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ABSTRACT

We show that compatible systems of ℓ -adic sheaves on a scheme of finite type over the ring of integers of a local field are compatible along the boundary up to stratification. This extends a theorem of Deligne on curves over a finite field. As an application, we deduce the equicharacteristic case of classical conjectures on ℓ -independence for proper smooth varieties over complete discrete valuation fields. Moreover, we show that compatible systems have compatible ramification. We also prove an analogue for integrality along the boundary.

1. Introduction

Let $S = \text{Spec}(\mathcal{O}_K)$ be the spectrum of an excellent Henselian discrete valuation ring \mathcal{O}_K of finite residue field $k = \mathbb{F}_q$ of characteristic p . Let K be the fraction field of \mathcal{O}_K . Given a scheme X of finite type over S and a prime $\ell \neq p$, we let $K(X, \overline{\mathbb{Q}}_\ell)$ denote the Grothendieck group of constructible $\overline{\mathbb{Q}}_\ell$ -sheaves on X , where $\overline{\mathbb{Q}}_\ell$ denotes an algebraic closure of \mathbb{Q}_ℓ . We fix a field Q , an index set I , and, for each $i \in I$, a prime number ℓ_i and an embedding $\iota_i: Q \rightarrow \overline{\mathbb{Q}}_{\ell_i}$. Let $|X|$ be the set of locally closed points of X . In other words, $|X| = |X_k| \cup |X_K|$ is the union of the sets of closed points of the two fibers. Note that the residue field of $x \in |X|$ is a finite extension of k or K , and the local Weil group $W(\bar{x}/x) \subseteq \text{Gal}(\bar{x}/x)$ is defined for any geometric point \bar{x} above x . We say that a system $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{Q}}_{\ell_i})$ is *compatible* if for every \bar{x} above $x \in |X|$, and for every $F \in W(\bar{x}/x)$, the local traces are compatible: there exists $a \in Q$ such that $\text{tr}(F, (L_i)_{\bar{x}}) = \iota_i(a)$ for all $i \in I$ [Zhe09, Définition 4.13].

In this paper, we study the compatibility of compatible systems along the boundary. We let $K_{\text{lisse}}(X, \overline{\mathbb{Q}}_\ell)$ denote the Grothendieck group of lisse $\overline{\mathbb{Q}}_\ell$ -sheaves on X .

DEFINITION 1.1. Let \bar{X} be a normal scheme of finite type over S and let X be a dense open subscheme. We say that $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbb{Q}}_{\ell_i})$ is *compatible on \bar{X}* if, for every $x \in |X|$, every geometric point \bar{a} of $X_{(x)} := \bar{X}_{(x)} \times_{\bar{X}} X$, and every $F \in W(X_{(x)}, \bar{a})$, $(\text{tr}(F, (L_i)_{\bar{a}}))_{i \in I}$ is compatible. Here $\bar{X}_{(x)}$ denotes the Henselization of \bar{X} at x , and $W(X_{(x)}, \bar{a})$ denotes the Weil group, namely the inverse image of $W(\bar{x}/x) \subseteq \text{Gal}(\bar{x}/x)$ by the surjective homomorphism $\pi_1(X_{(x)}, \bar{a}) \rightarrow \pi_1(\bar{X}_{(x)}, \bar{a}) \simeq \text{Gal}(\bar{x}/x)$.

In the case where \bar{X} is an integral smooth curve over k or K and $x \in \bar{X} - X$, $X_{(x)}$ is the spectrum of a field extension of the function field E of X and its fundamental group is the decomposition group of E at x , a subgroup of the Galois group of E .

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We call $X \subseteq \bar{X}$ a normal compactification over S if \bar{X} is normal, proper over S , and contains X as a dense open subscheme. Our first result is that compatible systems are compatible along the boundary up to stratification.

THEOREM 1.2. *Let X be a scheme of finite type over S and let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbb{Q}}_{\ell_i})$ be a compatible system with I finite. Then there exists a finite stratification $X = \bigcup_{\alpha} X_{\alpha}$ by normal subschemes such that each X_{α} admits a normal compactification \bar{X}_{α} over S such that $(L_i|_{X_{\alpha}})_{i \in I}$ is compatible on \bar{X}_{α} .*

We refer to Corollary 2.17 for the equivalent statement that compatible systems are compatible along the boundary up to modification. In the case of a curve over a finite field, we recover a theorem of Deligne [Del73, Théorème 9.8] (see Corollary 2.15). Takeshi Saito gave an example of a compatible system on a smooth surface X that is not compatible on a given smooth compactification \bar{X} (private communication with Hiroki Kato).

Theorem 1.2 implies the following valuative criterion for compatible systems, analogous to Gabber’s valuative criterion for Vidal’s ramified part of the fundamental group [Vid05, § 6.1].

COROLLARY 1.3. *Let X be a scheme of finite type over S and let $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{Q}}_{\ell_i})$. Consider commutative squares of schemes*

$$\begin{array}{ccc}
 \eta & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 V & \longrightarrow & S
 \end{array} \tag{1.1}$$

where $V = \text{Spec}(\mathcal{O}_L)$ with \mathcal{O}_L a Henselian valuation ring, and $\eta = \text{Spec}(L)$ is the generic point of V . Let $\bar{\eta} \rightarrow \eta$ be a geometric point and let $t \in V$ be the closed point.

- (1) $(L_i)_{i \in I}$ is a compatible system if and only if for every commutative square (1.1) with t quasi-finite over S , $(\text{tr}(F, (L_i)_{\bar{\eta}}))_{i \in I}$ is compatible for all $F \in W(\bar{\eta}/\eta)$.
- (2) If $(L_i)_{i \in I}$ is a compatible system and (1.1) is a commutative square with V strictly Henselian, then $(\text{tr}(F, (L_i)_{\bar{\eta}}))_{i \in I}$ is compatible for all $F \in \text{Gal}(\bar{\eta}/\eta)$.

Note that here we do not assume \mathcal{O}_L to be a discrete valuation ring or that $V \rightarrow S$ is local.

As an application, we deduce the equicharacteristic case of some classical conjectures by Serre on ℓ -independence (Conjectures C₄, C₅, and C₈ of [Ser70, § 2.3]; cf. [ST68, Appendix, Problems 1 and 2]).

THEOREM 1.4. *Let \mathcal{O}_L be a Henselian discrete valuation ring of characteristic $p > 0$, of fraction field L and residue field κ . Let X be a proper smooth scheme over L . Let \bar{L} be a separable closure of L and let $\bar{\kappa}$ be the residue field of \bar{L} . Let $X_{\bar{L}} = X \otimes_L \bar{L}$.*

- (1) For each m and each $F \in I(\bar{L}/L) := \text{Ker}(\text{Gal}(\bar{L}/L) \rightarrow \text{Gal}(\bar{\kappa}/\kappa))$, $\text{tr}(F, H^m(X_{\bar{L}}, \mathbb{Q}_{\ell}))$ is a rational integer independent of $\ell \neq p$.
- (2) (Cf. [Ter98, Theorem 3.3]) Assume that κ is a finite field. Then, for each m , each i , and each $F \in W(\bar{L}/L)$ whose image in $W(\bar{\kappa}/\kappa)$ is the n th power of the geometric Frobenius for $n \geq 0$, we have:
 - (2a) $\text{tr}(F, \text{gr}_i^M H^m(X_{\bar{L}}, \mathbb{Q}_{\ell}))$ is a rational integer independent of $\ell \neq p$, where M denotes the monodromy filtration; in particular,
 - (2b) $\text{tr}(F, H^m(X_{\bar{L}}, \mathbb{Q}_{\ell}))$ is a rational integer independent of $\ell \neq p$.

Part (2) was claimed in [CL18, Theorem 6.1], but the proof given there is incomplete.¹ A weaker form of (2) was proved by Terasoma [Ter98, Theorem 3.3].

Remark 1.5. Theorem 1.4(1) is the equicharacteristic $p > 0$ case of Serre's Conjecture C_4 . Theorem 1.4(2a) implies the equicharacteristic case of Conjecture C_5 (Remark 2.18(2)), while (2b) implies the equicharacteristic case of Conjecture C_8 . Parts (1) and (2b) of Theorem 1.4 hold more generally over a Henselian valuation field of characteristic $p > 0$ without assuming that the valuation is discrete (Remark 2.18(3)).

The alternating sum $\sum_m (-1)^m \text{tr}(F, H^m(X_{\bar{L}}, \mathbb{Q}_\ell))$ of the traces in (1) and (2b) was known to be a rational integer independent of $\ell \neq p$ more generally for X separated of finite type over L without the equicharacteristic assumption. See Vidal [Vid04, Proposition 4.2] (combined with Laumon [Lau81, Théorème 1.1]), Ochiai [Och99], and Zheng [Zhe08, Zhe09] (Theorems 2.3 and 3.5 below). Our valuative criterion allows us to further extend the results on the alternating sum to Henselian valuation fields [LZ19].

In the case where X is defined over a curve over a finite field, Theorem 1.4(2b) follows from Deligne's theorem for curves mentioned above and results of Weil II [Del80]. In the general case, after spreading out, the base becomes a variety over a finite field and we apply Corollary 1.3.

In §2, we give the proofs of Theorems 1.2 and 1.4. The proof of Theorem 1.2 relies on the preservation of compatible systems under direct images [Zhe09, Proposition 4.15]. Over a finite field the latter is a theorem of Gabber [Fuj02a, Theorem 2].

In §3, we study integrality along the boundary and prove an analogue of Theorem 1.2. This generalizes a theorem of Deligne on non-Archimedean absolute values of liftings of local Frobenius for curves over finite fields [Del80, Théorème 1.10.3].

Our original motivation for studying compatibility along the boundary is to understand the relationship between compatible systems of $\overline{\mathbb{Q}_\ell}$ -sheaves and systems of $\overline{\mathbb{F}_\ell}$ -sheaves with compatible wild ramification. The latter and variants were studied in recent work of Saito and Yatagawa [SY17, Yat18] and Guo [Guo18], generalizing earlier work of Deligne [Ill81] and Vidal [Vid04, Vid05]. In §4, we deduce from our valuative criterion for compatible systems that compatible systems have compatible ramification and, consequently, their reductions have compatible wild ramification. These notions are defined using Vidal's ramified part of the fundamental group, which involves images of local inertia groups at geometric points of compactifications \bar{X} of X . We define the decomposed part of the fundamental group by taking instead images of the local decomposition groups at $x \in |\bar{X}|$. We show, as another application of Theorem 1.2, that the union of the images of the local decomposition (or Weil) groups for $x \in |X|$ is dense in the decomposed part.

2. Compatible systems along the boundary

The strategy of the proof of Theorem 1.2 is to reduce to the case of lisse sheaves tamely ramified along a normal crossing divisor with unipotent local monodromy. For this we need to work with finite group actions. We now review the notion of compatible systems on Deligne–Mumford stacks [Zhe09, §5] and finite quotient stacks in particular. In this paper, Deligne–Mumford stacks are assumed to be quasi-separated with separated diagonal.

¹The authors of [CL18] have been made aware of this and have submitted a corrigendum.

Let \bar{k} be a separable closure of k . Each $F \in W(\bar{k}/k)$ is the n th power of the geometric Frobenius $\text{Fr}: a \mapsto a^{1/q}$ for some $n \in \mathbb{Z}$. We call n the *degree* of F . For an integer N , we let $W^{\geq N}(\bar{k}/k)$ denote the subset $\{\text{Fr}^n \mid n \geq N\}$.

Notation 2.1. For any connected Deligne–Mumford stack Y over S and any geometric point $\bar{a} \rightarrow Y$, we define the Weil group $W(Y, \bar{a})$ to be the inverse image of the Weil group $W(\bar{k}/k)$ by the homomorphism

$$r: \pi_1(Y, \bar{a}) \rightarrow \pi_1(S, \bar{a}) \simeq \text{Gal}(\bar{k}/k).$$

We define the *degree* of $F \in W(Y, \bar{a})$ to be the degree of $r(F)$. We let $W^{\geq N}(Y, \bar{a})$ denote the subset $r^{-1}(W^{\geq N}(\bar{k}/k))$ of elements of degree $\geq N$.

Let X be a Deligne–Mumford stack. For a point ξ of X , we let X_ξ denote the residual gerbe, which is necessarily a quotient stack $[x/G]$ by a finite group G of the spectrum of a field x (cf. [IZ13, p. 13]). For a geometric point \bar{x} above x , we have

$$\pi_1([x/G], \bar{x}) \simeq \text{Gal}(\bar{x}/y) \times_{\text{Gal}(x/y)} G,$$

where $y = x/G$.

Assume that X is of finite type over S . We let $|X|$ denote the set of locally closed points of X . For $\xi \in |X|$, x is quasi-finite over S , the spectrum of a finite field extension of k or K . The Weil group $W([x/G], \bar{x}) \subseteq \pi_1([x/G], \bar{x})$ is the inverse image of the Weil group $W(\bar{x}/y) \subseteq \text{Gal}(\bar{x}/y)$ by the homomorphism $\pi_1([x/G], \bar{x}) \rightarrow \text{Gal}(\bar{x}/y)$, which is surjective of kernel the inertia group.

DEFINITION 2.2. We say that $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{Q}_{\ell_i}})$ is *compatible* if it satisfies the following equivalent conditions.

- (1) For every $\xi \in |X|$, every geometric point \bar{x} above ξ , and every $F \in W(X_\xi, \bar{x})$, $(\text{tr}(F, (L_i)_{\bar{x}}))_{i \in I}$ is compatible.
- (2) For every quasi-finite morphism $f: x \rightarrow X$, where x is the spectrum of a field, $(f^*L_i)_{i \in I}$ is a compatible system on x (§ 1).
- (3) For every smooth morphism $f: Y \rightarrow X$ of finite type with Y a scheme, $(f^*L_i)_{i \in I}$ is a compatible system on Y (§ 1).

The implications (1) \Rightarrow (2) \Rightarrow (3) are trivial. (3) \Rightarrow (2) follows from the existence of smooth neighborhoods [LM00, Théorème 6.3]. (2) \Rightarrow (1) follows from [Zhe09, Proposition 5.6] applied to the quotient stack X_ξ , which is based on a method of Deligne and Lusztig [DL76, proof of Proposition 3.3].

In the case where $X = [Y/G]$ is a quotient stack of a scheme Y by a finite group, the residual gerbe at the image of $y \in Y$ is $[y/D(y)]$, where $D(y) < G$ is the decomposition group.

The main result of [Zhe09] can be stated as follows.

THEOREM 2.3. *Compatible systems on Deligne–Mumford stacks of finite type over S are stable under Grothendieck’s six operations and duality.*

This is stated for Deligne–Mumford stacks of finite type over k or K in [Zhe09, Proposition 5.8], but the same proof applies over S with [Zhe09, Théorème 1.16] replaced by the more general [Zhe09, Proposition 4.15]. The case of schemes of finite type over k is a theorem of Gabber [Fuj02a, Theorem 2].

We will only need the stability under Rj_* for an open immersion j .

Remark 2.4. Let x be quasi-finite over S and let $(L_i) \in \prod_{i \in I} K(x, \overline{\mathbb{Q}_{\ell_i}})$. If there exists an integer N such that $(\text{tr}(F, (L_i)_{\bar{x}}))_{i \in I}$ is compatible for all $F \in W^{\geq N}(x, \bar{x})$, then the same holds for all $F \in W(x, \bar{x})$ by [Zhe09, Proposition 1.15] (a consequence of Grothendieck’s arithmetic local monodromy theorem [ST68, Appendix] and a rationality lemma [Ill06, Lemma 8.1]).

In the regular case, compatibility of systems of unramified lisse sheaves extends to the boundary by the following variant of [Zhe09, Proposition 3.10].

PROPOSITION 2.5. *Let X be a regular Deligne–Mumford stack of finite type over S and let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbb{Q}_{\ell_i}})$. Assume that $(L_i|_U)_{i \in I}$ is compatible for some dense open substack $U \subseteq X$. Then $(L_i)_{i \in I}$ is compatible.*

Proof. The proof is very similar to that of [Zhe09, Proposition 3.10]. A related argument will be used in the proof of Proposition 2.10 below. By induction, we may assume that $D = X - U$ is regular and purely of codimension $d \geq 1$. Let $j: U \rightarrow X$ be the open immersion. By Theorem 2.3, $(Rj_*(L_i|_U))_{i \in I}$ is compatible. By the projection formula,

$$L_i \otimes_{\mathbb{Q}_{\ell_i}} Rj_* \mathbb{Q}_{\ell_i} \simeq Rj_*(L_i|_U).$$

Gabber’s absolute purity theorem (see [Fuj02b, Theorem 2.1.1] and [Rio14, Théorème 3.1.1]) extends to Deligne–Mumford stacks: the refined cycle class $\text{cl}_f \in H_D^{2d}(X, \mathbb{Q}_{\ell}(d))$ induces an isomorphism $\mathbb{Q}_{\ell} \xrightarrow{\sim} Rf^! \mathbb{Q}_{\ell}(d)[2d]$, where $f: D \rightarrow X$ denotes the closed immersion. Indeed, the definition of cl_f [Fuj02b, Definition 1.1.2] holds without change (with Chern classes defined by Grothendieck [Gro68, § 1]) and the fact that it induces an isomorphism reduces to the case of schemes. It follows that we have

$$R^m j_* \mathbb{Q}_{\ell} \simeq \begin{cases} \mathbb{Q}_{\ell}, & m = 0, \\ (\mathbb{Q}_{\ell})_D(-d), & m = 2d - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, we can reduce Proposition 2.5 to the case of schemes using Definition 2.2(3).

For $x \rightarrow D$ quasi-finite and $F \in W(\bar{x}/x)$ of degree n , $\text{tr}(F, (Rj_* \mathbb{Q}_{\ell})_{\bar{x}}) = 1 - q^{nd}$. Thus, for $n \neq 0$, $\text{tr}(F, (L_i)_{\bar{x}})$ can be recovered from $\text{tr}(F, (Rj_*(L_i|_U))_{\bar{x}})$. Therefore, $(L_i)_{i \in I}$ is compatible by Remark 2.4. □

Next we define compatibility on the boundary in the equivariant setting. Let \bar{X} be a scheme equipped with the action of a finite group G . For $x \in \bar{X}$, the decomposition group $D(x)$ acts on $\bar{X}_{(x)}$. For any geometric point \bar{x} above x , we have $\pi_1([x/D(x)], \bar{x}) \simeq \pi_1([\bar{X}_{(x)}/D(x)], \bar{x})$. For \bar{X} normal, $X \subseteq \bar{X}$ a G -stable dense open subscheme, and $\bar{a} \rightarrow X_{(x)}$ a geometric point, the homomorphism

$$\pi_1([X_{(x)}/D(x)], \bar{a}) \rightarrow \pi_1([\bar{X}_{(x)}/D(x)], \bar{a}) \simeq \pi_1([x/D(x)], \bar{x}) \tag{2.1}$$

is surjective.

DEFINITION 2.6. Let \bar{X} be a normal scheme of finite type over S equipped with an action of G by S -automorphisms and let X be a G -stable dense open subscheme. We say that $(L_i) \in \prod_{i \in I} K_{\text{lisse}}([X/G], \overline{\mathbb{Q}_{\ell_i}})$ is compatible on $[\bar{X}/G]$ if, for every $x \in \bar{X}$, every geometric point $\bar{a} \rightarrow X_{(x)}$, and every $F \in W([X_{(x)}/D(x)], \bar{a})$, $(\text{tr}(F, (L_i)_{\bar{a}}))_{i \in I}$ is compatible.

Remark 2.7. (1) $(L_i) \in \prod_{i \in I} K_{\text{lisse}}([X/G], \overline{\mathbb{Q}_{\ell_i}})$ is compatible on $[X/G]$ in the sense of Definition 2.6 if and only if it is compatible in the sense of Definition 2.2. This follows from the isomorphism in (2.1).

(2) Let $U \subseteq X \subseteq \bar{X}$ be a G -stable dense open subscheme. Then $(L_i) \in \prod_{i \in I} K_{\text{lisse}}([X/G], \overline{\mathbb{Q}_{\ell_i}})$ is compatible on $[\bar{X}/G]$ if and only if $(L_i|_{[U/G]})_{i \in I}$ is compatible on $[\bar{X}/G]$. This follows from the fact for $x \in \bar{X}$, the homomorphism $\pi_1([U(x)/D(x)], \bar{a}) \rightarrow \pi_1([X(x)/D(x)], \bar{a})$ is surjective.

(3) Assume that G acts freely on X . Let $Y = X/G$ and $\bar{Y} = \bar{X}/G$ be the quotient spaces. Then, for all $x \in \bar{X}$, if $y \in \bar{Y}$ denotes its image, then $[X(x)/D(x)] \simeq Y(y)$. Thus, in this case, $(L_i)_{i \in I}$ on X/G is compatible on $[\bar{X}/G]$ if and only if it is compatible on \bar{X}/G .

Remark 2.8. Let $x \in |\bar{X}|$ be a point that is not closed. The closure $Y = \overline{\{x\}} \subseteq \bar{X}$ admits a Zariski open cover by schemes finite over S . Thus, $Y = \bigcup_y Y(y)$, with y running through closed points of Y . We have $x \rightarrow Y(y) \rightarrow \bar{X}(y)$, which induces a morphism $X(x) \rightarrow X(y)$. If \bar{X} is separated, then $Y = Y(y)$ and $D(x) < D(y)$. Thus, in Definition 2.6, if \bar{X} is separated or $G = \{1\}$, then we may restrict to closed points of \bar{X} .

Remark 2.9. Given a point ξ of a Deligne–Mumford stack Y , one can define the Henselization of Y at ξ to be the limit of Deligne–Mumford stacks V for decompositions of the residual gerbe $Y_\xi \rightarrow Y$ into $Y_\xi \rightarrow V \xrightarrow{\phi} Y$ with ϕ representable and étale. Using [IZ13, Lemma 3.5], one can show that the Henselization of $[\bar{X}/G]$ at the image of $x \in \bar{X}$ is $[\bar{X}(x)/D(x)]$. Thus, Definition 2.6 depends only on the quotient stacks and can be extended to Deligne–Mumford stacks.

Let \bar{X} be a regular Deligne–Mumford stack and let $D \subseteq \bar{X}$ be a normal crossing divisor. We say that a lisse $\overline{\mathbb{Q}_\ell}$ -sheaf \mathcal{F} on $X = \bar{X} - D$ is *tamely ramified on \bar{X}* if for every geometric point \bar{x} above a generic point of D and every geometric point \bar{a} of $X(\bar{x}) := \bar{X}(\bar{x}) \times_{\bar{X}} X$, the wild inertia group of $X(\bar{x})$ acts trivially on $\mathcal{F}_{\bar{a}}$. Here $\bar{X}(\bar{x})$ denotes the strict Henselization. We say that $L \in K_{\text{lisse}}(X, \overline{\mathbb{Q}_\ell})$ is *tamely ramified on \bar{X}* if $L = [\mathcal{F}] - [\mathcal{G}]$ with \mathcal{F} and \mathcal{G} lisse and tamely ramified on \bar{X} .

PROPOSITION 2.10. *Let \bar{X} be a regular scheme of finite type over S equipped with an action of a finite group G by S -automorphisms. Let $D \subseteq \bar{X}$ be a normal crossing divisor such that $X = \bar{X} - D$ is G -stable. Let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}([X/G], \overline{\mathbb{Q}_{\ell_i}})$ be a compatible system. Assume that one of the following conditions holds.*

- (1) *For each i , there exist lisse $\overline{\mathbb{Z}_{\ell_i}}$ -sheaves \mathcal{F}_i and \mathcal{G}_i such that $L_i|_X = ([\mathcal{F}_i] - [\mathcal{G}_i]) \otimes_{\overline{\mathbb{Z}_{\ell_i}}} \overline{\mathbb{Q}_{\ell_i}}$, and $\mathcal{F}_i \otimes_{\overline{\mathbb{Z}_{\ell_i}}} \overline{\mathbb{Z}_{\ell_i}} / \ell_i^c \overline{\mathbb{Z}_{\ell_i}}$ and $\mathcal{G}_i \otimes_{\overline{\mathbb{Z}_{\ell_i}}} \overline{\mathbb{Z}_{\ell_i}} / \ell_i^c \overline{\mathbb{Z}_{\ell_i}}$ are constant for some rational number $c > 1/(\ell_i - 1)$. Here $\overline{\mathbb{Z}_{\ell_i}}$ denotes the ring of integers of $\overline{\mathbb{Q}_{\ell_i}}$.*
- (2) *$G = \{1\}$ and each L_i is tamely ramified on \bar{X} .*

Then $(L_i)_{i \in I}$ is compatible on $[\bar{X}/G]$.

For the proof of Theorem 1.2, we will only need part (1). For the proof of Proposition 2.10, we need a variant of Grothendieck’s arithmetic local monodromy theorem [ST68, Appendix]. We say that a family of matrices $\rho: E \rightarrow \text{GL}_n(\overline{\mathbb{Q}_\ell})$ is *quasi-unipotent* if each $\rho(g)$, $g \in E$, is quasi-unipotent. By Remark 2.11 below, a continuous representation $\rho: I \rightarrow \text{GL}_n(\overline{\mathbb{Q}_\ell})$ of a profinite group I is quasi-unipotent if and only if ρ is unipotent on an open subgroup $I_0 < I$.

Remark 2.11. Let $P_\ell^n \simeq \overline{\mathbb{Q}_\ell}^n$ be the space of monic polynomials of degree n . The subset $P_\ell^{n,\text{qu}} \subseteq P_\ell^n$ of polynomials whose roots are roots of unity is discrete and closed. This follows from continuity of roots and the fact that the only root of unity in $1 + \ell^c \overline{\mathbb{Z}_\ell}$ is 1, where $c > 1/(\ell - 1)$ is any rational number.

The function $M_n(\overline{\mathbb{Q}_\ell}) \rightarrow P_\ell^n$ carrying an $n \times n$ matrix to its characteristic polynomial is continuous. It follows that the subset $\text{QUnip}_n(\overline{\mathbb{Q}_\ell}) \subseteq M_n(\overline{\mathbb{Q}_\ell})$ of quasi-unipotent matrices is closed, and the subgroup of unipotent matrices $\text{Unip}_n(\overline{\mathbb{Q}_\ell}) < \text{QUnip}_n(\overline{\mathbb{Q}_\ell})$ is open.

LEMMA 2.12. *Consider short exact sequences of profinite groups*

$$1 \rightarrow P \rightarrow I \rightarrow I_\ell \rightarrow 1, \quad 1 \rightarrow I \rightarrow G \rightarrow G_s \rightarrow 1,$$

with P of supernatural order prime to ℓ and I_ℓ pro- ℓ Abelian. Assume that the conjugation action of G_s on I_ℓ is given by a character $\chi: G_s \rightarrow \mathbb{Z}_\ell^\times$ of infinite order. Then any continuous representation $\rho: G \rightarrow \text{GL}_n(\overline{\mathbb{Q}_\ell})$ is quasi-unipotent on I . Moreover, for any rational number $c > 1/(\ell - 1)$, we have

$$\text{QUnip}_n(\overline{\mathbb{Q}_\ell}) \cap (1 + \ell^c M_n(\overline{\mathbb{Z}_\ell})) \subseteq \text{Unip}_n(\overline{\mathbb{Q}_\ell}).$$

Proof. Let $U \in \text{QUnip}_n(\overline{\mathbb{Q}_\ell}) \cap (1 + \ell^c M_n(\overline{\mathbb{Z}_\ell}))$. Then U^a is unipotent for some integer $a > 0$, and $\log(U) = (1/a) \log(U^a)$ is nilpotent, so that $U = \exp(\log(U))$ is unipotent. This proves the second assertion.

The proof of the first assertion is identical to that of Grothendieck. Up to replacing G by an open subgroup, we may assume that ρ factors through the open subgroup $1 + \ell^c M_n(\overline{\mathbb{Z}_\ell})$. Then $\rho(P) = 1$. Take $g \in G_s$ such that $\chi(g)$ is not a root of unity. For $t \in I$, $\rho(t)$ is conjugate to $\rho(t)^{\chi(g)}$, so that $M = \log(\rho(t))$ is conjugate to $\log(\rho(t)^{\chi(g)}) = \chi(g) \log(\rho(t)) = \chi(g)M$. Thus, $\chi(g)^m \text{tr}(M^m) = \text{tr}(M^m)$, so that $\text{tr}(M^m) = 0$ for all $m \geq 1$. Therefore, M is nilpotent and $\rho(t) = \exp(M)$ is unipotent. \square

Proof of Proposition 2.10. The proof is similar to a part of Deligne’s proof of [Del73, Théorème 9.8].

We may assume that the index set I is finite. Let $L_i = [\mathcal{F}_i] - [\mathcal{G}_i]$ for \mathcal{F}_i and \mathcal{G}_i lisse on $[X/G]$. Let $x \in |D|$. Lemma 2.12 applies to the tame fundamental group $\pi_1^t([X_{(x)}/D(x)], \bar{a})$ (cf. [Del80, 1.7.12.1] in the case x above k). Indeed, by Abhyankar’s lemma [SGA1, XIII, Corollaire 5.3], we have a short exact sequence

$$1 \rightarrow I_t \rightarrow \pi_1^t([X_{(x)}/D(x)], \bar{a}) \xrightarrow{r} \pi_1([x/D(x)], \bar{x}) \rightarrow 1,$$

where $I_t = \prod_{\ell} \mathbb{Z}_\ell(1)^d$, d is the number of irreducible components of $D \times_{\bar{X}} \bar{X}_{(x)}$, and ℓ runs through primes different from the characteristic of x . Note that for any semisimple continuous representation $\rho: \pi_1^t([X_{(x)}/D(x)], \bar{a}) \rightarrow \text{GL}_n(\overline{\mathbb{Q}_\ell})$ and any $g \in I_t$ with $\rho(g)$ unipotent, we have $\rho(g) = 1$. Indeed, on each graded piece of the monodromy filtration given by the nilpotent operator $\log(\rho(g))$, g acts by 1.

In case (1) I_t acts unipotently on $(\mathcal{F}_i)_{\bar{a}}$ and $(\mathcal{G}_i)_{\bar{a}}$. In case (2) there exists an open subgroup I'_t of I_t that acts unipotently on $(\mathcal{F}_i)_{\bar{a}}$ and $(\mathcal{G}_i)_{\bar{a}}$. Each $g \in W(X_{(x)}, \bar{a})$ of degree 0 acts quasi-unipotently on $(\mathcal{F}_i)_{\bar{a}}$ and $(\mathcal{G}_i)_{\bar{a}}$. As in the proof of [Zhe09, Proposition 1.15], there exists a subgroup $G < W(X_{(x)}, \bar{a})$ of finite index such that the action of g commutes with that of G up to semisimplification. By [III06, Lemma 8.1], it suffices to consider $F \in W(X_{(x)}, \bar{a})$ of degree $\neq 0$. There exists an open subgroup $H < \pi_1^t(X_{(x)}, \bar{a})$ containing the image of F such that $H \cap I_t \subseteq I'_t$.

We may further assume that $H \cap I_t$ has the form NI_t for an integer $N > 0$ invertible on x . Let $\bar{Y}_{(y)}$ be the normalization of $\bar{X}_{(x)}$ in the pointed finite étale cover $(Y_{(y)}, \bar{b})$ of $(X_{(x)}, \bar{a})$ corresponding to H . Then $F \in W(Y_{(y)}, \bar{b})$. Moreover, $\bar{Y}_{(y)}$ is regular and the inverse image of D is a normal crossing divisor. Indeed, if $\bar{X}_{(\bar{x})}$ and $\bar{Y}_{(\bar{y})}$ denote the strict Henselizations and the irreducible components of $D \times_{\bar{X}} \bar{X}_{(\bar{x})}$ are defined by t_1, \dots, t_d , then $\bar{Y}_{(\bar{y})} \simeq \bar{X}_{(\bar{x})}[t_1^{1/N}, \dots, t_d^{1/N}]$. Therefore, up to replacing \bar{X} by \bar{Y} quasi-finite over \bar{X} giving rise to $\bar{Y}_{(y)}$, we may assume that I_t acts unipotently on $(\mathcal{F}_i)_{\bar{a}}$ and $(\mathcal{G}_i)_{\bar{a}}$.

Then the semisimplifications of $\mathcal{F}_i|_{[X_{(x)}/D(x)]}$ and $\mathcal{G}_i|_{[X_{(x)}/D(x)]}$ factor through r , so that $L_i|_{[X_{(x)}/D(x)]}$ is the pullback of $M_i \in K(\xi, \overline{\mathbb{Q}_\ell})$ via r , where $\xi = [x/D(x)]$. Let $j: [X/G] \rightarrow [\bar{X}/G]$ be the open immersion. By Theorem 2.3, $(Rj_*L_i)_{i \in I}$ is compatible. By the projection formula,

$$M_i \otimes_{\mathbb{Q}_{\ell_i}} (Rj_*\mathbb{Q}_{\ell_i})_{\xi} \simeq (Rj_*L_i)_{\xi}.$$

Gabber’s absolute purity theorem, extended to Deligne–Mumford stacks in the proof of Proposition 2.5, implies that (see [Ill02, Theorem 7.2] and [Rio14, Corollaire 3.1.4])

$$(R^m j_*\mathbb{Q}_{\ell})_{\xi} \simeq \begin{cases} \mathbb{Q}_{\ell}(-m) \binom{d}{m}, & 0 \leq m \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $F \in W(\xi, \bar{x})$ of degree n ,

$$\text{tr}(F, (Rj_*\mathbb{Q}_{\ell})_{\bar{x}}) = (1 - q^n)^d.$$

It follows that for $n \neq 0$, $\text{tr}(F, (M_i)_{\bar{x}})$ can be recovered from $\text{tr}(F, (Rj_*L_i)_{\bar{x}})$. Therefore, $(M_i)_{i \in I}$ is compatible by Remark 2.4. \square

PROPOSITION 2.13. *Let X be an integral normal scheme separated of finite type over S and let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbb{Q}_{\ell_i}})$ be a compatible system with I finite. Then there exist a proper morphism $f: X' \rightarrow X$ with X' integral normal inducing a universal homeomorphism $f^{-1}(U) \rightarrow U$ for some nonempty open $U \subseteq X$, and a normal compactification $X' \subseteq \bar{X}'$ over S , such that $(f^*L_i)_{i \in I}$ is compatible on \bar{X}' .*

Proof. We write $L_i = ([\mathcal{F}_i] - [\mathcal{G}_i]) \otimes_{\overline{\mathbb{Z}_{\ell_i}}} \overline{\mathbb{Q}_{\ell_i}}$. There exists a connected finite étale cover $Y \rightarrow X$, Galois of group G , such that $\mathcal{F}_i \otimes_{\overline{\mathbb{Z}_{\ell_i}}} \overline{\mathbb{Z}_{\ell_i}}/2\ell\overline{\mathbb{Z}_{\ell_i}}$ and $\mathcal{G}_i \otimes_{\overline{\mathbb{Z}_{\ell_i}}} \overline{\mathbb{Z}_{\ell_i}}/2\ell\overline{\mathbb{Z}_{\ell_i}}$ are constant.

Let S_0 be the closed point of S if X_K is empty and S otherwise. We apply Gabber’s refinement of de Jong’s equivariant alterations [deJ97] in the form of [Zhe09, Lemme 3.8] to the G -equivariant morphism $Y \rightarrow T$, where T is the normalization of S_0 in Y . There exist a Galois alteration $(Z, H) \rightarrow (Y, G)$ and an H -equivariant open immersion $Z \subseteq \bar{Z}$ with \bar{Z} regular and projective over S . Moreover, there exists an H -stable open subscheme $V \subseteq Z$ whose complement in \bar{Z} is a normal crossing divisor. Let $f: X' := Z/H \rightarrow Y/G \simeq X$. By the definition of Galois alteration, there exist a nonempty H -stable affine open subscheme $V_0 \subseteq V$ on which H acts freely and a nonempty open subscheme $U \subseteq X$ such that f induces a universal homeomorphism $f^{-1}(U) \rightarrow U$.

By Proposition 2.10, $(L_i|_{[V/H]})_{i \in I}$ is compatible on $[\bar{Z}/H]$. By Remark 2.7(3), $(L_i|_{V_0/H})_{i \in I}$ is compatible on $\bar{X}' := \bar{Z}/H$ and the proposition follows. \square

LEMMA 2.14. *Let $f: Y \rightarrow X$ be a universal homeomorphism between normal schemes separated of finite type over a Noetherian Nagata scheme T . Then, for any normal compactification \bar{Y} of*

Y over T , there exists a commutative diagram over T

$$\begin{array}{ccc} Y & \hookrightarrow & \bar{Y} \\ f \downarrow & & \downarrow \bar{f} \\ X & \hookrightarrow & \bar{X} \end{array}$$

where \bar{X} is a normal compactification of X over T and \bar{f} is a universal homeomorphism identifying \bar{Y} with the normalization of \bar{X} in Y .

Proof. We may assume that X is connected and that f is not an isomorphism. Let $K(X) \subseteq K(Y)$ be the fraction fields. There exists n such that $K(Y)^{p^n} \subseteq K(X)$, where $p > 0$ is the characteristic of $K(X)$. Up to replacing T by a closed subscheme, we may assume that $X \rightarrow T$ is dominant. The n th relative Frobenius factors as $Y \xrightarrow{f} X \rightarrow Y^{(p^n)}$. We take \bar{X} to be the normalization of $\bar{Y}^{(p^n)}$ in X . The morphism $\bar{f}: \bar{Y} \rightarrow \bar{X}$ is finite, surjective, and radicial and hence a universal homeomorphism. \square

Proof of Theorem 1.2. We may assume that X is reduced. By Proposition 2.13 and Lemma 2.14, there exist an integral normal open subscheme $X_0 \subseteq X$ and a normal compactification $X_0 \subseteq \bar{X}_0$ such that $(L_i|_{X_0})_{i \in I}$ is compatible on \bar{X}_0 . We conclude by Noetherian induction. \square

The theorem takes the following form in the case of curves, which is a theorem of Deligne [Del73, Théorème 9.8] in the case of curves over finite fields.

COROLLARY 2.15. *Let \bar{X} be a smooth curve over k or K and let $X \subseteq \bar{X}$ be a dense open subscheme. Then any compatible system $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbb{Q}}_{\ell_i})$ is compatible on \bar{X} .*

In this case, one may also directly adapt the proof of Proposition 2.10 with π_1^t replaced by π_1 .

Remark 2.16. Every pair of compactifications $Y \subseteq \bar{Y}_1$ and $Y \subseteq \bar{Y}_2$ over S (inclusions of dense open subschemes with \bar{Y}_1 and \bar{Y}_2 proper over S) are dominated by a third one: there exist a compactification $Y \subseteq \bar{Y}$ over S and morphisms $\bar{Y} \rightarrow \bar{Y}_1$ and $\bar{Y} \rightarrow \bar{Y}_2$ over S inducing the identity on Y . It suffices to take \bar{Y} to be the closure of the diagonal embedding $Y \subseteq \bar{Y}_1 \times_S \bar{Y}_2$. In the case where Y is normal, we may even take \bar{Y} to be normal by normalization.

It follows that in the situation of Theorem 1.2, every compactification $X_\alpha \subseteq \bar{X}'_\alpha$ is dominated by a normal compactification $X_\alpha \subseteq \bar{X}''_\alpha$ such that $(L_i|_{X_\alpha})_{i \in I}$ is compatible on \bar{X}''_α . This implies the following refinement of Proposition 2.13, which says that compatible systems are compatible along the boundary up to modification.

COROLLARY 2.17. *Let \bar{X} be a reduced scheme separated of finite type over S and let $X \subseteq \bar{X}$ be a dense open subscheme. Let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbb{Q}}_{\ell_i})$ be a compatible system with I finite. Then there exists a proper birational morphism $f: \bar{X}' \rightarrow \bar{X}$ with \bar{X}' normal such that $(f_X^* L_i)_{i \in I}$ is compatible on \bar{X}' . Here $f_X: f^{-1}(X) \rightarrow X$ is the restriction of f .*

Proof. Up to replacing \bar{X} by a compactification, we may assume that \bar{X} is proper over S . By Theorem 1.2, there exist a dense open subscheme U and a normal compactification $U \subseteq \bar{U}$ such that $(L_i|_U)_{i \in I}$ is compatible on \bar{U} . Let $U \subseteq \bar{X}'$ be a normal compactification dominating $U \subseteq \bar{X}$ and $U \subseteq \bar{U}$ and let $f: \bar{X}' \rightarrow \bar{X}$ be the morphism. Then $((f_X^* L_i)|_U)_{i \in I}$ is compatible on \bar{X}' . We conclude by Remark 2.7(2). \square

Proof of Corollary 1.3. The ‘if’ part of (1) follows from the definition. We prove (2) and the ‘only if’ part of (1). We may assume that I is finite. Up to replacing X by the closure of the image $\tau \in X$ of η , we may assume that X is irreducible of generic point τ . Up to shrinking X , we may assume that X is separated and $L_i \in K_{\text{lis}}(X, \overline{\mathbb{Q}}_{\ell_i})$ for all i . Let $X \subseteq \bar{X}$ be a compactification over S . We apply Corollary 2.17. Let $X' = f^{-1}(X)$. Note that (1.1) gives rise to a commutative square

$$\begin{array}{ccc} \eta & \longrightarrow & \bar{X}' \\ \downarrow & \nearrow g & \downarrow \\ V & \longrightarrow & S \end{array}$$

By the valuative criterion of properness, there exists a slashed arrow g as indicated, making the diagram commutative. In case (1), g induces $\eta \rightarrow X'_{(x)}$ and $W(\bar{\eta}/\eta) \rightarrow W(X'_{(x)}, \bar{\eta})$, where $x = g(t)$. In case (2), g induces $\eta \rightarrow X'_{(g(\bar{t}))} := \bar{X}'_{(g(\bar{t}))} \times_{\bar{X}'} X'$, where $\bar{X}'_{(g(\bar{t}))}$ denotes the strict Henselization of \bar{X}' at the geometric point $g(\bar{t})$, the image of the geometric point $\bar{t} = t$ of V under g . The geometric point $g(\bar{t})$ of \bar{X}' specializes to a geometric point \bar{x} above $x \in |X'|$. We have $\eta \rightarrow X'_{(g(\bar{t}))} \rightarrow X'_{(\bar{x})} \rightarrow X'_{(x)}$, which induces $\text{Gal}(\bar{\eta}/\eta) \rightarrow \pi_1(X'_{(\bar{x})}, \bar{\eta}) \subseteq W(X'_{(x)}, \bar{\eta})$. \square

Proof of Theorem 1.4. Let us first show (1) and (2b). We write $V_\ell = H^m(X_{\bar{L}}, \mathbb{Q}_\ell)$. By standard limit arguments, there exists a finitely generated subalgebra $R \subseteq L$ over \mathbb{F}_p such that X is defined over $B = \text{Spec}(R)$: there exists $f: \mathcal{X} \rightarrow B$ proper smooth such that $X \simeq \mathcal{X} \times_B \eta$, where $\eta = \text{Spec}(L)$. By the Grothendieck trace formula, the system $(Rf_* \overline{\mathbb{Q}}_\ell)_\ell$ on B is compatible. Each $R^m f_* \overline{\mathbb{Q}}_\ell$ is lisse and pure of weight m . It follows that $(R^m f_* \overline{\mathbb{Q}}_\ell)_\ell$ is compatible. By base change, $(R^m f_* \overline{\mathbb{Q}}_\ell)_{\bar{\eta}} \simeq V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$. Applying Corollary 1.3 to the composition of the commutative square

$$\begin{array}{ccc} \eta & \longrightarrow & B \\ \downarrow & & \downarrow \\ \text{Spec}(\mathcal{O}_L) & \longrightarrow & \text{Spec}(\mathbb{F}_p) \end{array}$$

and the closed immersion $\text{Spec}(\mathbb{F}_p) \rightarrow \text{Spec}(\mathcal{O}_K)$, where \mathcal{O}_K is any Henselian discrete valuation ring of residue field \mathbb{F}_p , we see that $\text{tr}(F, V_\ell)$ is a rational number independent of ℓ . In case (1), the eigenvalues are roots of unity by Grothendieck’s geometric local monodromy theorem [SGA7-1, Variante 1.3]. In case (2b), the eigenvalues are algebraic integers by a theorem of Ochiai [Och99, Proposition A]. It follows that in both cases $\text{tr}(F, V_\ell)$ is a rational integer independent of ℓ .

Part (2a) follows from (2b) and the monodromy weight conjecture, which is a theorem of Terasoma [Ter98, Lemma 1.2] and more generally Ito [Ito05, Proposition 7.1] in equal characteristic. Indeed, $\text{gr}_i^M V$ is pure of weight $m + i$, so that the characteristic polynomial of F on $\text{gr}_i^M V$ can be extracted from the characteristic polynomial of F on V . \square

The proof of Theorem 1.4 relies only on the special case of Theorem 1.2 with S replaced by $\text{Spec}(\mathbb{F}_p)$.

Remark 2.18. (1) Ito’s proof of the monodromy weight conjecture [Ito05] in equal characteristic and Grothendieck’s proof of the geometric local monodromy theorem both use Néron’s desingularization. For Theorem 1.4, the reduction to the tame case is more involved and Néron’s desingularization does not suffice.

(2) Theorem 1.4(2) implies that for each $F \in W^{\geq 0}(\bar{\kappa}/\kappa)$, $\text{tr}(F, H^m(X_{\bar{L}}, \mathbb{Q}_\ell)^I)$ is a rational integer independent of ℓ . Here $I = I(\bar{L}/L)$ denotes the inertia group. The eigenvalues being algebraic integers, it suffices to show that the trace is in \mathbb{Q} and independent of ℓ . Consider the primitive parts of $V_\ell = H^m(X_{\bar{L}}, \mathbb{Q}_\ell)$ defined by $P_i = \text{gr}_{-i}^M \text{Ker}(N)$ for $i \geq 0$. Here $N: V_\ell \rightarrow V_\ell(-1)$ is the logarithm of the unipotent part of the local monodromy. By the identity $\text{gr}_{-i}^M V_\ell = P_i \oplus \text{gr}_{-i-2}^M V_\ell(-1)$, $i \geq 0$, the primitive parts P_i and consequently $\text{Ker}(N)$ are compatible. Moreover, $V_\ell^I = \text{Ker}(N)^I$ and there exists an open subgroup U of I acting trivially on $\text{Ker}(N)$, so that $\text{tr}(F, V_\ell^I) = (1/[I : U]) \sum_{F'} \text{tr}(F', \text{Ker}(N)) \in \mathbb{Q}$ is independent of ℓ . Here F' runs through liftings of F in $\text{Gal}(\bar{L}/L)/U$.

(3) Theorem 1.4(1) and (2b) hold in fact without the assumption that the valuation on \mathcal{O}_L is discrete. The proof that $\text{tr}(F, H^m(X_{\bar{L}}, \mathbb{Q}_\ell))$ is rational and independent of ℓ is the same as above. For integrality, we apply Corollary 3.10 below.

3. Integrality along the boundary

Fix an integrally closed subring A of $\overline{\mathbb{Q}_\ell}$. A typical example is the integral closure of \mathbb{Z} in $\overline{\mathbb{Q}_\ell}$. Recall from [Zhe08, Variantes 5.11 and 5.13] that a $\overline{\mathbb{Q}_\ell}$ -sheaf \mathcal{F} on a scheme X of finite type over S is said to be *integral* if for every $x \in |X|$, the eigenvalues of $F \in W^{\geq 0}(\bar{x}/x)$ on $\mathcal{F}_{\bar{x}}$ belong to A . In this section, we study the integrality of integral sheaves on the boundary.

DEFINITION 3.1. Let \bar{X} be a normal scheme of finite type over S and let X be a dense open subscheme. Let \mathcal{F} be a lisse $\overline{\mathbb{Q}_\ell}$ -sheaf on X . We say that \mathcal{F} is *integral on \bar{X}* if for every $x \in |X|$ and every geometric point $\bar{a} \rightarrow X_{(x)}$, the eigenvalues of every $F \in W^{\geq 0}(X_{(x)}, \bar{a})$ on $\mathcal{F}_{\bar{a}}$ belong to A .

We have the following analogues of Remarks 2.7(1) and 2.8. A lisse $\overline{\mathbb{Q}_\ell}$ -sheaf \mathcal{F} on X is integral on X if and only if it is integral. Moreover, in Definition 3.1 we may restrict to x closed in \bar{X} .

Remark 3.2. Let $f: \bar{X} \rightarrow \bar{Y}$ be a finite surjective morphism of integral normal schemes of finite type over S and let $X \subseteq \bar{X}$, $Y \subseteq \bar{Y}$ be nonempty open subschemes satisfying $f(X) \subseteq Y$. Let $g: X \rightarrow Y$ be the restriction of f . Then a lisse $\overline{\mathbb{Q}_\ell}$ -sheaf \mathcal{F} on Y is integral on \bar{Y} if and only if $g^* \mathcal{F}$ is integral on \bar{X} .

The ‘only if’ part is obvious. For the ‘if’ part, up to shrinking X and Y as in Remark 2.7(2), we may assume that g is the composition of a universal homeomorphism with a finite étale morphism. In this case, for every $x \in \bar{X}$, $\pi_1(X_{(x)}, \bar{a})$ is an open subgroup of $\pi_1(Y_{(f(x))}, f(\bar{a}))$, say of index m . Then, for each eigenvalue λ of $F \in W^{\geq 0}(Y_{(f(x))}, f(\bar{a}))$ acting on $\mathcal{F}_{\bar{a}}$, we have $\lambda^m \in A$, so that $\lambda \in A$.

We have the following analogue of Theorem 1.2.

THEOREM 3.3. *Let X be a scheme of finite type over S and let \mathcal{F} be an integral lisse $\overline{\mathbb{Q}_\ell}$ -sheaf on X . Then there exists a finite stratification $X = \bigcup_\alpha X_\alpha$ by normal subschemes such that each X_α admits a normal compactification \bar{X}'_α over S such that $\mathcal{F}|_{X_\alpha}$ is integral on \bar{X}'_α .*

The theorem implies, by Remark 2.16, that every compactification $X_\alpha \subseteq \bar{X}'_\alpha$ is dominated by a normal compactification $X_\alpha \subseteq \bar{X}''_\alpha$ such that $\mathcal{F}|_{X_\alpha}$ is integral on \bar{X}''_α .

COROLLARY 3.4. *Let \bar{X} be a projective smooth curve over k or K and let $X \subseteq \bar{X}$ be a dense open subscheme. Then any integral lisse $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F} on X is integral on \bar{X} .*

The case of a curve over a finite field is a theorem of Deligne [Del80, Théorème 1.10.3].

The proof of Theorem 3.3 relies on the case $m = 0$ of the following theorem [Zhe08, Théorème 2.5 and Variantes 5.11 and 5.13].

THEOREM 3.5. *Let $f: X \rightarrow Y$ be a morphism of schemes of finite type over S . Let \mathcal{F} be an integral $\overline{\mathbb{Q}}_\ell$ -sheaf on X . Then $R^m f_* \mathcal{F}$ is integral for all m .*

The analogue for $R^m f_!$ was proved by Deligne and Esnault (see [DK73, XXI, Théorème 5.2.2] and [Esn06, Appendix, Theorem 0.2]). We refer to [Zhe08] for a description of the behavior of integral sheaves under other operations.

COROLLARY 3.6. *Let X be a scheme of finite type over S and let \mathcal{F} be a lisse $\overline{\mathbb{Q}}_\ell$ -sheaf on X . Assume that $\mathcal{F}|_U$ is integral for some dense open subscheme $U \subseteq X$. Then \mathcal{F} is integral.*

In the case X of finite type over k , this was noted in [Zhe19, Proposition 2.4].

Proof. Up to replacing X by its normalization, we may assume that X is normal. Let $j: U \rightarrow X$ be the open immersion. Then $\mathcal{F} \simeq j_*(\mathcal{F}|_U)$ is integral by Theorem 3.5. □

PROPOSITION 3.7. *Let \bar{X} be a regular scheme of finite type over S and let D be a normal crossing divisor. Let \mathcal{F} be an integral lisse $\overline{\mathbb{Q}}_\ell$ -sheaf on $X = \bar{X} - D$, tamely ramified on \bar{X} . Then \mathcal{F} is integral on \bar{X} . Moreover, $R^m j_* \mathcal{F}(m)$ is integral for all m , where $j: X \rightarrow \bar{X}$ is the open immersion.*

We will only need the first assertion. Some cases of the second assertion were proved in [Zhe09, Proposition 3.8 and Variantes 5.11 and 5.13].

Proof. We may assume that $D = \sum_{i \in I} D_i$ is a strict normal crossing divisor with D_i regular and defined globally by $t_i = 0$, and \mathcal{F} is L -ramified, where L is the set of prime numbers invertible on \bar{X} . We apply the construction of [Del80, 1.7.9]. For $J \subseteq I$, let $D_J^* = \bigcap_{j \in J} D_j \cap \bigcap_{i \in J^c} (\bar{X} - D_i)$. For each locally constant constructible sheaf of sets \mathcal{G} on X , L -ramified on \bar{X} , there exists an integer n invertible on \bar{X} such that \mathcal{G} extends to \mathcal{G}' on the cover $\bar{X}[t_i^{1/n}]_{i \in I}$ of \bar{X} , and we let $\mathcal{G}[D_J^*]$ denote the restriction of \mathcal{G}' to D_J^* , which is locally constant constructible. The action of μ_n^I on $\bar{X}[t_i^{1/n}]_{i \in I}$ induces an action of μ_n^J on $\mathcal{G}[D_J^*]$ (as μ_n^J acts trivially on D_J^*). Extending this construction to $\overline{\mathbb{Q}}_\ell$ -sheaves by taking limits, we obtain a lisse $\overline{\mathbb{Q}}_\ell$ -sheaf $\mathcal{F}[D_J^*]$ on D_J^* equipped with an action of I_L^J , where $I_L = \hat{\mathbb{Z}}_L(1)$.

Let us show that $\mathcal{F}[D_J^*]$ is integral. For $J \subseteq J' \subseteq I$, $\mathcal{F}[D_{J'}^*] = \mathcal{F}[D_J^*][D_{J'}^*]$. Thus, by induction we may assume that $\#J = 1$. Changing notation, it suffices to show that $\mathcal{F}[D]$ is integral for D a regular divisor defined by $t = 0$. By Grothendieck’s arithmetic local monodromy theorem applied to the Henselization of \bar{X} at the generic point of D , the action of $I_L = \hat{\mathbb{Z}}_L(1)$ on $\mathcal{F}[D]$ is quasi-unipotent. Up to replacing \bar{X} by $\bar{X}[t^{1/n}]$, we may assume that the action of I_L on $\mathcal{F}[D]$ is unipotent. Let $N: \mathcal{F}[D] \rightarrow \mathcal{F}[D](-1)$ be the logarithm of the action of I_L and let M be the local monodromy filtration on $\mathcal{F}[D]$. Then $\text{Ker}(N) = \mathcal{F}[D]^{I_L} \simeq (j_* \mathcal{F})|_D$, which is integral by Theorem 3.5. Thus, the primitive parts $P_i = \text{gr}_{-i}^M \text{Ker}(N)$ are integral. It follows that

$\mathrm{gr}_i^M \mathcal{F}[D] \simeq \bigoplus_j P_j(-(j+i)/2)$ is integral. Here j runs through integers $j \geq |i|$ satisfying $j \equiv i \pmod{2}$. Therefore, $\mathcal{F}[D]$ is integral.

Let $x \in D_j^*$. We have an exact sequence

$$1 \rightarrow I_t \rightarrow \pi_1^t(X_{(x)}, \bar{a}) \xrightarrow{r} \mathrm{Gal}(\bar{x}/x) \rightarrow 1. \tag{3.1}$$

By Lemma 2.12, there exists an open subgroup V of I_t acting unipotently on $\mathcal{F}_{\bar{a}}$. Assume that $x \in |D_j^*|$ and let \mathcal{F}' denote the semisimplification of $\mathcal{F}|_{X_{(x)}}$. Then V acts trivially on $\mathcal{F}'_{\bar{a}}$. The choice of a geometric point of $\lim_n X_{(x)}[t_i^{1/n}]_{i \in I}$ above \bar{a} gives a section s of r and $\mathcal{F}[D_j^*]_x$ corresponds to the action of $\mathrm{Gal}(\bar{x}/x)$ on $\mathcal{F}_{\bar{a}}$ via s . Then $U = V \cdot \mathrm{Im}(s)$ is an open subgroup of $\pi_1^t(X_{(x)}, \bar{a})$. For $F \in U$, the eigenvalues of F acting on $\mathcal{F}_{\bar{a}}$ are the same as the eigenvalues of $r(F)$ acting on $\mathcal{F}[D_j^*]_{\bar{x}}$, which belong to A if $F \in W^{\geq 0}$. It follows that \mathcal{F} is integral on \bar{X} .

For the second assertion of the proposition, note that the restriction of $R^m j_* \mathcal{F}$ to D_j^* is $H^m(I_L^J, \mathcal{F}[D_j^*])$. Since taking invariants $H^0(I_L, -) = (-)^{I_L}$ and co-invariants $H^1(I_L, -)(1) \simeq (-)_{I_L}$ preserve integral sheaves, the same holds for $H^m(I_L^J, -)(m) \simeq \bigoplus_K (-)_{I_L^K}^{J-K}$, where $K \subseteq J$ runs through subsets of cardinality m . \square

The rest of the proof of Theorem 3.3 is similar to that of Theorem 1.2. We proceed by Noetherian induction and reduce by Lemma 2.14 to proving the following.

PROPOSITION 3.8. *Let X be an integral normal scheme separated of finite type over S and let \mathcal{F} be an integral lisse $\overline{\mathbb{Q}_\ell}$ -sheaf on X . Then there exist a proper morphism $f: X' \rightarrow X$ with X' connected normal inducing a universal homeomorphism $f^{-1}(U) \rightarrow U$ for some nonempty open $U \subseteq X$, and a normal compactification $X' \subseteq \bar{X}'$ over S , such that $f^* \mathcal{F}$ is integral on \bar{X}' .*

Proof. The proof is similar to that of Proposition 2.13, except that here we do not need to work with stacks. We write $\mathcal{F} = (\mathcal{F}_0) \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Q}_\ell}$. There exists a finite étale cover $Y \rightarrow X$, Galois of group G , such that $\mathcal{F}_0 \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{Z}_\ell}/\ell \overline{\mathbb{Z}_\ell}$ is constant. We apply the second paragraph of the proof of Proposition 2.13. Since $\mathcal{F}|_V$ is tamely ramified on \bar{Z} , $\mathcal{F}|_V$ is integral on \bar{Z} by Proposition 3.7. Thus, by Remark 3.2, $f^* \mathcal{F}$ is integral on \bar{X}' and the proposition follows. \square

The same proof, with Proposition 3.7 replaced by Lemma 2.12 applied to (3.1), yields the following result on quasi-unipotence.

THEOREM 3.9. *Let X be a scheme of finite type over S and let \mathcal{F} be a lisse $\overline{\mathbb{Q}_\ell}$ -sheaf on X . Then there exists a finite stratification $X = \bigcup_\alpha X_\alpha$ by normal subschemes such that each X_α admits a normal compactification \bar{X}_α over S such that for every geometric point $\bar{x} \rightarrow \bar{X}$ and every geometric point $\bar{a} \rightarrow (X_\alpha)_{(\bar{x})} := (\bar{X}_\alpha)_{(\bar{x})} \times_{\bar{X}_\alpha} X_\alpha$, the action of $\pi_1((X_\alpha)_{(\bar{x})}, \bar{a})$ on $\mathcal{F}_{\bar{a}}$ is quasi-unipotent. Here $(\bar{X}_\alpha)_{(\bar{x})}$ denotes the strict Henselization.*

The analogues of Corollaries 2.17 and 1.3 hold with the same proofs. Let us state the analogue of Corollary 1.3.

COROLLARY 3.10. *Let X be a scheme of finite type over S and let \mathcal{F} be a $\overline{\mathbb{Q}_\ell}$ -sheaf on X . Then \mathcal{F} is integral if and only if for every commutative square (1.1) with t quasi-finite over S , the eigenvalues of every $F \in W^{\geq 0}(\bar{\eta}/\eta)$ acting on $\mathcal{F}_{\bar{\eta}}$ belong to A . Moreover, if (1.1) is a commutative square with V strictly Henselian, then the action of $\mathrm{Gal}(\bar{\eta}/\eta)$ on $\mathcal{F}_{\bar{\eta}}$ is quasi-unipotent.*

4. Ramified and decomposed parts of the fundamental group

In this section, we give applications related to Vidal’s ramified part of the fundamental group [Vid05, § 1.2]. We show that compatible systems have compatible ramification (Corollary 4.6) and, consequently, their reductions have compatible wild ramification (Corollary 4.14).

Let us first review the definition of the ramified part of the fundamental group. Let T be the spectrum of an excellent Henselian discrete valuation ring of residue characteristic exponent $p \geq 1$.

DEFINITION 4.1 (Vidal). Let X be an integral normal scheme separated of finite type over T and let \bar{a} be a geometric generic point of X . Let $X \subseteq \bar{X}$ be a normal compactification over T . Let $\bar{x} \rightarrow \bar{X}$ be a geometric point above $x \in \bar{X}$ and let $\bar{X}_{(\bar{x})}$ denote the strict Henselization. Let $\bar{b} \rightarrow X_{(\bar{x})} := X \times_{\bar{X}} \bar{X}_{(\bar{x})}$ be a geometric point above \bar{a} . We define the following closed subsets of $\pi_1(X, \bar{a})$, each of which is a union of subgroups.

- The subgroup $E_{X, \bar{X}, x, \bar{b}} = \text{Im}(\pi_1(X_{(\bar{x})}, \bar{b}) \rightarrow \pi_1(X, \bar{a}))$. See Remark 4.2(1) below for the justification of the subscript x instead of \bar{x} .
- $E_{X, \bar{X}}$, the closure of $\bigcup_{x, \bar{b}} E_{X, \bar{X}, x, \bar{b}}$, where x runs through points of \bar{X} and \bar{b} runs through geometric points above \bar{a} .
- The ramified part $E_{X/T} = \bigcap_{\bar{X}} E_{X, \bar{X}}$, where \bar{X} runs through normal compactifications of X over T .

The subsets $E_{X, \bar{X}}$ and $E_{X/T}$ are stable under conjugation.

Remark 4.2. (1) We have a short exact sequence

$$1 \rightarrow \pi_1(X_{(\bar{x})}, \bar{b}) \xrightarrow{i} \pi_1(X_{(x)}, \bar{b}) \xrightarrow{\rho} \pi_1(\bar{X}_{(x)}, \bar{b}) \rightarrow 1,$$

where $\pi_1(\bar{X}_{(x)}, \bar{b}) \simeq \text{Gal}(\bar{x}/x)$. The image of i depends on \bar{x} only via x and depends on \bar{b} as a geometric point of $X_{(x)}$.

(2) For any specialization $\bar{x} \rightarrow X_{(\bar{y})}$, we have $E_{X, \bar{X}, x, \bar{b}} \subseteq E_{X, \bar{X}, y, \bar{b}}$. Thus, in the definition of $E_{X, \bar{X}}$, we may restrict to closed points $x \in \bar{X}$.

(3) It follows from Gabber’s valuative criterion [Vid05, § 6.1] that for any finite stratification $X = \bigcup_{\alpha} X_{\alpha}$ into integral normal subschemes, $E_{X/T}$ is the closure of $\bigcup_{\alpha, \gamma_{\alpha}} \gamma_{\alpha}(E_{X_{\alpha}/T})$, where γ_{α} runs through paths from a geometric generic point $\bar{a}_{\alpha} \rightarrow X_{\alpha}$ to $\bar{a} \rightarrow X$.

Recall that S is the spectrum of an excellent Henselian discrete valuation ring with a finite residue field.

DEFINITION 4.3. Let X be a scheme of finite type over S . We say that a system $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{Q}_{\ell_i}})$ has compatible ramification if for every separated integral normal subscheme $Y \subseteq X$, $(\text{tr}(g, (L_i)_{\bar{a}}))_{i \in I_Y}$ is compatible for all $g \in E_{Y/T}$. Here \bar{a} is a geometric generic point of Y and $I_Y \subseteq I$ is the subset of i such that $L_i|_Y$ is in $K_{\text{lisse}}(Y, \overline{\mathbb{Q}_{\ell_i}})$.

Remark 4.4. Let X be an integral normal scheme separated of finite type over S and let \mathcal{F} be a lisse $\overline{\mathbb{Q}_{\ell}}$ -sheaf on X . Then the action of $E_{X/S}$ on $\mathcal{F}_{\bar{a}}$ is quasi-unipotent. This follows from Theorem 3.9, Remark 4.2(3), and the fact that quasi-unipotent matrices form a closed subset of $\text{GL}_n(\overline{\mathbb{Q}_{\ell}})$ (Remark 2.11).

Combining this with Gabber’s valuative criterion [Vid05, §6.1], we obtain the following valuative criterion for compatible ramification.

LEMMA 4.5. *Let X be a scheme of finite type over S . Then $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{Q}_{\ell_i}})$ has compatible ramification if and only if for every commutative square (1.1) with V strictly Henselian, $\text{tr}(F, (L_i)_{\bar{\eta}})_{i \in I}$ is compatible for all $F \in \text{Gal}(\bar{\eta}/\eta)$.*

Proof. We may assume that I is finite, X integral normal separated, and $L_i = [\mathcal{F}_i] - [\mathcal{G}_i]$ with \mathcal{F}_i and \mathcal{G}_i lisse, respectively, of ranks m_i and n_i . Consider the continuous map

$$\sigma: E_{X/S} \rightarrow C = \prod_{i \in I} (P_{\ell_i}^{m_i, \text{qu}} \times P_{\ell_i}^{n_i, \text{qu}})$$

carrying g to $(\det(T \cdot 1 - g, (\mathcal{F}_i)_{\bar{a}}), \det(T \cdot 1 - g, (\mathcal{G}_i)_{\bar{a}}))$, where $P_{\ell_i}^{r, \text{qu}}$ is as in Remark 2.11. Gabber’s criterion says that $E_{X/S}$ is the closure of the union of the images of $\text{Gal}(\bar{\eta}/\eta)$. Since C is discrete, $\sigma(E_{X/S})$ is the union of the images of $\text{Gal}(\bar{\eta}/\eta)$. □

Corollary 1.3(2) now takes the following form.

COROLLARY 4.6. *Let X be a scheme of finite type over S . Then any compatible system $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{Q}_{\ell_i}})$ has compatible ramification.*

Let P be a set of prime numbers. Given a profinite group G , we let $G^P \subseteq G$ denote the subset of elements g such that all prime factors of the supernatural order of g are contained in P . Note that G^P is a closed subset stable under conjugation, a union of subgroups of G . For a continuous homomorphism of profinite groups $\alpha: G \rightarrow H$, we have $\alpha(G^P) = \alpha(G) \cap H^P$.

We write $(p) = \{p\}$ for $p > 1$ and $(p) = \emptyset$ for $p = 1$. Then $G^{(p)}$ is the union of the p -Sylow subgroups of G .

Notation 4.7. In the situation of Definition 4.1, for $G = \pi_1(X, \bar{a})$, we write

$$E_{X, \bar{X}, x, \bar{b}}^P = E_{X, \bar{X}, x, \bar{b}} \cap G^P, \quad E_{X, \bar{X}}^P = E_{X, \bar{X}} \cap G^P, \quad E_{X/T}^P = E_{X/T} \cap G^P.$$

Remark 4.8. Alternatively, we can define these subsets as follows:

- $E_{X, \bar{X}, x, \bar{b}}^P = \text{Im}(\pi_1(X_{(\bar{x})}, \bar{b})^P \rightarrow \pi_1(X, \bar{a}))$;
- $E_{X, \bar{X}}^P$ is the closure of $\bigcup_{x, \bar{b}} E_{X, \bar{X}, x, \bar{b}}^P$, where x runs through points of \bar{X} and \bar{b} runs through geometric points above \bar{a} ;
- $E_{X/T}^P = \bigcap_{\bar{X}} E_{X, \bar{X}}^P$, where \bar{X} runs through normal compactifications of X over T .

For $P = (p)$, $E_{X/T}^{(p)}$ is called the *wildly ramified part* of the fundamental group and was defined by Vidal [Vid04, 2.1]. Our notation differs from that of Vidal, who writes $E'_{X/T}$ and $E_{X/T}$ for our $E_{X/T}$ and $E_{X/T}^{(p)}$, respectively.

Next we define compatible P -ramification for systems of $\overline{\mathbb{F}_{\ell}}$ -sheaves, where $\overline{\mathbb{F}_{\ell}}$ denotes an algebraic closure of \mathbb{F}_{ℓ} . For a profinite group G , an element $g \in G$ that is ℓ -regular (namely, of supernatural order prime to ℓ) and a virtual $\overline{\mathbb{F}_{\ell}}$ -representation M of G , the Brauer trace is defined by $\text{tr}^{\text{Br}}(g, M) = \sum_{\lambda} [\lambda]$, where λ runs through eigenvalues of g acting on M (with multiplicities) and $[\lambda]$ denotes the Teichmüller lift. Note that $\text{tr}^{\text{Br}}(g, M)$ is a sum of roots of unity (of order prime to ℓ) in $\overline{\mathbb{Q}_{\ell}}$.

Let X be a scheme of finite type over T . Let $K(X, \overline{\mathbb{F}_{\ell}})$ denote the Grothendieck group of constructible $\overline{\mathbb{F}_{\ell}}$ -sheaves. Recall that we fixed field embeddings $\iota_i: \mathbb{Q} \rightarrow \overline{\mathbb{Q}_{\ell_i}}$ for $i \in I$.

DEFINITION 4.9. Assume that P does not contain any ℓ_i . We say that a system $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{F}}_{\ell_i})$ has *compatible P -ramification* if for every separated integral normal subscheme $Y \subseteq X$, $(\text{tr}^{\text{Br}}(g, (L_i)_{\bar{a}}))_{i \in I_Y}$ is compatible for all $g \in E_{Y/T}^P$. Here \bar{a} is a geometric generic point of Y and $I_Y \subseteq I$ is the subset of i such that $L_i|_Y$ is in $K_{\text{lisse}}(Y, \overline{\mathbb{F}}_{\ell_i})$. We say that $(L_i)_{i \in I}$ has *compatible wild ramification* if it has compatible (p) -ramification.

In the special case $\ell_i = \ell$ and $Q = \overline{\mathbb{Q}}_{\ell}$, one recovers the notion of the same wild ramification of Deligne [Ill81] and Vidal [Vid04]. A weaker condition was recently studied by Saito and Yatagawa [SY17, Yat18]. In [Guo18], Guo showed that systems of compatible wild ramification in the sense of Definition 4.9 are preserved by Grothendieck’s six operations and duality.

Gabber’s valuative criterion [Vid05, §6.1] implies the following valuative criterion for compatible P -ramification.

LEMMA 4.10. *Let X be a scheme of finite type over T . Then $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{F}}_{\ell_i})$ has compatible P -ramification if and only if for every commutative square*

$$\begin{array}{ccc} \eta & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec}(\mathcal{O}_L) & \longrightarrow & T \end{array}$$

where \mathcal{O}_L is a strictly Henselian valuation ring of fraction field L and $\eta = \text{Spec}(L)$, $(\text{tr}^{\text{Br}}(g, (L_i)_{\bar{\eta}}))$ is compatible for all $g \in \text{Gal}(\bar{\eta}/\eta)^P$. Here $\bar{\eta} \rightarrow \eta$ is a geometric point.

Remark 4.11. Let X be an integral normal scheme separated of finite type over T and let $(L_i) \in \prod_{i \in I} K_{\text{lisse}}(X, \overline{\mathbb{F}}_{\ell_i})$ with I finite. Then $(L_i)_{i \in I}$ has compatible P -ramification if and only if there exists a normal compactification $X \subseteq \bar{X}$ over T such that $(\text{tr}^{\text{Br}}(g, (L_i)_{\bar{a}}))_{i \in I}$ is compatible for all $g \in E_{X, \bar{X}}^P$.

Indeed, for any finite quotient G of $\pi_1(X, \bar{a})$, if we let $E_{X, \bar{X}}^P(G)$ and $E_{X/T}^P(G)$ denote respectively the images of $E_{X, \bar{X}}^P$ and $E_{X/T}^P$ in G , then we have $E_{X/T}^P(G) = \bigcap_{\bar{X}} E_{X, \bar{X}}^P(G)$ by Lemma 4.12 below, and it follows that $E_{X/T}^P(G) = E_{X, \bar{X}}^P(G)$ for some \bar{X} . Here we used the fact that any pair of normal compactifications is dominated by a third one (cf. Remark 2.16).

LEMMA 4.12. *Let Π be a topological space and let B be a downward-directed set of closed subsets of Π : for $E_1, E_2 \in B$, there exists $E \in B$ such that $E \subseteq E_1 \cap E_2$. Let $\sigma: \Pi \rightarrow C$ be a map such that all fibers are compact. Then $\sigma(\bigcap_{E \in B} E) = \bigcap_{E \in B} \sigma(E)$.*

Proof. We have $\sigma(\bigcap_{E \in B} E) \subseteq \bigcap_{E \in B} \sigma(E)$. Conversely, let $g \in C - \bigcap_{E \in B} \sigma(E)$. Then $\sigma^{-1}(g) \cap \bigcap_{E \in B} E = \emptyset$. Since $\sigma^{-1}(g)$ is compact, there exist $E_1, \dots, E_n \in B$ such that $\sigma^{-1}(g) \cap \bigcap_{i=1}^n E_i = \emptyset$. Since B is downward directed, there exists $E \in B$ such that $\sigma^{-1}(g) \cap E = \emptyset$. In other words, $g \in C - \bigcap_{E \in B} \sigma(E)$. □

DEFINITION 4.13. Let E_{λ} be a finite extension of \mathbb{Q}_{ℓ} , of ring of integers \mathcal{O}_{λ} and residue field \mathbb{F}_{λ} . Consider the composition

$$K(X, E_{\lambda}) \xrightarrow[\sim]{(j^*)^{-1}} K(X, \mathcal{O}_{\lambda}) \xrightarrow{i^*} K(X, \mathbb{F}_{\lambda}),$$

where j^* is given by $-\otimes_{\mathcal{O}_\lambda} E_\lambda$ and i^* is given by $-\otimes_{\mathcal{O}_\lambda}^L \mathbb{F}_\lambda$. By [Zhe15, Proposition 9.4], j^* is an isomorphism and i^* is a surjection. Taking the colimit, we get the *decomposition map*

$$d_X : K(X, \overline{\mathbb{Q}_\ell}) \rightarrow K(X, \overline{\mathbb{F}_\ell}),$$

which is a surjection.

It follows from the definition that if $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{Q}_{\ell_i}})$ has compatible ramification, then $(d_X(L_i)) \in \prod_{i \in I} K(X, \overline{\mathbb{F}_{\ell_i}})$ has compatible P -ramification. Thus, Corollary 4.6 implies the following.

COROLLARY 4.14. *Let X be a scheme of finite type over S . Let $(L_i) \in \prod_{i \in I} K(X, \overline{\mathbb{Q}_{\ell_i}})$ be a compatible system. Then $(d_X(L_i)) \in \prod_{i \in I} K(X, \overline{\mathbb{F}_{\ell_i}})$ has compatible P -ramification, where P is the set of primes not equal to any ℓ_i . In particular, $(d_X(L_i))_{i \in I}$ has compatible wild ramification.*

Next we define the decomposed part of the fundamental group. The first three steps of the definition are analogous to Definition 4.1, with inertia groups (associated to strict Henselizations) replaced by decomposition groups (associated to Henselizations).

DEFINITION 4.15. Let X be an integral normal scheme separated of finite type over T and let \bar{a} be a geometric generic point of X . Let $X \subseteq \bar{X}$ be a normal compactification over T . Let $x \in \bar{X}$ be a point and let $\bar{b} \rightarrow X_{(x)}$ be a geometric point above \bar{a} . Let $X = \bigcup_\alpha X_\alpha$ be a finite stratification of X into integral normal subschemes. We define the following closed subsets of $\pi_1(X, \bar{a})$, each of which is a union of subgroups.

- The subgroup $D_{X, \bar{X}, x, \bar{b}} = \text{Im}(\pi_1(X_{(x)}, \bar{b}) \rightarrow \pi_1(X, \bar{a}))$.
- $D_{X, \bar{X}}$, the closure of $\bigcup_{x, \bar{b}} D_{X, \bar{X}, x, \bar{b}}$, where x runs through *locally closed* points of \bar{X} and \bar{b} runs through geometric points above \bar{a} .
- $D_{X/T}^{\text{naive}} = \bigcap_{\bar{X}} D_{X, \bar{X}}$, where \bar{X} runs through normal compactifications of X over T .
- $D_{X/T, (X_\alpha)}$, the closure of $\bigcup_{\alpha, \gamma_\alpha} \gamma_\alpha(D_{X_\alpha/T}^{\text{naive}})$, where γ_α runs through paths from a geometric generic point $\bar{a}_\alpha \rightarrow X_\alpha$ to $\bar{a} \rightarrow X$.
- The *decomposed part* $D_{X/T} = \bigcap D_{X/T, (X_\alpha)}$, where (X_α) runs through finite stratifications of X into integral normal subschemes.

Except for $D_{X, \bar{X}, x, \bar{b}}$, the above subsets are stable under conjugation.

The definition is functorial in an obvious sense, which we specify for $D_{X/T}^{\text{naive}}$ and $D_{X/T}$. Given a morphism $f : X \rightarrow Y$ of integral normal schemes of finite type over T and a path γ from $\bar{a} \rightarrow X$ to a geometric generic point $\bar{a}' \rightarrow Y$, the induced homomorphism $\gamma : \pi_1(X, \bar{a}) \rightarrow \pi_1(Y, \bar{a}')$ satisfies $\gamma(D_{X/T}^{\text{naive}}) \subseteq D_{Y/T}^{\text{naive}}$ and $\gamma(D_{X/T}) \subseteq D_{Y/T}$. We have $D_{X/T} \subseteq D_{X/T, (X_\alpha)} \subseteq D_{X/T}^{\text{naive}}$, where the second inclusion follows from the functoriality of $D_{X/T}^{\text{naive}}$.

Remark 4.16. (1) In the absence of a valuative criterion, we performed the last two steps in the definition to ensure that for any finite stratification (X_α) of X into integral normal subschemes, $D_{X/T}$ is the closure of $\bigcup_{\alpha, \gamma_\alpha} \gamma_\alpha(D_{X_\alpha/T})$, where γ_α runs through paths from a geometric generic point $\bar{a}_\alpha \rightarrow X_\alpha$ to $\bar{a} \rightarrow X$.

(2) If a denotes the generic point of X , then $D_{X, \bar{X}, a, \bar{a}} = \pi_1(X, \bar{a})$. On the other hand, for $x \in \bar{X}$ locally closed, the closure $\overline{\{x\}}$ is finite over T and if we let y denote the closed point of $\overline{\{x\}}$, then we have a canonical morphism $X_{(x)} \rightarrow X_{(y)}$ as in Remark 2.8, so that $D_{X, \bar{X}, x, \bar{b}} \subseteq D_{X, \bar{X}, y, \bar{b}}$. Thus, in the definition of $D_{X, \bar{X}}$, we may restrict to closed points $x \in \bar{X}$.

(3) For $x \in \bar{X}$ closed, the exact sequence in Remark 4.2(1) induces an exact sequence

$$1 \rightarrow E_{X, \bar{X}, x, \bar{b}} \rightarrow D_{X, \bar{X}, x, \bar{b}} \rightarrow \pi_1(\bar{X}_{(x)}, \bar{b}) \rightarrow 1.$$

Indeed, in the commutative square

$$\begin{array}{ccc} \pi_1(X_{(x)}, \bar{b}) & \xrightarrow{\rho} & \pi_1(\bar{X}_{(x)}, \bar{b}) \\ \tau \downarrow & & \downarrow \iota \\ \pi_1(X, \bar{a}) & \xrightarrow{\sigma} & \pi_1(T, \bar{a}) \end{array}$$

ι is an injection, so that $\text{Ker}(\tau) \subseteq \text{Ker}(\rho)$. Let $K = \text{Ker}(\sigma)$. Then

$$E_{X, \bar{X}, x, \bar{b}} = K \cap D_{X, \bar{X}, x, \bar{b}}, \quad E_{X, \bar{X}} \subseteq K \cap D_{X, \bar{X}}, \quad E_{X/T} \subseteq K \cap D_{X/T}.$$

(4) Assume that $T = S$. If X_k is geometrically unibranch, then $D_{X/S}$ contains the image of $\pi_1(X_k)$. Indeed, $D_{X_k/S} \subseteq \pi_1(X_k)$ contains the Frobenius element at every $x \in |X_k|$, so that $D_{X_k/S} = \pi_1(X_k)$ in this case by Chebotarev’s density theorem. If moreover X is proper over S , so that $\pi_1(X_k) \simeq \pi_1(X)$ [SGA4-3, XII, Théorème 5.9], then $D_{X/S} = \pi_1(X, \bar{a})$. The equality does not hold in general, even for X proper over S .

Theorem 1.2 implies the following density result.

COROLLARY 4.17. *Let X be an integral normal scheme separated of finite type over S . Then $D_{X/S}$ is the closure of $\bigcup_{\bar{x}, \gamma} \gamma(W(\bar{x}/x))$, where \bar{x} runs through geometric points of X above $x \in |X|$ and γ runs through paths from $\bar{x} \rightarrow x$ to $\bar{a} \rightarrow X$.*

In the corollary, we may replace $W(\bar{x}/x)$ by $W^{\geq N}(\bar{x}/x)$, which is a dense subset of $W(\bar{x}/x)$ for the profinite topology. Moreover, we may restrict to closed points $x \in X$ as in Remark 2.8.

Proof. Let C be the closure of $\bigcup_{\bar{x}, \gamma} \gamma(W(\bar{x}/x))$. We have $C \subseteq D_{X/S}$. Let G be a finite quotient of $\pi_1(X, \bar{a})$ and let $C(G)$ and $D(G)$ denote respectively the images of C and $D_{X/S}$ in G . It suffices to show that for any pair of $\overline{\mathbb{Q}_\ell}$ -characters χ and χ' of G satisfying $\chi|_{C(G)} = \chi'|_{C(G)}$, we have $\chi|_{D(G)} = \chi'|_{D(G)}$. Let \mathcal{F} and \mathcal{F}' be the corresponding lisse $\overline{\mathbb{Q}_\ell}$ -sheaves on X . Then \mathcal{F} and \mathcal{F}' are compatible (for $Q = \overline{\mathbb{Q}_\ell}$). We apply Theorem 1.2. Since $D_{X/S} \subseteq \bigcup_{\alpha, \gamma_\alpha} \gamma_\alpha(D_{X_\alpha/S}^{\text{naive}}) \subseteq \bigcup_{\alpha, \gamma_\alpha} \gamma_\alpha(D_{X_\alpha, \bar{X}_\alpha})$, every $g \in D(G)$ is in the image of some $\pi_1((X_\alpha)_{(x)}, \bar{b})$ for x closed in \bar{X}_α , which equals the image of $W((X_\alpha)_{(x)}, \bar{b})$. Thus, $\chi|_{D(G)} = \chi'|_{D(G)}$. \square

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