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A STUDY OF THE LENGTH FUNCTION OF GENERALIZED FRACTIONS OF MODULES

MARCEL MORALES[1,2] AND PHAM HUNG QUY[3]

[1] *Université de Grenoble I, Institut Fourier, UMR 5582, BP 74, 38402 Saint-Martin D'Hères, France* (morales@ujf-grenoble.fr)
[2] *ESPE, Université Lyon 1, 5 rue Anselme, 69317 Lyon Cedex, France*
[3] *Department of Mathematics, FPT University, 8 Ton That Thuyet, Hanoi, Vietnam* (quyph@fpt.edu.vn)

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Abstract Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module of dimension d. Let $\underline{x} = x_1, \dots, x_d$ be a system of parameters of M and let $\underline{n} = (n_1, \dots, n_d)$ be a d-tuple of positive integers. In this paper we study the length of generalized fractions $M(1/(x_1, \dots, x_d, 1))$, which was introduced by Sharp and Hamieh. First, we study the growth of the function

$$J_{\underline{x},M}(\underline{n}) = n_1 \cdots n_d e(\underline{x}; M) - \ell(M(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))).$$

Then we give an explicit calculation for the function $J_{\underline{x},M}(\underline{n})$ in the case in which M admits a certain Macaulay extension. Most previous results on this topic are improved in a clearly understandable way.

Keywords: system of parameters; generalized fractions; limit closure; local cohomology; Hilbert–Kunz function

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1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module of dimension d. Let $\underline{x} = x_1, \dots, x_d$ be a system of parameters of M. In this paper we study the length of generalized fractions $M(1/(x_1, \dots, x_d, 1))$, which was introduced by Sharp and Hamieh [**24**] and Sharp and Zakeri [**25**]. It has been proved [**7**, Lemma 2.3] that $M/((\underline{x})_M^{\lim})$ is isomorphic to $M(1/(x_1, \dots, x_d, 1))$, where

$$(\underline{x})_M^{\lim} = \bigcup_{n>0} ((x_1^{n+1}, \dots, x_d^{n+1})M : (x_1 \cdots x_d)^n).$$

We call $(\underline{x})_M^{\lim}$ the *limit closure* of the sequence $\underline{x}$ in M. It should be noted that the Hochster monomial conjecture is equivalent to the claim that $(\underline{x})_R^{\lim} \neq R$ for every system of parameters $\underline{x}$.

Let $\underline{n} = (n_1, \ldots, n_d)$ be a d-tuple of positive integers and let $\underline{x}^{\underline{n}} = x_1^{n_1}, \ldots, x_d^{n_d}$. We consider the functions in $\underline{n}$,

$$I_{\underline{x},M}(\underline{n}) = \ell(M/(\underline{x}^{\underline{n}})M) - e(\underline{x}^{\underline{n}}; M),$$
$$J_{\underline{x},M}(\underline{n}) = e(\underline{x}^{\underline{n}}, M) - \ell(M/(\underline{x}^{\underline{n}})^{\lim}_M),$$

where $e(\underline{x}; M)$ is the Serre multiplicity of M with respect to the sequence $\underline{x}$. In several papers N. T. Cuong *et al*. showed that the least degree of all polynomials in $\underline{n}$ bounding above $I_{\underline{x},M}(\underline{n})$ is independent of the choice of $\underline{x}$. It is called *the polynomial type* of M, and is denoted by $p(M)$. The invariant $p(M)$ keeps a lot of information about the structure of the module M. For example, M is Cohen–Macaulay if and only if $\ell(M/(\underline{x})M) = e(\underline{x}; M)$ for all systems of parameters $\underline{x}$ of M. This condition is equivalent to saying that $p(M) = -\infty$. Recall that a finitely generated R-module M is *generalized Cohen–Macaulay* if $H^i_{\mathfrak{m}}(M)$ has finite length for every $i < d$. We have that M is generalized Cohen–Macaulay if and only if $p(M) \leqslant 0$.

The behaviour of the function $J_{\underline{x},M}(\underline{n})$ was studied in several papers (see [**8**, **9**, **14**, **21**, **24**]). In [**24**], Sharp and Hamieh asked whether the length $\ell(M/(\underline{x}^{\underline{n}})^{\lim}_M)$ (equivalent to the function $J_{\underline{x},M}(\underline{n})$) becomes a polynomial when $n_i \gg 0$. The first counter-example for this question was given by N. T. Cuong *et al*. [**14**]. Therefore, in general, the function $J_{\underline{x},M}(\underline{n})$ is not a polynomial in $\underline{n}$. Fortunately, it has been shown that the least degree of polynomials bounding above $J_{\underline{x},M}(\underline{n})$ is independent of the choice of $\underline{x}$ (see [**13**, Theorem 4.4]). This invariant is called *the polynomial type of generalized fractions* of M, and is denoted by $pf(M)$. The invariant $pf(M)$ also carries the information of the structure of M. Under some mild conditions on the ring R, N. T. Cuong and Nhan showed that $pf(M) = -\infty$ (respectively, $pf(M) \leqslant 0$) if and only if $M/U_M(0)$ is Cohen–Macaulay (respectively, generalized Cohen–Macaulay), where $U_M(0)$ is the unmixed component of M (see [**9**, Theorem 3.1]).

The two functions $I_{\underline{x},M}(\underline{n})$ and $J_{\underline{x},M}(\underline{n})$ are closely related. While the polynomial type can be easily understood in terms of the annihilator of local cohomology and the dimension of the non-Cohen–Macaulay locus of M (see Proposition 2.3), not so much is known about the polynomial type of generalized fractions of M. In general it was proved in [**18**, Theorem 4.5] that $pf(M) \leqslant p(M)$. Our first result proves that if M is unmixed and $\underline{x}$ is a certain system of parameters, then $I_{\underline{x},M}(\underline{n}) \leqslant 2^{d-2} J_{\underline{x},M}(\underline{n})$, which implies that $pf(M) = p(M)$ (see Theorem 3.7). As a consequence we easily get a generalization of the result of N. T. Cuong and Nhan mentioned above (see Corollary 3.9).

Our second result consists of studying the function $J_{\underline{x},M}(\underline{n})$ in the case in which M admits a certain Macaulay extension and we can express $J_{\underline{x},M}(\underline{n})$ in terms of the non-Cohen–Macaulay locus of M (see Theorem 4.3). Based on this calculation, the counter-example of N. T. Cuong *et al*. for Sharp and Hamieh's question appears clearly as a particular case of a general statement (see Corollary 4.4).

Our third result concerns the question as to whether the function $J_{\underline{x},M}(\underline{n})$ can be defined by finitely many polynomials. Our counter-example for this question comes from the positive characteristic problems. We establish a connection between $J_{\underline{x},M}(\underline{n})$ and the Hilbert–Kunz function, and prove by using a recent result of Brenner [**1**] the existence of a local ring and a system of parameters such that the function $J_{\underline{x},M}(\underline{n})$, with $n = n_1 = \cdots = n_d$, cannot be defined by a finite set of polynomials (see Theorem 5.2).

2. Preliminaries

In this section we recall the definitions of the polynomial type and the polynomial type of generalized fractions, and their relations with some special systems of parameters. We also collect many results of [**10**, **12**] that play a key role in the proof of our first main result. It should be noted that the results of [**12**] can be found in the thesis of Quy [**22**, Chapter 3].

First we recall the notion of the *polynomial type* of a module. Let $(R, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated R-module of dimension d, let $\underline{x} = x_1, \dots, x_d$ be a system of parameters of M, and let $\underline{n} = (n_1, \dots, n_d)$ be a d-tuple of positive integers. We set $\underline{x}^{\underline{n}} = x_1^{n_1}, \dots, x_d^{n_d}$ and we consider the function in $\underline{n}$,

$$I_{\underline{x},M}(\underline{n}) = \ell(M/(\underline{x}^{\underline{n}})M) - e(\underline{x}^{\underline{n}}; M),$$

where $e(\underline{x}; M)$ is the Serre multiplicity of M with respect to the sequence $\underline{x}$. N. T. Cuong [**3**, Theorem 2.3] showed that the least degree of all polynomials in $\underline{n}$ bounding above $I_{\underline{x},M}(\underline{n})$ is independent of the choice of $\underline{x}$.

Definition 2.1. The least degree of all polynomials in $\underline{n}$ bounding above $I_{\underline{x},M}(\underline{n})$ is called *the polynomial type* of M and is denoted by $p(M)$.

The following basic properties of $p(M)$ can be found in [**3**].

Remark 2.2.

(i) We have $p(M) = p(\hat{M}) \leqslant d - 1$, where $\hat{M}$ is the $\mathfrak{m}$-adic completion of M.

(ii) An R-module M is Cohen–Macaulay if and only if $p(M) = -\infty$. Moreover, M is generalized Cohen–Macaulay if and only if $p(M) \leqslant 0$.

Let $\mathfrak{a}_i(M) = \operatorname{Ann} H^i_{\mathfrak{m}}(M)$ for $0 \leqslant i \leqslant d - 1$ and let $\mathfrak{a}(M) = \mathfrak{a}_0(M) \cdots \mathfrak{a}_{d-1}(M)$. We denote by $\mathrm{NC}(M)$ the non-Cohen–Macaulay locus of M, i.e. $\mathrm{NC}(M) = \{\mathfrak{p} \in \operatorname{supp}(M) \mid M_{\mathfrak{p}}$ is not Cohen–Macaulay$\}$. Recall that M is called *equidimensional* if $\dim M = \dim R/\mathfrak{p}$ for all minimal associated primes of M. The polynomial type of a module can be well understood by the annihilator of local cohomology as follows.

Proposition 2.3 (N. T. Cuong [2, Theorem 1.2]). *Suppose that R admits a dualizing complex. Then*

(i) $p(M) = \dim R/\mathfrak{a}(M)$*; and*

(ii) *if M is equidimensional, then* $p(M) = \dim(\mathrm{NC}(M))$.

Although the function $I_{\underline{x},M}(\underline{n})$ is not a polynomial in general, it has good behaviour for some special systems of parameters.

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Definition 2.4 (N. T. Cuong [4]). A system of parameters $x_1, \ldots, x_d$ of M is called *p-standard* if $x_d \in \mathfrak{a}(M)$ and $x_i \in \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M)$ for all $i = d-1, \ldots, 1$.

Definition 2.5 (Huneke [16], Goto and Yamagishi (personal communication, 1986)).

(i) A sequence in R, $\underline{x} = x_1, \ldots, x_s$, is called a *d-sequence* of M if $(x_1, \ldots, x_{i-1})M : x_j = (x_1, \ldots, x_{i-1})M : x_i x_j$ for all $i \leqslant j \leqslant s$.

(ii) A sequence $\underline{x} = x_1, \ldots, x_s$ is called a *strong d-sequence* if $\underline{x}^{\underline{n}} = x_1^{n_1}, \ldots, x_s^{n_s}$ is a d-sequence for all $\underline{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$.

For important properties of d-sequences, see [**16**, **26**].

Definition 2.6 (N. T. Cuong and D. T. Cuong [5]). A sequence of elements $\underline{x} = x_1, \ldots, x_s$ is called a *dd-sequence* of M if $\underline{x}$ is a strong d-sequence of M and the following conditions are satisfied:

(i) $s = 1$ or

(ii) $s > 1$ and $\underline{x}' = x_1, \ldots, x_{s-1}$ is a dd-sequence of M/x_s^n for all $n \geqslant 1$.

The function $I_{\underline{x},M}(\underline{n})$ is a polynomial for a p-standard system of parameters or a dd-sequence of parameters (see [**4**, Theorem 2.6 (ii)] and [**5**, Theorem 1.2]).

Proposition 2.7. *A system of parameters $\underline{x} = x_1, \ldots, x_d$ of M is a dd-sequence if and only if for all $n_1, \ldots, n_d > 0$ we have*

$$I_{\underline{x},M}(\underline{n}) = \sum_{i=0}^{p(M)} n_1 \cdots n_i e_i,$$

where $e_i = e(x_1, \ldots, x_i; 0 :_{M/(x_{i+2},\ldots,x_d)M} x_{i+1})$ and $e_0 = \ell(0 :_{M/(x_2,\ldots,x_d)M} x_1)$. Moreover, a p-standard system of parameters is a system of parameters that is a dd-sequence.

In order to introduce the notion of the *polynomial type of generalized fractions*, we recall the notion of the *limit closure* of a parameter ideal.

Definition 2.8. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of M. Then the *limit closure* of $\underline{x}$ in M is a submodule of M defined by

$$(\underline{x})_M^{\lim} = \bigcup_{n>0} ((x_1^{n+1}, \ldots, x_d^{n+1})M : (x_1 \cdots x_d)^n).$$

When $M = R$ we write $(\underline{x})^{\lim}$ for short.

For a detailed study of the limit closure, we refer the reader to [**10**].

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Remark 2.9.

(i) It is well known that $(\underline{x})M = (\underline{x})_M^{\lim}$ if and only if $\underline{x}$ is an M-sequence, i.e. M is Cohen–Macaulay.

(ii) The quotient $(\underline{x})_M^{\lim}/(\underline{x})M$ is the kernel of the canonical map

$$H^d(\underline{x}; M) \to H^d_{\mathfrak{m}}(M).$$

(iii) If $x_1, \dots, x_d$ is a dd-sequence, we have

$$(\underline{x})_M^{\lim} = \sum_{i=1}^{d} [(x_1, \dots, \widehat{x_i}, \dots, x_d)M :_M x_i] + (\underline{x})M$$

(see [**6**, Lemma 2.4]).

Similarly to the notion of polynomial type, we consider the function in $\underline{n}$,

$$J_{\underline{x},M}(\underline{n}) = e(\underline{x}^{\underline{n}}, M) - \ell(M/(\underline{x}^{\underline{n}})_M^{\lim}).$$

In general, $J_{\underline{x},M}(\underline{n})$ is not a polynomial in $\underline{n}$ (see [**14**]) but it is bounded by polynomials. Furthermore, the least degree of polynomials bounding above $J_{\underline{x},M}(\underline{n})$ is independent of the choice of $\underline{x}$ (see [**13**, Theorem 4.4]).

Definition 2.10. The least degree of all polynomials in $\underline{n}$ bounding above $J_{\underline{x},M}(\underline{n})$ is called *the polynomial type of generalized fractions* of M, and is denoted by $pf(M)$.

Now we recall the notion of an *unmixed component* of M, which is closely related to the limit closure and the polynomial type of generalized fractions.

Definition 2.11. The largest submodule of M of dimension less than d is called *the unmixed component* of M and it is denoted by $U_M(0)$.

It should be noted that if $\bigcap_{\mathfrak{p}\in \operatorname{Ass} M} N(\mathfrak{p}) = 0_M$ is a reduced primary decomposition of the zero submodule of M, then $U_M(0) = \bigcap_{\mathfrak{p}\in \operatorname{Assh} M} N(\mathfrak{p})$, where $\operatorname{Assh} M = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim R/\mathfrak{p} = \dim M\}$.

Remark 2.12.

(i) In [**10**, Theorem 4.1] it was proved that $U_M(0) = \bigcap_n (\underline{x}^{[n]})_M^{\lim}$ for any system of parameters $\underline{x}$ of M, where we define $\underline{x}^{[n]} = x_1^n, \dots, x_d^n$.

(ii) Suppose that R admits a dualizing complex. Then $pf(M) = -\infty$ (respectively, $pf(M) \leqslant 0$) if and only if $M/U_M(0)$ is Cohen–Macaulay (respectively, generalized Cohen–Macaulay); see [**9**, Theorem 3.1].

Recently, N. T. Cuong and Quy studied the splitting of local cohomology (see [**11**,**12**]); this will provide the main tool for the proof of our first result in this paper. We collect here some results that we need in what follows. Set

$$\mathfrak{b}(M) = \bigcap_{\underline{x};i=1}^{d} \operatorname{Ann}(0 : x_i)_{M/(x_1,\dots,x_{i-1})M},$$

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where $\underline{x} = x_1, \ldots, x_d$ runs over all systems of parameters of M. By [**23**, Satz 2.4.5] we have

$$\mathfrak{a}(M) \subseteq \mathfrak{b}(M) \subseteq \mathfrak{a}_0(M) \cap \cdots \cap \mathfrak{a}_{d-1}(M).$$

Therefore, we get the important point that for every parameter element $x \in \mathfrak{b}(M)$ we have $xH^i_{\mathfrak{m}}(M) = 0$ for all $i < d$. By using this fact and the guidelines of [**11**] the following splitting property of local cohomology was proved [**12**, Corollary 3.5]. It should be noted that a special case of it can be found in [**6**, Theorem 2.7].

Theorem 2.13. *Let $x \in \mathfrak{b}(M)^3$ be a parameter element of M. Let $U_M(0)$ be the unmixed component of M and set $\bar{M} = M/U_M(0)$. Then*

$$H^i_{\mathfrak{m}}(M/xM) \cong H^i_{\mathfrak{m}}(M) \oplus H^{i+1}_{\mathfrak{m}}(\bar{M})$$

for all $i < d - 1$.

We need the following lemma in what follows.

Lemma 2.14. *Let $N \subseteq H^0_{\mathfrak{m}}(M)$ be a submodule of finite length. Then $\mathfrak{b}(M) \subseteq \mathfrak{b}(M/N)$.*

Proof. Let $x_1, \ldots, x_d$ be an arbitrary system of parameters of M/N. It is also a system of parameters of M. By the definition of $\mathfrak{b}(M/N)$, we need only prove that

$$\mathfrak{b}(M) \subseteq \operatorname{Ann} \frac{[(x_1, \ldots, x_{i-1})M + N] : x_i}{(x_1, \ldots, x_{i-1})M + N}$$

for all $i \leqslant d$. Choose a positive integer n_0 such that $x_i^{n_0} N = 0$ and for all $n \geqslant n_0$ we have

$$\begin{aligned} (x_1, \ldots, x_{i-1})M : x_i^n &= (x_1, \ldots, x_{i-1})M : x_i^{n_0}, \\ [(x_1, \ldots, x_{i-1})M + N] : x_i^n &= [(x_1, \ldots, x_{i-1})M + N] : x_i^{n_0}. \end{aligned}$$

So

$$[(x_1, \ldots, x_{i-1})M + N] : x_i^{n_0} \subseteq (x_1, \ldots, x_{i-1})M : x_i^{2n_0} \subseteq [(x_1, \ldots, x_{i-1})M + N] : x_i^{2n_0}.$$

Hence, $(x_1, \ldots, x_{i-1})M : x_i^{2n_0} = [(x_1, \ldots, x_{i-1})M + N] : x_i^{2n_0}$ and we have

$$\begin{aligned} \operatorname{Ann} \frac{[(x_1, \ldots, x_{i-1})M + N] : x_i}{(x_1, \ldots, x_{i-1})M + N} &\supseteq \operatorname{Ann} \frac{[(x_1, \ldots, x_{i-1})M + N] : x_i^{2n_0}}{(x_1, \ldots, x_{i-1})M + N} \\ &= \operatorname{Ann} \frac{(x_1, \ldots, x_{i-1})M : x_i^{2n_0}}{(x_1, \ldots, x_{i-1})M + N} \\ &\supseteq \operatorname{Ann} \frac{(x_1, \ldots, x_{i-1})M : x_i^{2n_0}}{(x_1, \ldots, x_{i-1})M} \\ &\supseteq \mathfrak{b}(M). \end{aligned}$$

□

One key point of this paper is the introduction of a C-system of parameters. We call it a C-system of parameters in honour of Professor N. T. Cuong.

Definition 2.15. A system of parameters $x_1, \ldots, x_d$ is called a C-system of parameters of M if $x_d \in \mathfrak{b}(M)^3$ and $x_i \in \mathfrak{b}(M/(x_{i+1}, \ldots, x_d)M)^3$ for all $i = d-1, \ldots, 1$.

If $(R, \mathfrak{m})$ is the quotient of a Cohen–Macaulay ring, then we always have that $\dim R/\mathfrak{a}(M) < \dim M$ for every finitely generated R-module M (see [**6**, Corollary 5.8]). So every finitely generated R-module M admits a C-system of parameters. In the next lemma we collect some useful properties of a C-system of parameters.

Lemma 2.16. *Let $x_1, \ldots, x_d$ be a C-system of parameters of M. Then the following hold.*

(i) $x_1, \ldots, x_d$ *is a dd-sequence.*

(ii) $x_1^{n_1}, \ldots, x_d^{n_d}$ *is a C-system of parameters of M for all $n_1, \ldots, n_d \geqslant 1$.*

(iii) *For all $i \leqslant d$ we have that $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d$ is a C-system of parameters of M/x_iM.*

(iv) *Let $N \subseteq H^0_{\mathfrak{m}}(M)$ be a submodule of finite length. Then $x_1, \ldots, x_d$ is a C-system of parameters of M/N.*

Proof. Detailed proofs of (i)–(iii) are given in [**22**, Proposition 3.2.13, Corollary 3.2.12, Lemma 3.1.10] and will appear in [**12**]. The proofs of (i)–(iii) are based on the fact that the notions of a C-system of parameters, a p-standard system of parameters and a dd-sequence system of parameters are very closely related. Property (i) is similar to the last claim of Proposition 2.7. Property (ii) is similar to [**5**, Remark 3.3 (ii)] for a dd-sequence. The third property is similar to [**5**, Proposition 3.4] and can be proved by using the fact that $\mathfrak{b}(M) \subseteq \mathfrak{b}(M/xM)$ for every parameter element x.

(iv) For each $i \leqslant d$ we have that $M/((x_{i+1}, \ldots, x_d)M + N)$ is a quotient module of $M/(x_{i+1}, \ldots, x_d)M$ by a submodule of finite length. So $\mathfrak{b}(M/(x_{i+1}, \ldots, x_d)M) \subseteq \mathfrak{b}(M/((x_{i+1}, \ldots, x_d)M + N))$ by Lemma 2.14. Thus,

$$x_i \in \mathfrak{b}(M/(x_{i+1}, \ldots, x_d)M)^3 \subseteq \mathfrak{b}(M/((x_{i+1}, \ldots, x_d)M + N))^3. \qquad \square$$

3. On the polynomial type of generalized fractions

Since $p(M)$ and $pf(M)$ do not change after passing to the completion, in this section we assume that $(R, \mathfrak{m})$ is the image of a Cohen–Macaulay local ring. For each system of parameters $\underline{x} = x_1, \ldots, x_d$ set

$$I_{\underline{x},M} = \ell(M/(\underline{x})M) - e(\underline{x}; M)$$

and

$$J_{\underline{x},M} = e(\underline{x}; M) - \ell(M/(\underline{x})^{\lim}_M).$$

It should be noted that $I_{\underline{x},M}$ is much easier to understand than $J_{\underline{x},M}$.

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Lemma 3.1. *Let M be a generalized Cohen–Macaulay module and let $\underline{x} = x_1, \dots, x_d$ be a standard system of parameters of M. Then*

(i) $I_{\underline{x},M} = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(M))$;

(ii) $J_{\underline{x},M} = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell(H^i_{\mathfrak{m}}(M))$.

Proof. For the definition of a standard system of parameters and the proof of (i), see [**27**]; (ii) follows from [**13**, Theorem 5.1]. □

Lemma 3.2. *Let $\underline{x} = x_1, \dots, x_d$ be a system of parameters of M and let $U_M(0)$ be the unmixed component of M. Setting $\bar{M} = M/U_M(0)$, we have*

(i) $J_{\underline{x},M} = J_{\underline{x},\bar{M}}$,

(ii) $J_{\underline{x},M}(\underline{n}) = J_{\underline{x},\bar{M}}(\underline{n})$ *for all* $\underline{n}$,

(iii) $pf(M) = pf(\bar{M})$.

Proof. (i) Since $\dim U_M(0) < d$, we have $e(\underline{x}; M) = e(\underline{x}; \bar{M})$. For each $n \geqslant 1$ we set $\underline{x}^{[n]} = x_1^n, \dots, x_d^n$. By Remark 2.12 we have $U_M(0) = \bigcap_{n \geqslant 1} (\underline{x}^{[n]})^{\lim}_M$. By [**10**, Proposition 2.6] we have

$$\ell(M/(\underline{x})^{\lim}_M) = \ell(\bar{M}/(\underline{x})^{\lim}_{\bar{M}}).$$

Therefore, $J_{\underline{x},M} = J_{\underline{x},\bar{M}}$.

Part (ii) follows from (i), and (iii) follows from (ii). □

By the above lemma we can assume that M is unmixed (i.e. $U_M(0) = 0$) for the computation of either the function $J_{\underline{x},M}(\underline{n})$ or $pf(M)$. The following is important for our inductive technique.

Remark 3.3. Let M be an unmixed finitely generated R-module of dimension d. Then we have the following.

(i) $H^1_{\mathfrak{m}}(M)$ is finitely generated provided that $d \geqslant 2$ (see, for example, [**15**, Lemma 3.1]).

(ii) The set

$$\mathcal{F}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim M_{\mathfrak{p}} > 1 = \operatorname{depth} M_{\mathfrak{p}},\ \mathfrak{p} \neq \mathfrak{m}\}$$

is finite (see [**15**, Lemma 3.2]).

(iii) Let $\underline{x} = x_1, \dots, x_d$ be a C-system of parameters of M. Then

$$\mathcal{F}(M) = \operatorname{Ass} U_{M/x_dM}(0) \setminus \{\mathfrak{m}\}$$

and $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{F}(M)$. Hence, $\operatorname{Ass} M/x_1M \subseteq \operatorname{Assh} M/x_1M \cup \{\mathfrak{m}\}$, so $U_{M/x_1M}(0) \cong H^0_{\mathfrak{m}}(M/x_1M)$ (see [**12**, Proposition 4.11, Remark 4.12]).

Lemma 3.4. *Let M be an unmixed finitely generated R-module of dimension $d \geqslant 2$ and let $\underline{x} = x_1, \ldots, x_d$ be a C-system of parameters of M. Then $x_1 H^1_{\mathfrak{m}}(M) = 0$ and $\ell(H^1_{\mathfrak{m}}(M)) \leqslant I_{\underline{x},M}$.*

Proof. Set $M_d = M/x_d M$. Since M is unmixed, by Theorem 2.13 we have $H^1_{\mathfrak{m}}(M) \cong H^0_{\mathfrak{m}}(M_d)$. By Lemma 2.16 we have that $\underline{x}' = x_1, \ldots, x_{d-1}$ is a dd-sequence of M_d, so $H^0_{\mathfrak{m}}(M_d) = 0 :_{M_d} x_1$. Hence, $x_1.H^1_{\mathfrak{m}}(M) = 0$. Moreover, the properties of dd-sequences imply that $H^0_{\mathfrak{m}}(M_d) \cap (\underline{x}')M_d = 0$. Thus,

$$\begin{aligned}\ell(M_d/(\underline{x}')M_d) = \ell(H^0_{\mathfrak{m}}(M_d)) + \ell(\overline{M_d}/(\underline{x}')\overline{M_d}) &\geqslant \ell(H^1_{\mathfrak{m}}(M)) + e(\underline{x}'; \overline{M_d}) \\ &= \ell(H^1_{\mathfrak{m}}(M)) + e(\underline{x}'; M_d),\end{aligned}$$

where $\overline{M_d} = M_d/H^0_{\mathfrak{m}}(M_d)$. Therefore,

$$\ell(H^1_{\mathfrak{m}}(M)) \leqslant I_{\underline{x}',M_d} = I_{\underline{x},M}.$$

For the last equality notice that since x_d is M-regular we have $e(\underline{x}; M) = e(\underline{x}'; M_d)$. The proof is complete. □

Lemma 3.5. *Let M be an unmixed finitely generated R-module of dimension $d \geqslant 3$ and let $\underline{x} = x_1, \ldots, x_d$ be a C-system of parameters of M. Setting $M_1 = M/x_1 M$ and $\underline{x}' = x_2, \ldots, x_d$, we have $I_{\underline{x},M} \leqslant 2I_{\underline{x}',\bar{M}_1}$, where $\overline{M_1} = M_1/H^0_{\mathfrak{m}}(M_1)$.*

Proof. Since x_1 is M-regular, we have $e(\underline{x}; M) = e(\underline{x}'; M_1)$. So $I_{\underline{x},M} = I_{\underline{x}',M_1}$. By Lemma 2.16 we have that $\underline{x}' = x_2, \ldots, x_d$ is a C-system of parameters of M_1. Similar to the proof of the previous result, we have

$$I_{\underline{x}',M_1} = I_{\underline{x}',\bar{M}_1} + \ell(H^0_{\mathfrak{m}}(M_1)).$$

Thus, we need only prove that $\ell(H^0_{\mathfrak{m}}(M_1)) \leqslant I_{\underline{x}',\bar{M}_1}$. Consider the short exact sequence

$$0 \to M \xrightarrow{x_1 \cdot} M \to M_1 \to 0.$$

By Lemma 3.4 we have $x_1.H^1_{\mathfrak{m}}(M) = 0$. So by applying the local cohomology functor to the above short exact sequence we have $H^0_{\mathfrak{m}}(M_1) \cong H^1_{\mathfrak{m}}(M)$ and

$$0 \to H^1_{\mathfrak{m}}(M) \to H^1_{\mathfrak{m}}(M_1).$$

Thus,

$$\ell(H^0_{\mathfrak{m}}(M_1)) = \ell(H^1_{\mathfrak{m}}(M)) \leqslant \ell(H^1_{\mathfrak{m}}(M_1)).$$

On the other hand, by Remark 3.3 we have that $\overline{M_1}$ is unmixed, and $\underline{x}'$ is a C-system of parameters of $\overline{M_1}$ by Lemma 2.16. So

$$\ell(H^1_{\mathfrak{m}}(M_1)) = \ell(H^1_{\mathfrak{m}}(\bar{M}_1)) \leqslant I_{\underline{x}',\bar{M}_1}$$

by Lemma 3.4. Thus, $\ell(H^0_{\mathfrak{m}}(M_1)) \leqslant I_{\underline{x}',\bar{M}_1}$. The proof is complete. □

Proposition 3.6. *Let M be an unmixed finitely generated R-module of dimension d and let $\underline{x} = x_1, \dots, x_d$ be a C-system of parameters of M. Then $I_{\underline{x},M} \leqslant 2^{d-2} J_{\underline{x},M}$.*

Proof. We proceed by induction on d. The $d = 1$ case is trivial since M is Cohen–Macaulay. For $d = 2$, by Lemma 3.1 we have

$$I_{\underline{x},M} = \ell(H^1_{\mathfrak{m}}(M)) = J_{\underline{x},M}.$$

Assume that $d \geqslant 3$ and that the assertion was proved for $d - 1$. Setting $M_1 = M/x_1M$ and $\underline{x}' = x_2, \dots, x_d$, we have

$$\begin{aligned} I_{\underline{x},M} &\leqslant 2 I_{\underline{x}',\bar{M}_1} && \text{(by Lemma 3.5)} \\ &\leqslant 2^{d-2} J_{\underline{x}',\bar{M}_1} && \text{(by induction)} \\ &= 2^{d-2} J_{\underline{x}',M_1} && \text{(by Lemma 3.2).} \end{aligned}$$

Since x_1 is M-regular we have $e(\underline{x}; M) = e(\underline{x}'; M_1)$. On the other hand, we have

$$(\underline{x}')^{\lim}_{M_1} = \bigcup_n [(x_1, x_2^{n+1}, \dots, x_d^{n+1})M :_M (x_2, \dots, x_d)^n]/x_1M \subseteq (\underline{x})^{\lim}_M / x_1 M.$$

So $\ell(M/(\underline{x})^{\lim}_M) \leqslant \ell(M_1/(\underline{x}')^{\lim}_{M_1})$. Thus, $J_{\underline{x}',M_1} \leqslant J_{\underline{x},M}$. Therefore, we get the assertion $I_{\underline{x},M} \leqslant 2^{d-2} J_{\underline{x},M}$. □

Theorem 3.7. *Let $(R, \mathfrak{m})$ be the image of a Cohen–Macaulay local ring and let M be an unmixed finitely generated R-module of dimension d. Then $pf(M) = p(M)$. Moreover, $pf(M) = \dim R/\mathfrak{a}(M)$.*

Proof. By [**18**, Theorem 4.5], we have $pf(M) \leqslant p(M)$. Thus, we need only prove that $pf(M) \geqslant p(M)$. Let $\underline{x} = x_1, \dots, x_d$ be a C-system of parameters of M. By Lemma 2.16, for all d-tuples of positive integers $\underline{n} = (n_1, \dots, n_d)$ we have that $\underline{x}^{\underline{n}} = x_1^{n_1}, \dots, x_d^{n_d}$ is also a C-system of parameters. By Proposition 3.6 we have

$$I_{\underline{x},M}(\underline{n}) = I_{\underline{x}^{\underline{n}},M} \leqslant 2^{d-2} J_{\underline{x}^{\underline{n}},M} = 2^{d-2} J_{\underline{x},M}(\underline{n})$$

for all $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$. Thus, $p(M) \leqslant pf(M)$. The last assertion follows from Proposition 2.3. The proof is complete. □

The next result is a consequence of the above theorem and Lemma 3.2.

Corollary 3.8. *Let $(R, \mathfrak{m})$ be the image of a Cohen–Macaulay local ring and let M be a finitely generated R-module with the unmixed component $U_M(0)$. Then*

$$pf(M) = p(M/U_M(0)).$$

Recall that an R-module M is called *pseudo-Cohen–Macaulay (respectively, pseudo-generalized Cohen–Macaulay)* if $pf(M) = -\infty$ (respectively, $pf(M) \leqslant 0$). As a consequence of Corollary 3.8 we get a generalization of the main result of [**9**].

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Corollary 3.9. *Let $(R, \mathfrak{m})$ be the image of a Cohen–Macaulay local ring and let M be a finitely generated R-module with the unmixed component $U_M(0)$. Then M is pseudo-Cohen–Macaulay (respectively, pseudo-generalized Cohen–Macaulay) if and only if $M/U_M(0)$ is Cohen–Macaulay (respectively, generalized Cohen–Macaulay).*

It is natural to raise the following question.

Question 3.10. Let M be an unmixed finitely generated R-module of dimension d and let $\underline{x} = x_1, \ldots, x_d$ be a C-system of parameters of M. Is the function $J_{\underline{x},M}(\underline{n})$ a polynomial in $\underline{n}$ when $n_1, \ldots, n_d \gg 0$?

It should be noted that [**8**, Theorem 4.5] gives an affirmative answer for this question in the $pf(M) \leqslant 1$ case.

4. The case in which M admits a Macaulay extension

Definition 4.1. Let M be a finitely generated R-module of dimension d. We say that M admits a *Macaulay extension* M' if we have an exact sequence

$$0 \to M \to M' \to N \to 0,$$

where M' is a finitely generated Cohen–Macaulay R-module and $\dim N \leqslant d-2$.

Remark 4.2 (see, for example, [20, 23]). Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and let M be a finitely generated R-module of dimension d. We recall that if M is unmixed, the module $D^d(D^d(M))$ (where $D^d(M)$ is the Matlis dual of $H^d_{\mathfrak{m}}(M)$) satisfies condition S_2 and we have an exact sequence

$$0 \to M \to D^d(D^d(M)) \to N \to 0$$

with $\dim N \leqslant d-2$. Moreover, if there exist a finitely generated R-module M' of dimension d satisfying condition S_2, and an exact sequence

$$0 \to M \to M' \to M'/M \to 0$$

with $\dim M'/M \leqslant d-2$, then $M' \cong D^d(D^d(M))$. That is, if an unmixed module M admits Macaulay extensions M' and M'', then $M' \cong M''$. In this case, $\operatorname{Supp}(M'/M)$ is the non-Cohen–Macaulay locus of M.

We can now state the main result of this section.

Theorem 4.3. *Let M be a finitely generated R-module of dimension d. Suppose that M has a Macaulay extension M'. Let $\underline{x} = x_1, \ldots, x_d$ be an arbitrary system of parameters of M. Set $N = M'/M$. Then*

$$J_{\underline{x},M}(\underline{n}) = \ell(N/(\underline{x}^{\underline{n}})N)$$

for all d-tuples $\underline{n} = (n_1, \ldots, n_d)$.

Proof. For any system of parameters $\underline{y} = y_1, \dots, y_d$, the short exact sequence

$$0 \to M \to M' \to N \to 0$$

induces the following commutative diagram with the last two columns exact:

$$\begin{array}{ccccccc}
 & & & & H^{d-1}(\underline{y}; N) & \longrightarrow & H^{d-1}_{\mathfrak{m}}(N) = 0 \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & (\underline{y})^{\lim}_M/(\underline{y})M & \xrightarrow{\beta} & H^d(\underline{y}; M) & \longrightarrow & H^d_{\mathfrak{m}}(M) \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow \\
 & & 0 & \longrightarrow & H^d(\underline{y}; M') & \longrightarrow & H^d_{\mathfrak{m}}(M') \\
 & & & & \downarrow & & \downarrow \\
 & & & & H^d(\underline{y}; N) & \longrightarrow & H^d_{\mathfrak{m}}(N) = 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

Both the second row and the third row are exact by Remark 2.9. Therefore, we have $\alpha \circ \beta = 0$. Thus, we have the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & M/(\underline{y})^{\lim}_M & \xrightarrow{\pi} & H^d_{\mathfrak{m}}(M) \\
 & & \downarrow{\scriptstyle \bar{\alpha}} & & \downarrow{\scriptstyle \sigma} \\
0 & \longrightarrow & M'/(\underline{y})M' & \xrightarrow{\tau} & H^d_{\mathfrak{m}}(M') \\
 & & \downarrow & & \\
 & & N/(\underline{y})N & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

with the middle column exact. Moreover, we have that both π and τ are injective and σ is bijective. Therefore, $\tau \circ \bar{\alpha} = \sigma \circ \pi$ is injective and so is $\bar{\alpha}$. Hence, we have the short exact sequence

$$0 \to M/(\underline{y})^{\lim}_M \to M'/(\underline{y})M' \to N/(\underline{y})N \to 0.$$

Thus,

$$\ell(M/(\underline{y})^{\lim}_M) = \ell(M'/(\underline{y})M') - \ell(N/(\underline{y})N).$$

Now, for each $\underline{n} = (n_1, \dots, n_d)$, applying the above assertion for the system of parameters $\underline{x}^{\underline{n}} = x_1^{n_1}, \dots, x_d^{n_d}$, we have

$$\ell(M/(\underline{x}^{\underline{n}})^{\lim}_M) = \ell(M'/(\underline{x}^{\underline{n}})M') - \ell(N/(\underline{x}^{\underline{n}})N).$$

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Since M' is Cohen–Macaulay, we have

$$\ell(M'/(\underline{x}^{\underline{n}})M') = e(\underline{x}^{\underline{n}}; M') = e(\underline{x}^{\underline{n}}; M).$$

Therefore, $J_{\underline{x},M}(\underline{n}) = \ell(N/(\underline{x}^{\underline{n}})N)$ for all d-tuples $\underline{n} = (n_1, \ldots, n_d)$. The proof is complete. □

The length $\ell(N/(\underline{x}^{\underline{n}})N)$ is much easier to understand than the function $J_{\underline{x},M}(\underline{n})$. In many cases we can see that it coincides with a polynomial or a finite number of polynomials for $\underline{n} \gg 0$. The following corollary, the proof of which is short and conceptual, extends [**14**, Lemma 2.4].

Corollary 4.4. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geqslant 3$ and let $x_1, \ldots, x_d$ be a system of parameters of R. Let $M = (x_1, \ldots, x_{d-v})$, $v \leqslant d-2$. Then for the system of parameters $\underline{x} = x_1 + x_d, x_2, \ldots, x_d$ of M we have*

$$J_{\underline{x},M}(\underline{n}) = \ell(R/(x_1, \ldots, x_d))n_{d-v+1} \cdots n_{d-1} \min\{n_1, n_d\}$$

for all $n_1, \ldots, n_d \geqslant 1$. Therefore, $J_{\underline{x},M}(\underline{n})$ is not a polynomial.

Proof. Since $\dim R/M \leqslant d-2$, R is a Macaulay extension of M. By Theorem 4.3 we have

$$J_{\underline{x},M}(\underline{n}) = \ell(R/(x_1, \ldots, x_{d-v}, (x_1 + x_d)^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}))$$

for all $n_1, \ldots, n_d \geqslant 1$. Hence,

$$\begin{aligned} J_{\underline{x},M}(\underline{n}) &= \ell(R/(x_1, \ldots, x_{d-v}, x_{d-v+1}^{n_{d-v+1}}, \ldots, x_{d-1}^{n_{d-1}}, x_d^{\min\{n_1,n_d\}})) \\ &= \ell(R/(x_1, \ldots, x_d))n_{d-v+1} \cdots n_{d-1} \min\{n_1, n_d\} \end{aligned}$$

for all $n_1, \ldots, n_d \geqslant 1$. □

The next result follows from Theorem 4.3 and Proposition 2.7.

Corollary 4.5. *Let M be a finitely generated R-module of dimension d. Suppose that M has a Macaulay extension M' with $\dim M'/M = t \leqslant d-2$. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of M such that $x_1, \ldots, x_t$ forms a dd-sequence of $N = M'/M$ and $x_{t+1}, \ldots, x_d \in \operatorname{Ann} N$. Then $J_{\underline{x},M}(\underline{n})$ is a polynomial in $\underline{n}$ for all $n_1, \ldots, n_d \geqslant 1$. Moreover,*

$$J_{\underline{x},M}(\underline{n}) = n_1 \cdots n_t e(x_1, \ldots, x_t; N) + \sum_{i=0}^{t-1} n_1 \cdots n_i e_i,$$

where $e_i = e(x_1, \ldots, x_i; 0 :_{N/(x_{i+2}, \ldots, x_t)N} x_{i+1})$ and $e_0 = \ell(0 :_{N/(x_2, \ldots, x_t)N} x_1)$.

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5. Relation with the Hilbert–Kunz function

By considering all explicit examples, it can be expected that $J_{\underline{x},M}(\underline{n})$ coincides with finitely many polynomials in $\underline{n}$ (see [**14**, **21**]). As we will see, this is not always the case. More precisely, we will give an example in characteristic p such that the function $J_{\underline{x},M}(\underline{n})$ cannot be controlled by finitely many polynomials. This question is closely related to the Hilbert–Kunz function.

Let $(A, \mathfrak{n})$ be a Noetherian local ring containing a field of positive characteristic p. Let I be an ideal of A and let $q = p^e$ be a power of p. We define $I^{[q]} = (f^q \mid q \in I)$ as the eth Frobenius power of I. If I is an $\mathfrak{n}$-primary ideal, we always have that $A/I^{[q]}$ has finite length. So we have a function

$$f_{\mathrm{HK}}(I)\colon q \mapsto \ell(A/I^{[q]}),$$

called the Hilbert–Kunz function, which was first studied by Kunz in [**17**]. In [**19**] Monsky proved that the limit

$$e_{\mathrm{HK}}(I) = \lim_{q\to\infty} \frac{\ell(A/I^{[q]})}{q^{\dim A}}$$

exists as a real number; it is called the Hilbert–Kunz multiplicity of I, and the Hilbert–Kunz multiplicity of $\mathfrak{n}$ is called the Hilbert–Kunz multiplicity of A. It is natural to ask whether the Hilbert–Kunz multiplicity of an $\mathfrak{n}$-primary ideal is always a rational number. There are many positive partial answers to this question. However, recently Brenner disproved this question by way of the following celebrated result.

Theorem 5.1 (Brenner [1, Theorem 8.3]). *There exists a Noetherian local domain whose Hilbert–Kunz multiplicity is an irrational number.*

We are ready to prove the main result of this section.

Theorem 5.2. *There exist a regular local ring $(R, \mathfrak{m})$ of dimension d with $\mathfrak{m}$ generated by a regular system of parameters $\underline{x} = x_1, \ldots, x_d$ and a finitely generated R-module M, $\dim M = d$, such that the function $J_{\underline{x},M}(n) = n^d e(\underline{x}; M) - \ell(M/(\underline{x}^{[n]})^{\lim}_M)$ cannot be represented by finitely many polynomials in n, where $\underline{x}^{[n]} = x_1^n, \ldots, x_d^n$.*

Proof. Let $(A, \mathfrak{n})$ be the ring of characteristic p whose Hilbert–Kunz multiplicity is irrational, as in Brenner's result. Replacing A by its completion (notice that the Hilbert–Kunz multiplicity does not change), we can assume that $(A, \mathfrak{n})$ is complete. By the Cohen structure theorem we have that A is the image of a regular local ring $(R, \mathfrak{m})$ of dimension d. Since $e_{\mathrm{HK}}(A)$ is irrational, we have that A is not regular and so $\dim R - \dim A \geqslant 1$. If $\dim R - \dim A = 1$, we replace R by $R[X]_{(\mathfrak{m},X)R[X]}$. Henceforth we can assume that $\dim R - \dim A \geqslant 2$. Letting the R-module M be the kernel of the canonical map $R \to A$, we have that $\dim M = d$. Choose a regular system of parameters $\underline{x} = x_1, \ldots, x_d$ that generates $\mathfrak{m}$. By Theorem 4.3 we have

$$J_{\underline{x},M}(n) = \ell(A/(\underline{x}^{[n]})A)$$

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for all $n \geqslant 1$. For all $i = 1, \ldots, d$ we denote by a_i the image of x_i in A. We have that the sequence $\underline{a} = a_1, \ldots, a_d$ generates the maximal ideal $\mathfrak{n}$ of A. Now we assume that there are only finitely many polynomials $P_1(n), \ldots, P_r(n)$ such that for each $n \geqslant 1$ we have $J_{\underline{x},M}(n) = P_i(n)$ for some i and find a contradiction. We consider the case in which n is a prime power $q = p^e$ and we have

$$J_{\underline{x},M}(q) = \ell(A/(\underline{a})^{[q]}) = \ell(A/\mathfrak{n}^{[q]}).$$

Since there are infinitely many q, we must have a polynomial, say $P_1(n)$, such that

$$\ell(A/\mathfrak{n}^{[q]}) = P_1(q)$$

for infinitely many $q = p^e$. It should be noted that if a polynomial takes integer values at infinitely many integer numbers, then all of its coefficients are rational. Thus, the leading coefficient of $P_1(n)$ is a rational number and $\deg P_1(n) = \dim A$. So

$$e_{\mathrm{HK}}(A) = \lim_{q\to\infty} \frac{\ell(A/\mathfrak{n}^{[q]})}{q^{\dim A}} = \lim_{q\to\infty} \frac{P_1(q)}{q^{\dim A}}$$

is a rational number, which contradicts our assumption about A. The proof is complete. □

For the next result we need the concept of the principle of *idealization*. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module. We make the Cartesian product $R \times M$ into a commutative ring with respect to component-wise addition and multiplication defined by $(r, m)\cdot(r', m') = (rr', rm' + r'm)$. We call this the idealization of M (over R) and denote it by $R \ltimes M$. The idealization $R \ltimes M$ is a Noetherian local ring with identity $(1, 0)$, its maximal ideal is $\mathfrak{m} \times M$ and its Krull dimension is $\dim R$. If $\underline{x} = x_1, \ldots, x_d$ is a system of parameters of R, then $(\underline{x,0}) = (x_1, 0), \ldots, (x_d, 0)$ is a system of parameters of the idealization $R \ltimes M$.

Lemma 5.3 (N. T. Cuong *et al.* [14, Lemma 2.6]). *Let* $\dim M = \dim R = d$ *and* $S = R \ltimes M$. *Let* $\underline{x} = x_1, \ldots, x_d$ *be a system of parameters of* R. *Then we have*

$$\ell(S/(\underline{x,0})_S^{\lim}) = \ell(R/(\underline{x})_R^{\lim}) + \ell(M/(\underline{x})_M^{\lim}).$$

Now we prove the last result of this paper.

Corollary 5.4. *There exists a Noetherian local ring* $(S, \mathfrak{n})$ *of dimension* d *and a system of parameters* $\underline{y} = y_1, \ldots, y_d$ *such that the function* $J_{\underline{y},S}(n)$ *cannot be represented by finitely many polynomials in* n.

Proof. We choose $(R, \mathfrak{m})$ and M as in Theorem 5.2. Let $\underline{x} = x_1, \ldots, x_d$ be a regular system of parameters of R. Let $S = R \ltimes M$ and $\underline{y} = (x_1, 0), \ldots, (x_d, 0)$. We can check that $e(\underline{y}; S) = e(\underline{x}; R) + e(\underline{x}; M)$. Since R is regular we have $(\underline{x}^{[n]})_R^{\lim} = (\underline{x}^{[n]})$ for all $n \geqslant 1$. So

$$\ell(R/(\underline{x}^{[n]})_R^{\lim}) = \ell(R/(\underline{x}^{[n]})) = n^d e(\underline{x}; R).$$

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Combining with Lemma 5.3 we have

$$\begin{aligned} J_{\underline{y},S}(n) &= \ell(S/(\underline{y}^{[n]})^{\lim}_{S}) - n^d e(\underline{y};S) \\ &= (\ell(R/(\underline{x}^{[n]})^{\lim}_{R}) + \ell(M/(\underline{x}^{[n]})^{\lim}_{M})) - n^d(e(\underline{x};R) + e(\underline{x};M)) \\ &= \ell(M/(\underline{x}^{[n]})^{\lim}_{M}) - n^d e(\underline{x};M) \\ &= J_{\underline{x},M}(n). \end{aligned}$$

The assertion now follows from Theorem 5.2. The proof is complete. □

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