally marketable indemnities in a multivariate setting. Such an extension is not as straightforward as it seems at first sight and involves more than cosmetic adjustments due to two significant challenges:

1. There are several possible notions of the non-decreasing and 1-Lipschitz condition for multivariate functions.
2. There is a need to take into account the dependence structure between the different losses in a given portfolio. This issue is absent in the univariate

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framework. As we will see, the characterizations of multivariate universally marketable indemnities vary critically with the dependence structure governing the random losses under consideration.

We first put forward the definition of a multivariate universally marketable indemnity. To avoid over-generalizations, we have relaxed the requirement of being universally marketable to being marketable only among a collection of random vectors.

Definition 2.3. *Let $I:\mathbb{R}^n \to \mathbb{R}$ be a multivariate insurance indemnity function and $\mathcal{X}$ be a collection of integrable n-dimensional random vectors. Then I is said to be* universally marketable over $\mathcal{X}$ *if for any $\mathbf{X}=(X_1,\ldots,X_n)^\top \in \mathcal{X}$ and any risk functional π that is defined on $\mathcal{X}$ and preserves the convex order, we can find a premium $P \geq \mathbb{E}[I(\mathbf{X})]$ such that*

$$\pi\left((X_1+\cdots+X_n)-I(\mathbf{X})+P\right) \leq \pi(X_1+\cdots+X_n). \tag{2.2}$$

Two remarks about this multivariate definition are in order:

1. We have chosen to work with the sum $X_1+\cdots+X_n$, arguably the most commonly used aggregating functional, due to its natural financial interpretation as the total loss borne by a policyholder as well as its mathematical tractability. Apart from the replacement of the single loss X to the aggregate loss $X_1+\cdots+X_n$, the mutually acceptable spirit of the multivariate definition parallels that in Definition 2.1, that is, a premium for the indemnity can be set in a way attractive to a wide variety of policyholders as well as the insurer.
2. Definition 2.3, unlike Definition 2.1, is stated in terms of a collection of random vectors rather than the whole collection of random vectors. This collection of random vectors has a large role to play in characterizing universal marketability in the multivariate setting. Naturally, the larger this collection, the more restrictive the requirement of being universally marketable becomes. We will see this more clearly in the rest of this article.

In the single-risk setting, it has been established in Theorem 2.2 that the universal marketability of a univariate insurance indemnity is equivalent to the universally risk-reducing property as well as the non-decreasing and 1-Lipschitz condition. Proposition 2.4 below shows that the first equivalence continues to hold true in the multiple-risk setting. Its proof can be given by going along the same line in the corresponding part of the proof of Theorem 2.2, and we omit it here for brevity.

Proposition 2.4. *Let $I:\mathbb{R}^n \to \mathbb{R}$ be a multivariate insurance indemnity function and $\mathcal{X}$ be a collection of integrable random vectors. The following statements are equivalent:*

(a) I is universally marketable over $\mathcal{X}$;

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(b) I is universally risk-reducing over $\mathcal{X}$ with respect to convex order, that is,

$$(X_1 + \cdots + X_n) - I(\mathbf{X}) \leq_{\mathrm{cx}} (X_1 + \cdots + X_n) - \mathbb{E}[I(\mathbf{X})], \tag{2.3}$$

for any random vector $\mathbf{X} = (X_1, \ldots, X_n)^\top \in \mathcal{X}$.

To develop the multivariate counterpart of Statement (c) in Theorem 2.2 requires the extension of the non-decreasing and 1-Lipschitz condition for multivariate functions, which is a more delicate issue. To this end, consider the following three sets of multivariate indemnities, all of which generalize the univariate non-decreasing and 1-Lipschitz condition in one way or another. For any generic vector $\mathbf{x} = (x_1, \ldots, x_n)^\top$ in $\mathbb{R}^n$, define

$$\begin{aligned}
\mathcal{I}_1 &= \left\{ I : \mathbb{R}^n \to \mathbb{R} \;\middle|\; I(\mathbf{x}) \text{ is non-decreasing and 1-Lipschitz in } \sum_{k=1}^{n} x_k \right\}, \\
\mathcal{I}_2 &= \{ I : \mathbb{R}^n \to \mathbb{R} \mid I(\mathbf{x}) = I_1(x_1) + \cdots + I_n(x_n), \text{ where each } I_k \text{ is univariate} \\
&\qquad\qquad \text{non-decreasing and 1-Lipschitz for } k = 1, \ldots, n \}, \\
\mathcal{I}_3 &= \{ I : \mathbb{R}^n \to \mathbb{R} \mid I \text{ is non-decreasing and componentwise 1-Lipschitz} \}.
\end{aligned}$$

Among these three sets, both $\mathcal{I}_1$ and $\mathcal{I}_2$ are proper subsets of $\mathcal{I}_3$; for example, for $d \in \mathbb{R}$ and $n = 2$, $I(x, y) := (x + y/2 - d)_+ \in \mathcal{I}_3 \setminus (\mathcal{I}_1 \cup \mathcal{I}_2)$. Indemnities in $\mathcal{I}_1$, each as a function of the sum of its arguments, are essentially univariate, meaning that the insurer is hedging against the aggregate loss it faces. Indemnities in $\mathcal{I}_2$ arise when the insurer hedges each loss on a standalone basis with the use of univariate non-decreasing and 1-Lipschitz functions. $\mathcal{I}_3$ is the largest set, where the non-decreasing and 1-Lipschitz condition plays its role on a componentwise basis. For both theoretical and practical reasons, it is desirable to have a set of indemnities that not only includes as many commonly used insurance policies in practice as possible but is also marketable on as large a collection of random losses as possible. In this subsection and the next section, we shall show that each of these three sets produces marketable indemnities within a certain collection of random losses.

Arguably the most natural conjecture when generalizing Statement (c) of Theorem 2.2 to the multiple-risk setting is to impose the non-decreasing and 1-Lipschitz condition on a multivariate indemnity function I in a componentwise fashion, that is, we replace Statement (c) in Theorem 2.2 by

(c)′ I is a non-decreasing and componentwise 1-Lipschitz function, that is, $I \in \mathcal{I}_3$.

While this condition is necessary for a multivariate indemnity function to be universally marketable, it turns out to be not sufficient. Here is a simple counter-example.

Example 2.5. Let $n = 2$, $X_1 = Z$ and $X_2 = -Z/2$, where Z is a standard normal random variable. Define $I(x_1, x_2) = x_1/4 + x_2$, which is a non-decreasing and componentwise 1-Lipschitz function. However,

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$$X_1 + X_2 - I(X_1, X_2) = \frac{3Z}{4} \nleq_{cx} \frac{Z}{2} = X_1 + X_2 - \mathbb{E}[I(X_1, X_2)].$$

By Proposition 2.4, I is not universally marketable.

Note that the two losses X_1 and X_2 in Example 2.5 are, by construction, counter-monotonic. Here X_2 may be interpreted as a strategy adopted by the policyholder to hedge against X_1. The given indemnity function, which entails a high coverage on X_2 and a low coverage on X_1, turns out to weaken the risk-reducing capacity of X_2 as a hedging strategy. This explains the increase of risk faced by the policyholder after the insurance is purchased. To ensure that relation (2.3) holds for all portfolios of random losses, including those with extreme negative dependence structures, Theorem 2.6 below shows that the non-decreasing and componentwise 1-Lipschitz condition has to be strengthened to the condition that the indemnity is a non-decreasing and 1-Lipschitz function of the aggregate loss. For completeness, this result is stated in the form of a full characterization of universal marketability in the multivariate framework.

Theorem 2.6. *Let $I : \mathbb{R}^n \to \mathbb{R}$ be a multivariate indemnity function. The following statements are equivalent:*

(a) I is universally marketable (over the set of all integrable random vectors);
(b) I is universally risk-reducing with respect to convex order, that is,

$$(X_1 + \cdots + X_n) - I(\mathbf{X}) \leq_{\text{cx}} (X_1 + \cdots + X_n) - \mathbb{E}[I(\mathbf{X})], \tag{2.4}$$

for any integrable random vector $\mathbf{X} = (X_1, \ldots, X_n)^\top$;
(c) $I(\mathbf{x})$ is a non-decreasing and 1-Lipschitz function of $\sum_{k=1}^n x_k$ for any $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$, that is, $I \in \mathcal{I}_1$.

The message from Theorem 2.6 is that a multivariate universally marketable indemnity must be constructed through the aggregate loss and is essentially no different from a univariate strategy. The cause of this unanticipated phenomenon is the need to accommodate losses with all possible dependence structures, including extreme negative dependence structures which are mainly in a theoretical vacuum of little practical interest.

In a similar spirit, if we restrict *a priori* attention to only additively separable indemnities, meaning that they can be written as the sum of univariate indemnities, that is, $I(\mathbf{x}) = \sum_{k=1}^n I_k(x_k)$, and consider universally marketable indemnities within this set in the hope that $\mathcal{I}_2$ is the appropriate set, then the answer is again unexpected: The univariate indemnities are all constrained to be coinsurance strategies with the same quota share.

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Corollary 2.7. *Let $I:\mathbb{R}^n \to \mathbb{R}$ be a multivariate insurance indemnity function such that $I(\mathbf{x}) = I_1(x_1) + \cdots + I_n(x_n)$ for some univariate functions $I_1, \ldots, I_n$ and $I_k(0) = 0$ for all $k = 1, \ldots, n$. The following statements are equivalent:*

(a) I is universally marketable (over the set of all integrable random vectors);
(b) I is universally risk-reducing with respect to convex order, that is,

$$(X_1 + \cdots + X_n) - I(\mathbf{X}) \leq_{\mathrm{cx}} (X_1 + \cdots + X_n) - \mathbb{E}[I(\mathbf{X})],$$

for any integrable random vector $\mathbf{X} = (X_1, \ldots, X_n)^\top$;
(c) Each I_k, with $k = 1, \ldots, n$, takes the form $I_k(x) = cx$ for some common constant $c \in [0, 1]$.

If we confine our study of multivariate universally marketable indemnities to dependence structures which arise naturally in practical situations, then we may expect that the resulting collection of indemnity policies will be larger and include genuinely multivariate functions which insurance companies may provide in their policy menus. This will be confirmed in the next section.

3. Universally marketable insurance under mixtures

In view of the somewhat disappointing results in Section 2.2, in this section we introduce a mixture model which accommodates a wide variety of dependence structures commonly encountered in practice. We show that when the universal marketability requirement is imposed only on losses within the mixture model, it is necessary and sufficient that the multivariate insurance indemnity concerned is a non-decreasing and componentwise 1-Lipschitz function, which is a substantially more general condition than being a non-decreasing and 1-Lipschitz function of the sum of its arguments.

3.1. The mixture structure

In an attempt to develop a rich class of reasonable dependence structures, we reason that insurance losses from different lines of business in practice coexist in a stochastic environment that is vulnerable to changes in universal macroeconomic factors and impacts of economy-wide exogenous events. These losses are correlated in such a way that they are governed by certain common risk factors while subject to their own idiosyncratic risk factors. Motivated by these considerations, we introduce the following *mixture structure* for an n-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_n)^\top$:

$$\mathbf{X} = \mathbf{Y} + G(\mathbf{Z}). \tag{3.1}$$

In this model, $\mathbf{Y} = (Y_1, \ldots, Y_n)^\top$ and $\mathbf{Z} = (Z_1, \ldots, Z_m)^\top$ are two risk vectors, with the components $Y_1, \ldots, Y_n, Z_1, \ldots, Z_m$ assumed to be mutually independent, and $G:\mathbb{R}^m \to \mathbb{R}^n$ is a function assumed to satisfy $G(\mathbf{z_1}) \leq G(\mathbf{z_2})$ for any $\mathbf{z_1}, \mathbf{z_2} \in \mathbb{R}^m$ with $\mathbf{z_1} \leq \mathbf{z_2}$, where the order $\leq$ is understood componentwise.

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The mixture structure (3.1) is a flexible modeling vehicle on both actuarial and financial grounds. From an actuarial perspective, by suitably specifying the function G and risk vectors $\mathbf{Y}$ and $\mathbf{Z}$, we can easily achieve a wide variety of commonly used dependence structures including mutual independence (by setting G to the zero function), comonotonicity (by assuming that $m = 1$ and $\mathbf{Y}$ is deterministic; see Dhaene *et al.* (2002) and Puccetti and Scarsini (2010) for more information about comonotonicity), and hybrid structures with positive dependence (by assuming that $\mathbf{Y}$ and $\mathbf{Z}$ are non-deterministic and G is non-zero). The use of (3.1) therefore offers a unifying treatment of universal marketability for random losses governed by various positive dependence structures.

From a financial perspective, structure (3.1) also lends itself to modeling idiosyncratic and common risk factors. On a componentwise basis, each individual loss X_k can be decomposed into two parts:

$$X_k = Y_k + G_k(\mathbf{Z}), \qquad k = 1, \ldots, n,$$

in which the first part Y_k represents X_k's idiosyncratic risk (which, by definition, is specific to X_k and independent of other losses) while the second part $G_k(\mathbf{Z})$, the kth component of $G(\mathbf{Z})$, captures the impacts of the risk vector $\mathbf{Z}$, which is common across all of the n random losses and induces their interdependence. The common risk factors hosted by $\mathbf{Z}$ can be configured to encompass different sources (hence the independence of $Z_1, \ldots, Z_m$) of systematic as well as systemic risks, which are the main drivers of the interdependence among losses a policyholder faces. Systematic risk is inherent in the market and cannot be eliminated through diversification, whereas systemic risk is the possibility of the collapse of the entire financial system due to certain external events (a typical example being the bankruptcy of Lehman Brothers in 2008). Mathematically, the function G quantifies the impact of $\mathbf{Z}$ on the individual losses and can be designed to retrieve many models of practical interest. In the important special case where each Z_j impacts on X_k on a standalone basis so that G can be identified with an $n \times m$ matrix of non-decreasing deterministic functions g_{kj} for $k = 1, \ldots, n$ and $j = 1, \ldots, m$, the mixture structure (3.1) reduces to the additive model

$$X_k = Y_k + g_{k1}(Z_1) + \cdots + g_{km}(Z_m),$$

where the functions g_{kj} can be further specified to reflect the loadings of the individual losses to the common risk factors. In this form, the mixture model covers, for instance, the following linear factor model as a special case upon rescaling:

$$X_k = \rho_k Y_k + \sqrt{1 - \rho_k^2} \boldsymbol{\beta}_k^\top \mathbf{Z}, \qquad k = 1, \ldots, n,$$

where $\rho_k \in (0, 1)$ is a deterministic coefficient adjusting the relative weights of idiosyncratic and common risk factors, while $\boldsymbol{\beta}_k \in \mathbb{R}_+^m$ is a deterministic loading vector capturing the sensitivity of X_k to the common risk vector $\mathbf{Z}$. In finance

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in general and in credit risk analysis in particular, the loading to the systematic risk factor is often termed the market beta. See Chapter 9 of Cochrane (2005) for the applications of this classic model in financial asset pricing.

3.2. Main results

Theorem 3.1 presents a complete characterization of universal marketability within the mixture structure (3.1).

Theorem 3.1. *Let $I:\mathbb{R}^n \to \mathbb{R}$ be a multivariate indemnity function. Restricted to the mixture structure* (3.1)*, the following statements are equivalent:*

(a) I is universally marketable;
(b) I is universally risk-reducing with respect to convex order, that is,

$$(X_1 + \cdots + X_n) - I(\mathbf{X}) \le_{\mathrm{cx}} (X_1 + \cdots + X_n) - \mathbb{E}[I(\mathbf{X})], \tag{3.2}$$

for any integrable random vector $\mathbf{X} = (X_1, \ldots, X_n)^\top$ of form (3.1)*;*
(c) I is a non-decreasing and componentwise 1-Lipschitz function, that is, $I \in \mathcal{I}_3$.

Comparing Theorems 2.6 and 3.1, we conclude that when the set $\mathcal{X}$ in Definition 2.3 is relaxed from the set of all integrable random losses to the set of random losses lying in the mixture model (3.1), the permissible multivariate indemnity functions are no longer confined to be essentially univariate functions non-decreasing and 1-Lipschitz in the sum of the arguments ($\mathcal{I}_1$), but are allowed to be genuinely multivariate functions which only have to be non-decreasing and componentwise 1-Lipschitz ($\mathcal{I}_3$). Given the versatility of the mixture model as discussed above, this enlargement of the set of permissible indemnity functions is mathematically and practically important. By excluding pathological dependence structures that are of little practical interest, we have established a much wider collection of marketable multivariate indemnity functions which insurance companies can profitably offer and which can serve as the set of candidate functions for subsequent investigation in optimal multivariate insurance studies.

In parallel with Corollary 2.7, if we only consider multivariate indemnity functions that can be written as the sum of univariate indemnity functions, then an equivalent condition for this multivariate function I to be universally marketable within the mixture structure is that I is a member of $\mathcal{I}_2$, that is, each individual univariate indemnity function is non-decreasing and 1-Lipschitz.

Corollary 3.2. *Let $I:\mathbb{R}^n \to \mathbb{R}$ be a multivariate insurance indemnity function such that $I(\mathbf{x}) = I_1(x_1) + \cdots + I_n(x_n)$ for some univariate functions $I_1, \ldots, I_n$. Restricted to the mixture structure* (3.1)*, the following statements are equivalent:*

(a) I is universally marketable;

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(b) I is universally risk-reducing with respect to convex order, that is,

$$(X_1 + \cdots + X_n) - I(\mathbf{X}) \leq_{\mathrm{cx}} (X_1 + \cdots + X_n) - \mathbb{E}[I(\mathbf{X})], \tag{3.3}$$

for any integrable random vector $\mathbf{X} = (X_1, \ldots, X_n)^\top$ *of form* (3.1)*;*

(c) Each I_k *is a non-decreasing and 1-Lipschitz function, that is,* $I \in \mathcal{I}_2$*.*

Corollary 3.2 can be easily proved by applying Theorem 3.1 and the simple fact that $I(\mathbf{x}) = \sum_{k=1}^n I_k(x_k)$ is non-decreasing and componentwise 1-Lipschitz if and only if each I_k is non-decreasing and 1-Lipschitz. Hence, we omit the proof here.

We remark that in the special but important case when $\mathbf{X}$ is a random vector with comonotonic or independent components, the implication (c) $\Rightarrow$ (b) of Corollary 3.2 can be proved in a much simpler way without relying on Theorem 3.1. Notice that when $I \in \mathcal{I}_2$, each I_k is non-decreasing and 1-Lipschitz. By Theorem 2.2, for each X_k, we have $X_k - I_k(X_k) \leq_{cx} X_k - \mathbb{E}[I_k(X_k)]$. Then relation (3.3) can be established by the additive property of convex order. For comonotonic random vectors, the additive property is proved in Corollary 1 of Dhaene *et al.* (2002), while for random vectors with independent components, the additive property is proved in Theorem 3.A.12 of Shaked and Shanthikumar (2007).

A result in the same vein as Corollary 3.2 for a different class of random vectors can be formulated as an application of Theorem 3.4 of Cai and Wei (2012). We first recall the concept of positive dependence through stochastic ordering. A random vector $\mathbf{X}$ is said to be positively dependent through stochastic ordering (PDS) if, for $k = 1, \ldots, n$,

$$\mathbb{E}[f(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n) | X_k = x_k],$$

is non-decreasing in x_k for any non-decreasing function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that the above conditional expectation exists. For more discussions on PDS and other related positive dependence structures, we refer the reader to Block *et al.* (1985), Section 2.1 of Joe (1997), and Section 3.10 of Müller and Stoyan (2002). In their Theorem 3.4, Cai and Wei (2012) showed that for a PDS random vector $\mathbf{X}$, if there exist non-decreasing functions g_k and h_k such that $g_k(X_k) \leq_{cx} h_k(X_k)$ for each $k = 1, \ldots, n$, then $\sum_{k=1}^n g_k(X_k) \leq_{cx} \sum_{k=1}^n h_k(X_k)$. Consider a multivariate indemnity function of the form $I(\mathbf{x}) = \sum_{k=1}^n I_k(x_k)$. We first assume that $I \in \mathcal{I}_2$ and define non-decreasing functions $g_k = \mathrm{Id} - I_k$ and $h_k = \mathrm{Id} - \mathbb{E}[I_k(X_k)]$ for each k, in which Id represents the identity function. By Theorem 2.2 above and Theorem 3.4 of Cai and Wei (2012), it is not hard to see that relation (3.3) holds for any PDS random vector $\mathbf{X}$. To show the converse, for each k, we follow the proof of Theorem 2.6 to set all elements in $\mathbf{X}$ but X_k to be deterministic and apply Theorem 2.2. Consequently, relation (3.3) holds for any PDS random vector $\mathbf{X}$, or, equivalently, the indemnity $I(\mathbf{x}) = \sum_{k=1}^n I_k(x_k)$ is universally marketable among all n-dimensional PDS random vectors if and only if each univariate indemnity function I_k is non-decreasing and 1-Lipschitz.

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Remark 3.3. *In this article, we analyze multivariate universally marketable indemnity schedules by extending the non-decreasing and 1-Lipschitz condition to the multivariate case. There are alternative multivariate extensions that could be explored, one of which is by means of the notion of comonotonic allocations. To see this, observe from Theorem 2.2 that a univariate indemnity is universally marketable if and only if the random vector $(X - I(X), I(X))^\top$ is comonotonic for any integrable loss X. In the language of bilateral risk-sharing problems, $(X - I(X), I(X))^\top$ is a feasible allocation of the aggregate risk X, and $X - I(X)$ and $I(X)$ are the policyholder's and insurer's allocations, respectively. Thus, one way to characterize a univariate universally marketable indemnity is that it gives rise to a comonotonic allocation regardless of the risk borne by the policyholder.*

In a similar spirit, one may pursue a characterization of multivariate universal marketability in the following form:

> *Let $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$ be a multivariate indemnity function and $\mathcal{X}$ be a collection of integrable random vectors. The following statements are equivalent:*
>
> *(a) I is universally marketable over $\mathcal{X}$.*
> *(d) The allocation $(X_1 + \cdots + X_n - I(\mathbf{X}), I(\mathbf{X}))^\top$ is comonotonic for any $\mathbf{X} = (X_1, \ldots, X_n)^\top \in \mathcal{X}$.*[5]

We distinguish two cases.

- *When $\mathcal{X}$ is the entire collection of integrable random vectors, the two statements are indeed equivalent. By Theorem 2.6, it is enough to prove that (d) is true if and only if:*

  (c) I is a non-decreasing and 1-Lipschitz function of the sum of its arguments.

  The implication (c) $\Rightarrow$ (d) is obvious. Now assume (d). If I is not a function of the sum of its arguments, then there exist $\mathbf{x}_1 = (x_{11}, \ldots, x_{1n})^\top \in \mathbb{R}^n$ and $\mathbf{x}_2 = (x_{21}, \ldots, x_{2n})^\top \in \mathbb{R}^n$ such that $\sum_{k=1}^n x_{1k} = \sum_{k=1}^n x_{2k}$ but $I(\mathbf{x}_1) \neq I(\mathbf{x}_2)$. This violates (d) and so $I(\mathbf{x})$ must be a function of $\sum_{k=1}^n x_k$ for all $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$. Then the comonotonicity of $\left(\sum_{k=1}^n X_k - I(\sum_{k=1}^n X_k), I(\sum_{k=1}^n X_k)\right)^\top$ for all integrable $X_1, \ldots, X_n$ implies that (indeed, is equivalent to the statement that) I is non-decreasing and 1-Lipschitz in $\sum_{k=1}^n x_k$.
- *When $\mathcal{X}$ is the collection of random vectors that follow the mixture structure (3.1), the equivalence between (a) and (d) no longer holds. In fact, the same proof above can be used to show that the following statements are equivalent:*

  (c) I is a non-decreasing and 1-Lipschitz function of the sum of its arguments.
  (d)$'$ The allocation $(X_1 + \cdots + X_n - I(\mathbf{X}), I(\mathbf{X}))^\top$ is comonotonic for all $\mathbf{X}$ within the mixture structure.

  Since (c) is much stronger than the non-decreasing and componentwise 1-Lipschitz condition in Theorem 3.1 (c) and essentially strips I of its

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multivariate character, studying multivariate universal marketability via the comonotonicity of $(X_1 + \cdots + X_n - I(\mathbf{X}), I(\mathbf{X}))^\top$ *may not produce fruitful results. That is why in this article, we choose to characterize universal marketability (univariate and multivariate) in terms of the non-decreasing and 1-Lipschitz condition, which is a simple, concrete, and easily verifiable structural property describing the behavior of an indemnity function.*

4. Concluding remarks

This paper provides a quantitative treatment of universally marketable indemnities, which not only appeal to any policyholders whose risk preferences respect the convex order, but are also profitable to insurers relying on the law of large numbers to evaluate insurance premiums. We start with liberating the definition of universal marketability proposed in Cheung *et al.* (2014) from the standard expected utility framework to a more general setting which only assumes that policyholders' risks are quantified by a convex-order-preserving functional. Probabilistic and structural characterizations of universal marketability are then developed. In the single-risk setting, it is shown that an insurance indemnity is universally marketable if and only if it is a non-decreasing and 1-Lipschitz function. In the multivariate framework, however, the need to accommodate risk vectors with arbitrary dependence structures makes universally marketable indemnities confined to functions that are non-decreasing and 1-Lipschitz in the sum of the arguments, meaning that the policyholder can only hedge against his/her risks by purchasing insurance on the ground-up loss. As the highlight of this article, we propose a multivariate mixture structure which effectively captures policyholders' exposure to systematic and systemic risks and considerably enlarges the set of multivariate universally marketable indemnities. Within this mixture structure, we show that the necessary and sufficient condition for a multivariate insurance indemnity to be universally marketable can be relaxed to non-decreasing monotonicity and componentwise 1-Lipschitzity.

We end this paper by discussing two future research directions.

1. In this paper, we assume that policyholders' risk preferences are quantified by convex-order-preserving functionals. This implies that policyholders are risk-averse (in the sense of strong risk aversion) and excludes functionals that are not law-invariant such as robust shortfall, general (non-law-invariant) coherent risk measures, and general convex risk measures. It will be of both theoretical and practical interest to extend the analysis of this paper to such risk functionals, although the concomitant technical challenges are likely significant.
2. As an application of Theorem 3.1, one may formulate the following multivariate optimal insurance problem for a policyholder:

$$\inf_{I \in \mathcal{I}_3} \pi\left(X_1 + \cdots + X_n - I(\mathbf{X}) + P\right), \tag{4.1}$$

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where the loss vector $\mathbf{X} = (X_1, \ldots, X_n)^\top$ is of the mixture structure (3.1), π is the convex-order-preserving risk functional adopted by the policyholder, and P (dependent on $I(\mathbf{X})$) is the premium charged by the insurer. Theorem 3.1 guarantees that every $I \in \mathcal{I}_3$ is universally marketable in the sense that P can be calibrated in a way acceptable to the insurer and the risk faced by the policyholder drops following the purchase of I. We leave Problem (4.1) as a future research problem.

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Notes

1. In Definition 2.1, we could have defined univariate universal marketability over a class of random variables to be prudent and gradually make this class bigger and bigger. In the light of the results in Cheung *et al*. (2014), such a restriction is unnecessary in the univariate case, but this approach is warranted in the multivariate case.
2. In inequality (2.1), we are comparing the risk exposure of a policyholder before and after the purchase of insurance without taking into account his/her initial wealth. This does not lead to any loss of generality. To account for the initial wealth w explicitly, one can consider the translated risk functional $\pi^w(Y) := \pi(Y - w)$, which quantifies the risk exposure net of the initial wealth, and all the results in this paper can be stated equivalently in terms of π^w.
3. A universally marketable indemnity makes a given policyholder and the insurer both better off, but it may not be a Pareto-optimal indemnity.
4. Although Theorem 2.2 (b) can be written equivalently as $X - I(X) + \mathbb{E}[I(X)] \leq_{\mathrm{cx}} X$, it should be pointed out that the premium charged by the insurer is, according to the definition of universal marketability, some value larger than or equal to $\mathbb{E}[I(X)]$, not necessarily $\mathbb{E}[I(X)]$. The term $\mathbb{E}[I(X)]$ is there just to ensure that both sides have the same expected value, a precondition for the use of convex order.
5. We use (d) to distinguish this statement from the three statements in Theorems 2.6 and 3.1.

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APPENDIX A. PROOFS

A.1. Proof of Theorem 2.2

To see that (a) implies (b), we fix any integrable random variable X and take $\pi(X) = \mathbb{E}[v(X)]$ for any fixed non-decreasing convex function v (such that all expectations that follow are well-defined). By the definition of universal marketability, there exists $P \geq \mathbb{E}[I(X)]$ such that

$$\begin{aligned}\mathbb{E}[v(X - I(X) + \mathbb{E}[I(X)])] &= \pi(X - I(X) + \mathbb{E}[I(X)]) \\ &\leq \pi(X - I(X) + P) \\ &\leq \pi(X) \\ &= \mathbb{E}[v(X)],\end{aligned}$$

where the first inequality is due to the monotonicity of v. Since $X - I(X) + \mathbb{E}[I(X)]$ and X share the same expected value and convex order is closed under shifts, we have

$$X - I(X) \leq_{\text{cx}} X - \mathbb{E}[I(X)].$$

The converse is obvious since Statement (b) implies that for any integrable random variable X and any π that preserves the convex order, we have

$$\pi(X - I(X) + \mathbb{E}[I(X)]) \leq \pi(X),$$

which means that (2.1) is true with $P = \mathbb{E}[I(X)]$.

To show that (c) implies (b), we assume that I is a non-decreasing and 1-Lipschitz function. Then $(X - I(X), I(X))^\top$ is a comonotonic random vector for any integrable random variable X. Because convex order is preserved under comonotonic addition, we have

$$[X - I(X)] + \mathbb{E}[I(X)] \leq_{\text{cx}} [X - I(X)] + I(X) = X,$$

or $X - I(X) \leq_{\text{cx}} X - \mathbb{E}[I(X)]$.

Finally, we prove by contraposition that (b) implies (c). First, we suppose that I is not non-decreasing. Then there exist $a \in \mathbb{R}$ and $\eta > 0$ such that $I(a + \eta) - I(a) < 0$. For any $p \in (0, 1)$, define a discrete random variable $\tilde{X}$ by

$$\tilde{X} = \begin{cases} a, & \text{with probability } p, \\ a + \eta, & \text{with probability } 1 - p. \end{cases}$$

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The maximum value that $\tilde{X} - \mathbb{E}[I(\tilde{X})]$ can attain equals $a + \eta - \mathbb{E}[I(\tilde{X})]$. Since $[a + \eta - I(a + \eta)] - [a - I(a)] = \eta + I(a) - I(a + \eta) > 0$, the maximum value that $\tilde{X} - I(\tilde{X})$ can attain is $a + \eta - I(a + \eta)$. Notice that

$$\begin{aligned} a + \eta - \mathbb{E}[I(\tilde{X})] &= a + \eta - [pI(a) + (1 - p)I(a + \eta)] \\ &= a + \eta - I(a + \eta) + p[I(a + \eta) - I(a)] \\ &< a + \eta - I(a + \eta), \end{aligned}$$

which means that $\tilde{X} - I(\tilde{X}) \nleq_{\text{cx}} \tilde{X} - \mathbb{E}[I(\tilde{X})]$. Second, we assume that I is not 1-Lipschitz. Then there exist $b \in \mathbb{R}$ and $\zeta > 0$ such that $I(b + \zeta) - I(b) > \zeta$. Considering a similar discrete random variable $\hat{X}$ defined by

$$\hat{X} = \begin{cases} b, & \text{with probability } q, \\ b + \zeta, & \text{with probability } 1-q, \end{cases}$$

for some $q \in (0, 1)$, we can easily show that the maximum value of $\hat{X} - I(\hat{X})$ is $b - I(b)$, which is greater than the maximum value of $\hat{X} - \mathbb{E}[I(\hat{X})] = b - I(b) + \zeta - (1 - q)[I(b + \zeta) - I(b)]$ whenever q is strictly positive but sufficiently small. Again we arrive at $\hat{X} - I(\hat{X}) \nleq_{\text{cx}} \hat{X} - \mathbb{E}[I(\hat{X})]$.

A.2. Proof of Theorem 2.6

The equivalence between Statements (a) and (b) is a consequence of Proposition 2.4, so it suffices to show that Statements (b) and (c) are equivalent.

To show that (c) implies (b), we assume that $I(\mathbf{x})$ is a non-decreasing and 1-Lipschitz function of $\sum_{k=1}^{n} x_k$ for any $\mathbf{x} \in \mathbb{R}^n$. Then relation (2.4) holds for any integrable random vector $\mathbf{X}$ by applying Theorem 2.2 to the random variable $\sum_{k=1}^{n} X_k$.

Conversely, we now assume that (2.4) holds for any integrable random vector $\mathbf{X}$. In particular, for a fixed k, it is true for the random vector $(x_1, \ldots, x_{k-1}, X_k, x_{k+1}, \ldots, x_n)^\top$, where X_k is any given integrable random variable and $\mathbf{x}_{(k)} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)^\top$ is a vector of any given real constants. If we write $I_k(X_k; \mathbf{x}_{(k)})$ for $I(x_1, \ldots, x_{k-1}, X_k, x_{k+1}, \ldots, x_n)$, then relation (2.4) implies that

$$X_k - I_k(X_k; \mathbf{x}_{(k)}) \leq_{\text{cx}} X_k - \mathbb{E}[I_k(X_k; \mathbf{x}_{(k)})],$$

which is true for any random variable X_k. By Theorem 2.2, $I_k(\cdot\,; \mathbf{x}_{(k)})$ is non-decreasing and 1-Lipschitz. Thus, by the arbitrariness of k, the function I is non-decreasing and componentwise 1-Lipschitz.

To further show that I is a non-decreasing and 1-Lipschitz function of the sum of its arguments, we take $X_1, \ldots, X_{n-1}$ to be any $n - 1$ random variables and set $X_n := s - \sum_{k=1}^{n-1} X_k$ for an arbitrarily given $s \in \mathbb{R}$. Then (2.4) reduces to

$$I(\mathbf{X}) \leq_{\text{cx}} \mathbb{E}[I(\mathbf{X})],$$

which implies that $I(\mathbf{X})$ is a constant almost surely. From the arbitrariness of the random variables $X_1, \ldots, X_{n-1}$ and real number s, we deduce that $I(\mathbf{x})$ depends on the values of $\mathbf{x}$ only through the sum $\sum_{k=1}^{n} x_k$, for any real vector $\mathbf{x}$. It follows that $I(\mathbf{x}) = I\left(0, \ldots, 0, \sum_{k=1}^{n} x_k\right)$, which in turn is a non-decreasing and 1-Lipschitz function of the sum $\sum_{k=1}^{n} x_k$ because I is non-decreasing and componentwise 1-Lipschitz.

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A.3. Proof of Corollary 2.7

The equivalence between Statements (a) and (b) follows from Proposition 2.4, so it suffices to show that Statements (b) and (c) are equivalent. By Theorem 2.6, this is further equivalent to proving that $I(\mathbf{x}) = I_1(x_1) + \cdots + I_n(x_n)$ is a non-decreasing and 1-Lipschitz function of $\sum_{k=1}^n x_k$ for any $\mathbf{x} \in \mathbb{R}^n$ if and only if $I_k(x) = cx$ for all $k = 1, \ldots, n$ and some common constant $c \in [0, 1]$.

We first assume that each I_k takes the form $I_k(x) = cx$ for some $c \in [0, 1]$. Then the function I becomes

$$I(\mathbf{x}) = I_1(x_1) + \cdots + I_n(x_n) = c(x_1 + \cdots + x_n),$$

which is obviously a non-decreasing and 1-Lipschitz function of $\sum_{k=1}^n x_k$.

To show the converse, we now assume that $I(\mathbf{x}) = I_1(x_1) + \cdots + I_n(x_n)$ is a non-decreasing and 1-Lipschitz function of $\sum_{k=1}^n x_k$ for any $\mathbf{x} \in \mathbb{R}^n$. This makes it legitimate to write $I(\mathbf{x}) = I(\sum_{k=1}^n x_k)$. For a fixed $k \in \{1, \ldots, n\}$, taking all elements of $\mathbf{x}$ except x_k to be zero and making use of the fact that $I_k(0) = 0$ for all $k = 1, \ldots, n$, we get $I(x_k) = I_k(x_k)$ for all $x_k \in \mathbb{R}$, which implies that each I_k is identical to I. Thus, for any $\mathbf{x} \in \mathbb{R}^n$, we have

$$I(x_1 + \cdots + x_n) = I(x_1) + \cdots + I(x_n).$$

Combining the continuity (which is a consequence of 1-Lipschitzity) of I with the above relation, we deduce from a well-known result on Cauchy's functional equation (see, e.g., Section 5.2 of Kuczma (2009)) that I must be a linear function and null at zero. Since I is non-decreasing and 1-Lipschitz, we conclude that $I_k(x) = I(x) = cx$ for all $k = 1, \ldots, n$ and some common constant $c \in [0, 1]$.

A.4. Proof of Theorem 3.1

We first prepare a lemma for proving Theorem 3.1.

Lemma A.1. *If I is non-decreasing and componentwise 1-Lipschitz, then relation (3.2) holds for any mutually independent random variables $X_1, \ldots, X_n$.*

Proof. We first show that, for each $j = 1, \ldots, n$,

$$\sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})|X_j, \ldots, X_n] \leq_{\text{cx}} \sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})|X_{j+1}, \ldots, X_n], \tag{A.1}$$

in which $\mathbb{E}[I(\mathbf{X})|X_j, \ldots, X_n] = I(\mathbf{X})$ when $j = 1$, while $\mathbb{E}[I(\mathbf{X})|X_{j+1}, \ldots, X_n] = \mathbb{E}[I(\mathbf{X})]$ when $j = n$. For any real vector $(x_{j+1}, \ldots, x_n)^\top$, notice that

$$\mathbb{E}[I(\mathbf{X})|X_{j+1} = x_{j+1}, \ldots, X_n = x_n] = \mathbb{E}[\mathbb{E}[I(\mathbf{X})|X_j, X_{j+1} = x_{j+1}, \ldots, X_n = x_n]],$$

and that $\mathbb{E}[I(\mathbf{X})|X_j, X_{j+1} = x_{j+1}, \ldots, X_n = x_n]$ is a non-decreasing and 1-Lipschitz function of X_j. Thus, by Theorem 2.2,

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$$X_j + \sum_{1\le k\le n, k\neq j} x_k - \mathbb{E}[I(\mathbf{X})|X_j, X_{j+1}=x_{j+1},\ldots,X_n=x_n]$$
$$\le_{\mathrm{cx}} X_j + \sum_{1\le k\le n, k\neq j} x_k - \mathbb{E}[I(\mathbf{X})|X_{j+1}=x_{j+1},\ldots,X_n=x_n],$$

holds for any real vector $(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)^\top$. Integrating both sides above with respect to the joint distribution of $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n$ yields relation (A.1). By making use of the transitive property of the convex order, we have

$$\begin{aligned}\sum_{k=1}^n X_k - I(\mathbf{X}) &\le_{\mathrm{cx}} \sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})|X_2,\ldots,X_n]\\ &\le_{\mathrm{cx}} \sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})|X_3,\ldots,X_n]\\ &\;\;\vdots\\ &\le_{\mathrm{cx}} \sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})|X_n]\\ &\le_{\mathrm{cx}} \sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})].\end{aligned}$$

This ends the proof. □

Proof of Theorem 3.1 By Proposition 2.4, it then suffices to prove that Statements (b) and (c) are equivalent, that is, relation (3.2) holds for all mixtures of form (3.1) if and only if I is non-decreasing and componentwise 1-Lipschitz.

We first assume that relation (3.2) holds for any random vector of form (3.1). In particular, it is true for any deterministic vector $\mathbf{Z}$, in which case the random vector $\mathbf{X}$ has mutually independent components. From the proof of Theorem 2.6, we know that I must be non-decreasing and componentwise 1-Lipschitz.

To show the converse, we now assume that I is non-decreasing and componentwise 1-Lipschitz and that $\mathbf{X}$ is any random vector of form (3.1). Relation (3.2) can be proved by the following two-step procedure:

$$\sum_{k=1}^n X_k - I(\mathbf{X}) \le_{\mathrm{cx}} \sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})|\mathbf{Z}] \le_{\mathrm{cx}} \sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})]. \tag{A.2}$$

For this purpose, denote by v any convex function and by $\mathbf{1}\in\mathbb{R}^n$ a vector with all elements equal to one. Keeping in mind that the random vectors $\mathbf{Y}$ and $\mathbf{Z}$ are independent, we deduce that

$$\begin{aligned}\mathbb{E}\left[v\left(\sum_{k=1}^n X_k - I(\mathbf{X})\right)\right] &= \int_{\mathbb{R}^m} \mathbb{E}\left[v\left([\mathbf{Y}+G(\mathbf{z})]^\top\mathbf{1} - I(\mathbf{Y}+G(\mathbf{z}))\right)\right]\mathrm{d}F_{\mathbf{Z}}(\mathbf{z})\\ &\le \int_{\mathbb{R}^m} \mathbb{E}\left[v\left([\mathbf{Y}+G(\mathbf{z})]^\top\mathbf{1} - \mathbb{E}[I(\mathbf{Y}+G(\mathbf{z}))]\right)\right]\mathrm{d}F_{\mathbf{Z}}(\mathbf{z})\\ &= \mathbb{E}\left[v\left([\mathbf{Y}+G(\mathbf{Z})]^\top\mathbf{1} - \mathbb{E}[I(\mathbf{Y}+G(\mathbf{Z}))|\mathbf{Z}]\right)\right]\\ &= \mathbb{E}\left[v\left(\sum_{k=1}^n X_k - \mathbb{E}[I(\mathbf{X})|\mathbf{Z}]\right)\right],\end{aligned}$$

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where $F_{\mathbf{Z}}$ is the distribution of $\mathbf{Z}$ and the inequality is due to Lemma A.1. Hence, the first convex order inequality in (A.2) is proved.

We now show the second inequality in relation (A.2). In view of the fact that both sides have the same expectation, it suffices to prove that, for any $d \in \mathbb{R}$,

$$\mathbb{E}\left[\left(\sum_{k=1}^{n} X_k - \mathbb{E}[I(\mathbf{X})|\mathbf{Z}] - d\right)_+\right] \leq \mathbb{E}\left[\left(\sum_{k=1}^{n} X_k - \mathbb{E}[I(\mathbf{X})] - d\right)_+\right]. \tag{A.3}$$

It is easy to see that, for any $x, y, d \in \mathbb{R}$,

$$(x-d)_+ - (y-d)_+ \leq 1_{\{x>d\}} \times (x-y),$$

in which the indicator function $1_{\{x>d\}}$ equals 1 when $x > d$ and 0 otherwise. Thus,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^{n} X_k - \mathbb{E}[I(\mathbf{X})|\mathbf{Z}] - d\right)_+\right] - \mathbb{E}\left[\left(\sum_{k=1}^{n} X_k - \mathbb{E}[I(\mathbf{X})] - d\right)_+\right] \\
&\quad \leq \mathbb{E}\left[1_{\left\{\sum_{k=1}^{n} X_k - \mathbb{E}[I(\mathbf{X})|\mathbf{Z}] > d\right\}} \times \left(\mathbb{E}[I(\mathbf{X})] - \mathbb{E}[I(\mathbf{X})|\mathbf{Z}]\right)\right] \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[1_{\{[\mathbf{Y}+G(\mathbf{Z})]^\top \mathbf{1} - \mathbb{E}[I(\mathbf{Y}+G(\mathbf{Z}))|\mathbf{Z}]>d\}} \,\Big|\, \mathbf{Z}\right] \times \left(\mathbb{E}[I(\mathbf{Y}+G(\mathbf{Z}))] - \mathbb{E}[I(\mathbf{Y}+G(\mathbf{Z}))|\mathbf{Z}]\right)\right].
\end{aligned}$$

By the independence between the random vectors $\mathbf{Y}$ and $\mathbf{Z}$, we rewrite the right-hand side above as

$$\mathbb{E}\left[h_1(\mathbf{Z})(\mathbb{E}[h_2(\mathbf{Z})] - h_2(\mathbf{Z}))\right], \tag{A.4}$$

in which the functions h_1 and h_2 are defined as

$$\begin{aligned}
h_1(\mathbf{z}) &= \mathbb{E}\left[1_{\{h_3(\mathbf{Y},\mathbf{z})>d\}}\right], \\
h_2(\mathbf{z}) &= \mathbb{E}\left[I(\mathbf{Y}+G(\mathbf{z}))\right], \\
h_3(\mathbf{y},\mathbf{z}) &= [\mathbf{y}+G(\mathbf{z})]^\top \mathbf{1} - \mathbb{E}\left[I(\mathbf{Y}+G(\mathbf{z}))\right].
\end{aligned}$$

To prove inequality (A.3), it then suffices to show that (A.4) is non-positive.

It is clear that h_2 is non-decreasing. Now we show that h_1 has the same property. Notice that $h_3(\mathbf{y},\mathbf{z})$ is non-decreasing in each $z_i, i = 1, \ldots, m$. To see this, for each i, fix any real vectors $\mathbf{y} \in \mathbb{R}^n$ and $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)^\top \in \mathbb{R}^{m-1}$. For any two real numbers $s \leq t$, write

$$\mathbf{z_1} = (z_1, \ldots, z_{i-1}, s, z_{i+1}, \ldots, z_m)^\top \qquad \text{and} \qquad \mathbf{z_2} = (z_1, \ldots, z_{i-1}, t, z_{i+1}, \ldots, z_m)^\top$$

Then we have

$$\begin{aligned}
h_3(\mathbf{y},\mathbf{z_2}) - h_3(\mathbf{y},\mathbf{z_1}) &= [G(\mathbf{z_2}) - G(\mathbf{z_1})]^\top \mathbf{1} + \mathbb{E}\left[I(\mathbf{Y}+G(\mathbf{z_1}))\right] - \mathbb{E}\left[I(\mathbf{Y}+G(\mathbf{z_2}))\right] \\
&\geq [G(\mathbf{z_2}) - G(\mathbf{z_1})]^\top \mathbf{1} + [G(\mathbf{z_1}) - G(\mathbf{z_2})]^\top \mathbf{1} \\
&= 0,
\end{aligned}$$

where the inequality is due to the non-decreasing and componentwise 1-Lipschitz property of indemnity I as well as the monotonicity of the function G. Since the indicator function

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is non-decreasing, this componentwise monotonicity of $h_3(\mathbf{y}, \mathbf{z})$ in each z_i implies that h_1 is non-decreasing.

Hence, we have

$$\mathbb{E}[h_1(\mathbf{Z})(\mathbb{E}[h_2(\mathbf{Z})] - h_2(\mathbf{Z}))] = -\mathrm{Cov}(h_1(\mathbf{Z}), h_2(\mathbf{Z})) \leq 0,$$

in which the inequality follows from Theorem 2.1 of Esary *et al*. (1967). This completes the proof.

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