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ON THE QUASI-ASYMPTOTICALLY LOCALLY EUCLIDEAN GEOMETRY OF NAKAJIMA'S METRIC

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Abstract We show that on the Hilbert scheme of n points on $\mathbb{C}^2$, the hyperkähler metric constructed by Nakajima via hyperkähler reduction is the quasi-asymptotically locally Euclidean (QALE) metric constructed by Joyce.

Keywords: hyperkähler manifold; QALE metric; hyperkähler reduction

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1. Introduction

The Hilbert scheme (or Douady scheme) of n points on $\mathbb{C}^2$, $\mathrm{Hilb}_0^n(\mathbb{C}^2)$, is a crepant resolution of

$$(\mathbb{C}^2)_0^n/S_n = \left\{ q \in (\mathbb{C}^2)^n,\ \sum_j q_j = 0 \right\} \Big/ S_n,$$

where the symmetric group S_n acts by permutation of the indices:

$$\sigma \cdot q = (q_{\sigma^{-1}(1)}, q_{\sigma^{-1}(2)}, \dots, q_{\sigma^{-1}(n)}).$$

Hence we have a map

$$\pi : \mathrm{Hilb}_0^n(\mathbb{C}^2) \to (\mathbb{C}^2)_0^n/S_n.$$

The complex manifold $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ carries a natural complex symplectic structure that comes from the S_n invariant one of $(\mathbb{C}^2)_0^n$. A compact Kähler manifold admitting a complex symplectic form carries in its Kähler class a hyperkähler metric; this is now a well-known consequence of the solution of the Calabi conjecture by Yau (see [**3**]). However, $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ is non-compact: for instance, $\mathrm{Hilb}_0^2(\mathbb{C}^2) = T^*\mathbb{P}^1(\mathbb{C})$. There are many extensions of Yau's result to non-compact manifolds (see, for example, [**2**, **27**, **28**]) and in 1999 Joyce introduced quasi-asymptotically locally Euclidean (QALE) manifolds, which are a generalization of asymptotically locally Euclidean (ALE) manifolds. When $\Gamma \subset \mathrm{O}(d)$ is a finite subgroup acting freely on $\mathbb{S}^{d-1}$, a complete ALE manifold (M^d, g) asymptotic to $\mathbb{R}^d/\Gamma$ is such that outside a compact set, M is diffeomorphic to $(\mathbb{R}^d \setminus \mathbb{B})/\Gamma$ and the metric is asymptotic to the Euclidean metric (the precise definition requires estimates of the difference between g and the Euclidean metric). When X^m is a crepant resolution of $\mathbb{C}^m/\Gamma$ for $\Gamma \subset \mathrm{SU}(m)$ a finite group, then roughly speaking a QALE Kähler metric on X^m is such that firstly, on pieces of X^m that (up to a finite ambiguity) are diffeomorphic to a subset of $X_A \times \mathrm{Fix}(A)$, where A is a subgroup of Γ and X_A is a crepant resolution of $\mathrm{Fix}(A)^{\perp}/A$, the metric is asymptotic to the sum of a QALE Kähler metric on X_A and a Euclidean metric on $\mathrm{Fix}(A)$ and secondly the metric is asymptotic to the Euclidean metric elsewhere. Joyce has proven the following theorem (see [**16**] and [**17**, Theorems 9.3.3 and 9.3.4]).

Theorem 1.1. *If $\Gamma \subset \mathrm{SU}(m)$ is a finite group and if $X^m \to \mathbb{C}^m/\Gamma$ is a crepant resolution, then in any Kähler class of QALE metric there is a unique QALE Kähler–Ricci flat metric. Moreover, if $\Gamma \subset \mathrm{Sp}(m/2)$, then this metric is hyperkähler.*

In particular, up to scaling, $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ carries a unique hyperkähler metric that is asymptotic to $(\mathbb{C}^2)_0^n/S_n$.

Another fruitful construction of hyperkähler metrics is the hyperkähler quotient construction of Hitchin *et al*. [**14**]. In fact, in 1999 Nakajima constructed a hyperkähler metric on $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ as a hyperkähler quotient [**25**]. Moreover, Nakajima asked whether this metric could be recovered via a resolution of the Calabi conjecture; Joyce also said that it is likely that QALE hyperkähler metrics can be explicitly constructed using the hyperkähker quotient, but outside the case of $\Gamma \subset \mathrm{SU}(2) = \mathrm{Sp}(1)$ treated by Kronheimer [**18**] he has no examples. The main result of this paper is the following theorem.

Theorem 1.2. *On $\mathrm{Hilb}_0^n(\mathbb{C}^2)$, up to a scaling, Joyce's metric and Nakajima's metric coincide.*

It should be noted that a given complex manifold can carry two very different hyperkähler metrics. For instance, it has been clearly explained by Lebrun that $\mathbb{C}^2$ carries two quite different Kähler–Ricci flat metrics: the Euclidean one and the Taub–Nut metric, which has cubic volume growth [**20**].

The main straightforward idea of the proof of this result is to study the asymptotics of Nakajima's metric; however, in order to use Joyce's uniqueness result, we also need asymptotics on the derivatives of Nakajima's metric. This study could probably be done at the cost of lengthy computations. Our analysis of the asymptotics of Nakajima's metric tells us that Joyce's metric and Nakajima's metric differ by $O(\rho^{-2}\sigma^{-2})$, where ρ is the distance to a fixed point and σ is a regularized version of the distance to the singular set. In order to use the classical argument of Yau that gives the uniqueness of the solution to the Calabi conjecture, we need to find a function φ that vanishes at infinity such that the difference between the two Kähler forms of Nakajima's metric and Joyce's metric is $\mathrm{i}\partial\bar{\partial}\varphi$. Joyce has developed elaborate tools to solve equations of the type $\Delta u = f$ on QALE manifolds, but the decay $O(\rho^{-2}\sigma^{-2})$ is critical for this analysis. We have circumvented this difficulty with the Li–Yau estimates of the Green kernel of a manifold with non-negative curvature [**21**], and we have obtained the following result, which has independent interest and which can be generalized to other QALE manifolds.

Theorem 1.3. *If f is a locally bounded function on $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ that satisfies, for some $\varepsilon > 0$,*

$$f = O\left(\frac{1}{\rho^\varepsilon \sigma^2}\right),$$

then the equation $\Delta u = f$ has a unique solution such that

$$u = O\left(\frac{\log(\rho+2)}{\rho^\varepsilon}\right).$$

For further more profound results about the analysis on QALE space, see the very interesting work by Degeratu and Mazzeo [**7**].

In the physics literature, $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ is associated with the moduli space of instantons on non-commutative $\mathbb{R}^4$ [**26**]. Our motivation for the study of the asymptotic geometry of the Nakajima metric comes from a question of Vafa and Witten about the space of L^2 harmonic forms on $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ endowed with the Nakajima metric. Let $\mathcal{H}^k$ be the space of L^2 harmonic k-forms on $\mathrm{Hilb}_0^n(\mathbb{C}^2)$:

$$\mathcal{H}^k = \{\alpha \in L^2(\Lambda^k T^* \mathrm{Hilb}_0^n(\mathbb{C}^2)),\ \mathrm{d}\alpha = \mathrm{d}^*\alpha = 0\}.$$

The following question is posed in [**29**] (see also the nice survey of Hausel [**11**]).

Conjecture 1.4.

$$\mathcal{H}^k = \begin{cases} \{0\} & \text{if } k \neq 2(n-1) = \dim_{\mathbb{C}} \mathrm{Hilb}_0^n(\mathbb{C}^2), \\ \mathrm{Im}(H_c^k(\mathrm{Hilb}_0^n(\mathbb{C}^2)) \to H^k(\mathrm{Hilb}_0^n(\mathbb{C}^2))) & \text{if } k = 2(n-1). \end{cases}$$

However, Vafa and Witten stated: 'we do not understand the predictions of S-duality on non-compact manifolds precisely enough to fully exploit them'. The paper [**12**] contains some results related to other string theory predictions on L^2 harmonic forms.

In fact, Hitchin has shown that the vanishing of the space of L^2 harmonic k-forms (for $k \neq 2(n-1)$) is a general fact for hyperkähler reduction of the flat quaternionic space $\mathbb{H}^m$ by a compact subgroup of $\mathrm{Sp}(m)$ [**13**]; he obtained this result with a generalization of an idea of Gromov (see [**9**] and also related works by Jost and Zuo [**15**] and by McNeal [**22**]). For the degree $k = 2(n-1)$, the cohomology of $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ is well known:

$$\begin{aligned} H_c^{2(n-1)}(\mathrm{Hilb}_0^n(\mathbb{C}^2)) &\simeq \mathrm{Im}(H_c^{2(n-1)}(\mathrm{Hilb}_0^n(\mathbb{C}^2)) \to H^{2(n-1)}(\mathrm{Hilb}_0^n(\mathbb{C}^2))) \\ &\simeq H^{2(n-1)}(\mathrm{Hilb}_0^n(\mathbb{C}^2)) \simeq \mathbb{R}, \end{aligned}$$

and a dual class to the generator is $\pi^{-1}\{0\}$. Moreover, a general result of Anderson says that the image of the cohomology with compact support in the cohomology always injects inside the space of L^2 harmonic forms [**1**]. Hence for the Hilbert scheme of n points in $\mathbb{C}^2$ endowed with Nakajima's metric we always have

$$\dim \mathcal{H}^{2(n-1)} \geqslant 1$$

and Conjecture 1.4 predicts the equality $\dim \mathcal{H}^{2(n-1)} = 1$.

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There are many results on the topological interpretation of the space of L^2 harmonic forms on non-compact manifolds but all of them require a little knowledge of asymptotic geometry (see [**6**] for a list of such results); the rough idea is that this asymptotic geometry would impose a certain behaviour of L^2 harmonic forms (decay, polyhomogeneity in a good compactification) and that would imply a topological interpretation of this space as a cohomology of a compactification. With our paper [**5**], our main result implies the following theorem.

Theorem 1.5. *The Vafa–Witten Conjecture (Conjecture 1.4) is true when* $n = 3$.

The case $n = 2$ can be treated by explicit computation (see [**13**] for clever computations).

As the Vafa–Witten Conjecture is in fact more general and concerns the quivers varieties constructed by Nakajima [**24**], a natural perspective is to understand the asymptotic geometry of the quivers varieties; the class of quasi-asymptotically conical manifolds introduced by Mazzeo should also be useful [**23**]. In a different direction it would be good to develop suitable QALE tolls to settle the status of the Vafa–Witten Conjecture.

2. Nakajima's metric

2.1. The quotient construction

In [**25**], Nakajima has shown that the Hilbert scheme of n points in $\mathbb{C}^2$ carries a natural hyperkähler metric; this metric is obtained through the hyperkähler quotient construction of Hitchin *et al*. [**14**]: the complex vector space

$$\mathbb{M}_n = \mathbb{M} := \{(A, B, x, y) \in \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{M}_n(\mathbb{C}) \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^*,\ \operatorname{tr} A = \operatorname{tr} B = 0\}$$

has a complex structure

$$J(A, B, x, y) = (B^*, -A^*, y^*, -x^*);$$

if we let $K = \mathrm{i}J$, then $(\mathbb{M},\ I = \mathrm{i},\ J, K := \mathrm{i}J)$ becomes a quaternionic vector space. Moreover, the unitary group $\mathrm{U}(n)$ acts linearly on $\mathbb{M}$: if $g \in \mathrm{U}(n)$ and $z = (A, B, x, y) \in \mathbb{M}$, then

$$g \cdot z = (gAg^{-1}, gBg^{-1}, gx, yg^{-1}).$$

The real moment map of this $\mathrm{U}(n)$-action is

$$\mu(A, B, x, y) = \frac{1}{2\mathrm{i}}([A, A^*] + [B, B^*] + x \otimes x^* - y^* \otimes y) \in \mathfrak{u}(n).$$

If $h \in \mathfrak{u}(n)$ and $z = (A, B, x, y) \in \mathbb{M}$, we let

$$l_z(h) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{th} \cdot z = ([h, A], [h, B], hx, -yh).$$

By definition, we have for $z \in \mathbb{M}$, $\delta z \in T_z\mathbb{M} \simeq \mathbb{M}$:

$$\langle \mathrm{d}\mu(z)(\delta z), h \rangle = \langle \mathrm{i}l_z(h), \delta z \rangle.$$

The action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{M}$ preserves the complex symplectic form

$$\omega_{\mathbb{C}}(z, z') = \operatorname{tr}(A \cdot B' - B \cdot A') + y'(x) - y(x'),$$

and the associated complex moment map is

$$\mu_{\mathbb{C}}(A, B, x, y) = [A, B] + x \otimes y \in \mathcal{M}_n(\mathbb{C}).$$

If $t > 0$, we define

$$\mathbb{L}_t(n) = \mathbb{L}_t := \mu^{-1}\left\{\frac{t}{2\mathrm{i}}\right\} \cap \mu_{\mathbb{C}}^{-1}\{0\},$$

then the map

$$\boldsymbol{\mu} := (\mu, \mu_{\mathbb{C}}) : \mathbb{M} \to \mathfrak{u}(n) \oplus \mathcal{M}_n(\mathbb{C})$$

is a submersion near $\mathbb{L}_t$ and $\mathrm{U}(n)$ acts freely on it; hence the quotient $\mathbb{H}_t := \mathbb{L}_t / \mathrm{U}(n)$ is a smooth manifold and this manifold is endowed with the Riemannian metric g_N, which makes the submersion $\mathbb{L}_t \to \mathbb{L}_t / \mathrm{U}(n)$ Riemannian. By definition, the tangent space of $\mathrm{U}(n) \cdot z$ is naturally isometric to the orthogonal of the space

$$\operatorname{Im} l_z \oplus I \operatorname{Im} l_z \oplus J \operatorname{Im} l_z \oplus K \operatorname{Im} l_z.$$

In particular, $\mathbb{H}_t$ is endowed with a quaternionic structure that is in fact integrable; therefore, the metric g_N is hyperkähler, and hence Kähler and Ricci flat.

2.2. Some remarks

Because for $\lambda > 0$ we have $\lambda \mathbb{L}_t = \mathbb{L}_{\lambda^2 t}$, all the spaces $\{\mathbb{H}_t\}_{t>0}$ are homeomorphic and their Riemannian metrics are proportional.

For $t = 0$ the quotient $\mathbb{L}_0 / \mathrm{U}(n)$ is not a smooth manifold. It is easy to show that

$$(A, B, x, y) \in \mathbb{L}_0 \Leftrightarrow (x = 0,\ y = 0,\ [A, B] = [A, A^*] = [B, B^*] = 0);$$

hence, if $(A, B, 0, 0) \in \mathbb{L}_0$, we can define the joint spectrum of (A, B):

$$S_n \cdot ((\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_n, \mu_n)) \in (\mathbb{C}^2)^n_0 / S_n,$$

where $(\mathbb{C}^2)^n_0 := \{(q_1, \ldots, q_n) \in (\mathbb{C}^2)^n,\ \sum_j q_j = 0\}$ and the symmetric group S_n acts on $(\mathbb{C}^2)^n_0$ by permutation of the indices. We get a homeomorphism (in fact an isometry)

$$\mathbb{L}_0 / \mathrm{U}(n) \simeq (\mathbb{C}^2)^n_0 / S_n.$$

In fact, for $t > 0$ we still have

$$(A, B, x, y) \in \mathbb{L}_t \Rightarrow y = 0. \tag{2.1}$$

Hence, if $z := (A, B, x, 0) \in \mathbb{L}_t$, we have

$$[A, B] = 0$$

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and therefore there is a $g \in \mathrm{U}(n)$ such that gAg^{-1} and gBg^{-1} are upper triangular matrices:

$$gAg^{-1} = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \quad \text{and} \quad gBg^{-1} = \begin{pmatrix} \mu_1 & * & \cdots & * \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \mu_n \end{pmatrix}.$$

The joint spectrum of (A, B) is still defined and we can define

$$\pi(\mathrm{U}(n) \cdot z) = S_n \cdot ((\lambda_1, \mu_1), (\lambda_2, \mu_2), \dots, (\lambda_n, \mu_n)) \in (\mathbb{C}^2)^n_0 / S_n.$$

$\mathbb{H}_t$ is homeomorphic to $\mathrm{Hilb}^n_0(\mathbb{C}^2)$, the Hilbert scheme of n points (with the centre of mass removed) in $\mathbb{C}^2$. The map $\pi : \mathrm{Hilb}^n_0(\mathbb{C}^2) \to (\mathbb{C}^2)^n_0/S_n$ is in fact a crepant resolution of $(\mathbb{C}^2)^n_0/S_n$.

Remark 2.1. We also remark that if $v = (\delta A, \delta B, \delta x, 0) \in T_\zeta \mathbb{L}_t$ is orthogonal to the range of l_ζ, then Jv is also in $T_\zeta \mathbb{L}_t$, and hence $\delta x = 0$.

2.3. The geometry of $\mathrm{Hilb}^2_0(\mathbb{C}^2)$

As an example we look at the geometry of $\mathrm{Hilb}^2_0(\mathbb{C}^2)$. Let $z = (A, B, x, 0) \in \mathcal{M}_2(\mathbb{C}) \oplus \mathcal{M}_2(\mathbb{C}) \oplus \mathbb{C}^2 \oplus (\mathbb{C}^2)^*$ such that $\operatorname{tr} A = \operatorname{tr} B = 0$ and

$$[A, A^*] + [B, B^*] + xx^* = t\,\mathrm{Id}, \qquad [A, B] = 0.$$

When $\det A \neq 0$ or $\det B \neq 0$ we can find a $\gamma \in \mathrm{U}(2)$ such that

$$\gamma A \gamma^{-1} = \begin{pmatrix} \lambda & a \\ 0 & -\lambda \end{pmatrix}, \qquad \gamma B \gamma^{-1} = \begin{pmatrix} \mu & b \\ 0 & -\mu \end{pmatrix}$$

and then let $\gamma(x) = (x_1, x_2)$. The equation $[A, B] = 0$ implies that there is a number ρ such that $a = \lambda\rho$ and $b = \mu\rho$. The remaining equations for $R^2 := |\lambda|^2 + |\mu|^2$ are then

$$|\rho|^2 R^2 + |x_1|^2 = t, \qquad -|\rho|^2 R^2 + |x_2|^2 = t, \qquad -2R^2\rho + x_1\bar{x}_2 = 0.$$

We can always choose γ such that $x_1, x_2 \in \mathbb{R}_+$, in which case we obtain

$$\rho^2 = \sqrt{4 + \frac{t^2}{R^4}} - 2.$$

Hence

$$\rho = \frac{t}{2R^2} + O\biggl(\frac{1}{R^6}\biggr);$$

if $x_1 = \sqrt{2t}\sin(\phi)$, $x_2 = \sqrt{2t}\cos(\phi)$, then $t\cos(2\phi) = \rho^2 R^2$ and $t\sin(2\phi) = 2\rho R^2$. Hence

$$\phi = \tfrac{1}{4}\pi + O\biggl(\frac{1}{R}\biggr)$$

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and

$$x_1 = \sqrt{\tfrac{1}{2}t} + O\left(\frac{1}{R}\right), \qquad x_2 = \sqrt{\tfrac{1}{2}t} + O\left(\frac{1}{R}\right).$$

Hence, for $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\} \simeq (\mathbb{C}^2)^2_0$ we have found

$$z(\lambda, \mu) = \left(\begin{pmatrix} \lambda & \lambda\rho(R) \\ 0 & -\lambda \end{pmatrix}, \begin{pmatrix} \mu & \mu\rho(R) \\ 0 & -\mu \end{pmatrix}, x(R), 0 \right) \in \mathbb{L}_t.$$

Moreover, $z(\lambda, \mu)$ and $z(\lambda', \mu')$ are in the same $\mathrm{U}(2)$ orbit if and only if $(\lambda, \mu) = \pm(\lambda', \mu')$; hence we have a map

$$z : (\mathbb{C}^2 \setminus \{0\})/\{\pm \mathrm{Id}\} \to \mathbb{L}_t / \mathrm{U}(2).$$

We can show that

$$z^* g_N = 2[|\mathrm{d}\lambda|^2 + |\mathrm{d}\mu|^2] + O\left(\frac{1}{R^4}\right).$$

This shows that $(\mathrm{Hilb}^2_0(\mathbb{C}^2), g_N)$ is an ALE hyperkähler metric asymptotic to $\mathbb{C}^2/\{\pm\mathrm{Id}\}$. These manifolds have been classified by Kronheimer [**19**], so that in this case Nakajima's metric is the Eguchi–Hansen metric on $T^*\mathbb{P}^1(\mathbb{C})$.

2.4. A last useful remark

A priori, it is not clear whether the above map z is holomorphic; the following useful lemma implies that this is the case.

Lemma 2.2. *Suppose that a compact Lie group G acts on $\mathbb{H}^m$ by quaternionic linear maps and let $\boldsymbol{\mu} : \mathbb{H}^m \to \mathfrak{g}^* \otimes \mathrm{Im}\,\mathbb{H}$ be the associated moment map. Assume that for some $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in \mathfrak{g}^* \otimes \mathrm{Im}\,\mathbb{H}$ the hyperkähler quotient $Q := \boldsymbol{\mu}^{-1}\{\zeta\}/G$ is well defined. If X is a complex manifold and $\Psi : X \to \boldsymbol{\mu}^{-1}\{\zeta\}$ is a smooth map such that locally*

$$\Psi(x) = g(x)\tilde{\Psi}(x),$$

where $g : X \to G^{\mathbb{C}}$ is smooth and $\tilde{\Psi} : X \to \mu_{\mathbb{C}}^{-1}\{\zeta_{\mathbb{C}}\}$ is holomorphic, then the induced map $\bar{\Psi} : X \to Q$ is also holomorphic.

Proof. If $q \in \mathbb{H}^m$, then let P_q be the orthogonal projection onto the orthogonal of $\mathrm{Im}\, l_q \oplus I\,\mathrm{Im}\, l_q \oplus J\,\mathrm{Im}\, l_q \oplus K\,\mathrm{Im}\, l_q = \mathrm{Im}\, l_q^{\mathbb{C}} \oplus J\,\mathrm{Im}\, l_q^{\mathbb{C}}$, where $l_q : \mathfrak{g} \to \mathbb{H}^q$ is defined as before by

$$l_q(h) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{e}^{th} \cdot q = h \cdot q.$$

We must show that if $x \in X$, then, for $q := \Psi(x)$,

$$P_q(\mathrm{d}\Psi(x)(Iv)) = IP_q(\mathrm{d}\Psi(x)(v)).$$

If $\dot{g}(x) = \mathrm{d}g(x)(Iv)g^{-1}(x) \in \mathfrak{g}^{\mathbb{C}}$, we then have

$$\mathrm{d}\Psi(x)(Iv) = \dot{g}(x) \cdot q + g(x) \cdot \mathrm{d}\tilde{\Psi}(x)(Iv) = l_q^{\mathbb{C}}(\dot{g}(x)) + g(x) \cdot \mathrm{d}\tilde{\Psi}(x)(Iv).$$

By definition, $P_q(l_q^{\mathbb{C}}(\dot{g}(x))) = 0$ and $g(x)$ and P_q are complex linear; hence

$$P_q(\mathrm{d}\Psi(x)(Iv)) = P_q(g(x) \cdot I\mathrm{d}\tilde{\Psi}(x)(v)) = IP_q(\mathrm{d}\Psi(x)(v)).$$

$\square$

3. Joyce's metric

In [**16**,**17**], Joyce built many new Kähler–Ricci flat metrics on some crepant resolution of quotients of $\mathbb{C}^m$ by a finite subgroup of $\mathrm{SU}(m)$; his construction relies on the resolution of a Calabi–Yau problem on QALE manifolds. We will follow the presentation given in [**17**, §§ 9.1 and 9.2] for the QALE geometry of the Hilbert scheme of n points on $\mathbb{C}^2$.

3.1. The local product resolution of $\mathrm{Hilb}_0^n(\mathbb{C}^2)$

If $\mathfrak{p} = (I_1, I_2, \dots, I_k)$ is a partition of $\{1, 2, \dots, n\}$*, the I_l are called the clusters of $\mathfrak{p}$. We define

$$V_\mathfrak{p} = \{q \in (\mathbb{C}^2)_0^n,\ \forall l \in \{1, \dots, k\},\ \forall i, j \in I_l : q_i = q_j\}$$

and $A_\mathfrak{p} = \{\gamma \in S_n,\ \gamma q = q,\ \forall q \in V_\mathfrak{p}\} \simeq S_{n_1} \times S_{n_2} \times \dots \times S_{n_k}$, where $n_l = \#I_l$. Then

$$W_\mathfrak{p} = V_\mathfrak{p}^\perp \simeq \bigoplus_{l=1}^k (\mathbb{C}^2)_0^{n_l}.$$

Let $m_\mathfrak{p} = \mathrm{codim}_\mathbb{C} V_\mathfrak{p} = \dim_\mathbb{C} W_\mathfrak{p} = 2(n - l(\mathfrak{p}))$, where $l(\mathfrak{p}) = k$. The set $\mathcal{P}_n$ of partitions of $\{1, 2, \dots, n\}$ has the following partial order:

$$\mathfrak{p} \leqslant \mathfrak{q} \Leftrightarrow V_\mathfrak{q} \subset V_\mathfrak{p} \Leftrightarrow W_\mathfrak{p} \subset W_\mathfrak{q}.$$

Hence $\mathfrak{p} \leqslant \mathfrak{q}$ if and only if $\mathfrak{p}$ is a refinement of $\mathfrak{q}$, i.e. if $\mathfrak{q} = (J_1, J_2, \dots, J_r)$, then there are partitions $(I_{l,1}, I_{l,2}, \dots, I_{l,n_l})$ of $J_l = I_{l,1} \cup \dots \cup I_{l,n_l}$ such that the clusters of $\mathfrak{p}$ are the $I_{l,j}$. The smallest partition is $\mathfrak{p}_0 = \{1\} \cup \{2\} \cup \dots \cup \{n\}$ with $V_{\mathfrak{p}_0} = (\mathbb{C}^2)_0^n$; the largest partition is $\mathfrak{p}_\infty = \{1, 2, \dots, n\}$ with $V_{\mathfrak{p}_\infty} = \{0\}$.

The fundamental partitions are the $\mathfrak{p}_{i,j}$ defined by

$$\mathfrak{p}_{i,j} = (\{i, j\}, \{k_1\}, \{k_2\}, \dots, \{k_{n-2}\})$$

with $\{1, 2, \dots, n\} \setminus \{i, j\} = \{k_1, k_2, \dots, k_{n-2}\}$; we also have

$$V_{i,j} := V_{\mathfrak{p}_{i,j}} = \{q \in (\mathbb{C}^2)_0^n,\ q_i = q_j\}.$$

And, for any partition $\mathfrak{p} \neq \mathfrak{p}_0$,

$$V_\mathfrak{p} = \bigcap_{\mathfrak{p}_{i,j} \leqslant \mathfrak{p}} V_{i,j}.$$

We will also define $\Delta_\mathfrak{p} = \{(i, j) \in \{1, 2, \dots, n\}^2,\ \mathfrak{p}_{i,j} \not\leqslant \mathfrak{p}\}$ and $\Delta_\mathfrak{p}^c = \{(i, j) \in \{1, 2, \dots, n\}^2,\ \mathfrak{p}_{i,j} \leqslant \mathfrak{p}\}$. The singular locus of $(\mathbb{C}^2)_0^n / S_n$ is the quotient of the generalized diagonal

$$S = \Big(\bigcup_{\mathfrak{p} \neq \mathfrak{p}_0} V_p\Big) \Big/ S_n = \Big(\bigcup_{i,j} V_{i,j}\Big) \Big/ S_n.$$

* The I_l are disjoint and their union is $\{1, 2, \dots, n\}$.

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Finally, let

$$S_{\mathfrak{p}} = \left(\bigcup_{(i,j)\in\Delta_{\mathfrak{p}}} V_{i,j} \right) \Big/ A_{\mathfrak{p}}$$

and, for $R > 0$, let $T_{\mathfrak{p}}$ be the R-neighbourhood of $S_{\mathfrak{p}}$ in $(\mathbb{C}^2)^n_0/A_{\mathfrak{p}}$:

$$T_{\mathfrak{p}} := \{q \in (\mathbb{C}^2)^n_0,\ \exists (i,j) \in \Delta_{\mathfrak{p}},\ |q_i - q_j| < R\}/A_{\mathfrak{p}}.$$

The resolution $\pi : \mathrm{Hilb}^n_0(\mathbb{C}^2) \to (\mathbb{C}^2)^n_0/S_n$ is a local product resolution; indeed there is a resolution of $W_{\mathfrak{p}}/A_{\mathfrak{p}}$, namely

$$\pi_{\mathfrak{p}} : \mathrm{Hilb}^{\mathfrak{p}}_0(\mathbb{C}^2) := \prod_{l=1}^{k} \mathrm{Hilb}^{n_l}_0(\mathbb{C}^2) \to W_{\mathfrak{p}}/A_{\mathfrak{p}},$$

such that, for $U_{\mathfrak{p}} = (\pi_{\mathfrak{p}} \times \mathrm{Id})^{-1}(T_{\mathfrak{p}}) \subset \mathrm{Hilb}^{\mathfrak{p}}_0(\mathbb{C}^2) \times V_{\mathfrak{p}}$ and $\phi_{\mathfrak{p}} : (\mathbb{C}^2)^n_0/A_{\mathfrak{p}} \to (\mathbb{C}^2)^n_0/S_n$, there is a local biholomorphism onto its image $\psi_{\mathfrak{p}} : U_{\mathfrak{p}} \to \mathrm{Hilb}^n_0(\mathbb{C}^2)$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hilb}^{\mathfrak{p}}_0(\mathbb{C}^2) \times V_{\mathfrak{p}} \setminus U_{\mathfrak{p}} & \xrightarrow{\psi_{\mathfrak{p}}} & \mathrm{Hilb}^n_0(\mathbb{C}^2) \\ \downarrow{\scriptstyle \pi_{\mathfrak{p}} \times \mathrm{Id}} & & \downarrow{\scriptstyle \pi} \\ (\mathbb{C}^2)^n_0/A_{\mathfrak{p}} \setminus T_{\mathfrak{p}} & \xrightarrow{\phi_{\mathfrak{p}}} & (\mathbb{C}^2)^n_0/S_n \end{array}$$

The local biholomorphism $\psi_{\mathfrak{p}}$ can be defined with the hyperkähler quotient description of $\mathrm{Hilb}^{\mathfrak{p}}_0(\mathbb{C}^2)$ and $\mathrm{Hilb}^n_0(\mathbb{C}^2)$. We will identify $V_{\mathfrak{p}}$ with $(\mathbb{C}^2)^k_0$.

We consider

$$\zeta = ((A_1, B_1, x_1, 0), (A_2, B_2, x_2, 0), \ldots, (A_k, B_k, x_k, 0)) \in \prod_{j=1}^{k} \mathbb{L}_t(n_j)$$

and

$$\eta = ((\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_k, \mu_k)) \in (\mathbb{C}^2)^k_0 \setminus U_{\mathfrak{p}},$$

such that

$$i \neq j \Rightarrow |\lambda_i - \lambda_j|^2 + |\mu_i - \mu_j|^2 \neq 0.$$

We associate with (ζ, η) the vector $(A, B, x, 0) \in \mathbb{M}(n)$ such that A and B are block diagonal matrices with respective main diagonal blocks

$$(A_1 + \lambda_1, A_2 + \lambda_2, \ldots, A_k + \lambda_k) \quad \text{and} \quad (B_1 + \mu_1, B_2 + \mu_2, \ldots, B_k + \mu_k)$$

and $x = (x_1, x_2, \ldots, x_k)$. Then $\psi_{\mathfrak{p}}((\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_k)) \cdot \zeta, \eta)$ is the set of points in the $\mathrm{GL}_n(\mathbb{C})$-orbit of $(A, B, x, 0)$ satisfying the real moment map equation (see § 4 for more details).

3.2. The QALE metric on $\mathrm{Hilb}_0^n(\mathbb{C}^2)$

We introduce several distance-type functions on $(\mathrm{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2)\times V_{\mathfrak{p}})\setminus U_{\mathfrak{p}}$. If $z \in \mathrm{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2)\times V_{\mathfrak{p}} \setminus U_{\mathfrak{p}}$ and $v = (\pi_{\mathfrak{p}} \times \mathrm{Id})(z)$, we define

$$\mu_{\mathfrak{p},\mathfrak{q}}(z) = \inf_{\gamma\in A_{\mathfrak{p}}} \mathrm{d}(\gamma \cdot v, V_{\mathfrak{q}}) = \mathrm{d}(v, (A_{\mathfrak{p}} V_{\mathfrak{q}})/A_{\mathfrak{p}})$$

and

$$\nu_{\mathfrak{p}}(z) = 1 + \inf_{p\neq p_0} \mu_{\mathfrak{p},\mathfrak{q}}(z).$$

A Riemannian metric g on $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ is QALE (asymptotic to $(\mathbb{C}^2)_0^n/S_n$) if, for each partition $\mathfrak{p}$, there is a metric $g_{\mathfrak{p}}$ on $\mathrm{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2)$ such that, for all $l \in \mathbb{N}$,

$$\nabla^l(\psi_{\mathfrak{p}}^* g - (g_{\mathfrak{p}} + \mathrm{eucl}_{V_{\mathfrak{p}}})) = \sum_{\mathfrak{q}\not\leqslant\mathfrak{p}} O\biggl(\frac{1}{\nu_{\mathfrak{p}}^{2+l}\mu_{\mathfrak{p},\mathfrak{q}}^{2m_{\mathfrak{q}}-2}}\biggr). \tag{3.1}$$

However, if $\mathfrak{q} \not\leqslant \mathfrak{p}$ there is always an $(i,j) \in \Delta_{\mathfrak{p}}$ such that $\mathfrak{p}_{i,j} \not\leqslant \mathfrak{p}$ and $\mathfrak{p}_{i,j} \leqslant \mathfrak{q}$; therefore,

$$\mu_{\mathfrak{p},\mathfrak{p}_{i,j}}^{2m_{\mathfrak{p}_{i,j}}-2} = \mu_{\mathfrak{p},\mathfrak{p}_{i,j}}^2 \leqslant \mu_{\mathfrak{p},\mathfrak{q}}^{2m_{\mathfrak{q}}-2}.$$

If we introduce $\rho_{\mathfrak{p}}(z) = \inf_{(i,j)\in\Delta_{\mathfrak{p}}} \mu_{\mathfrak{p},\mathfrak{p}_{i,j}}(z)$, then, for $v = (\pi_{\mathfrak{p}} \times \mathrm{Id})(z) \in (\mathbb{C}^2)_0^n/A_{\mathfrak{p}}$, we have

$$\rho_{\mathfrak{p}}(z) = \inf_{(i,j)\in\Delta_{\mathfrak{p}}} |v_i - v_j|.$$

The asymptotics (3.1) are equivalent to

$$\nabla^l(\psi_{\mathfrak{p}}^* g - (g_{\mathfrak{p}} + \mathrm{eucl}_{V_{\mathfrak{p}}})) = O\biggl(\frac{1}{\nu_{\mathfrak{p}}^{2+l}\rho_{\mathfrak{p}}^2}\biggr). \tag{3.2}$$

We introduce two other distance-type functions: when $z \in \mathrm{Hilb}_0^n(\mathbb{C}^2)$ and $\pi(z) = (v_1, v_2, \dots v_n) \in (\mathbb{C}^2)_0^n/S_n$, we let

$$\rho(z) = \sqrt{\sum_{i<j} |v_i - v_j|^2}$$

and

$$\sigma(z) = \inf_{i\neq j}\{|v_i - v_j|\} + 1.$$

If $\mathfrak{p}$ is a partition of $\{1, 2, \dots, n\}$ and ϵ, τ, R are positive real numbers, then we introduce

$$\check{\mathcal{C}}_{\mathfrak{p}_0} = \{(v_1, \dots, v_n) \in (\mathbb{C}^2)_0^n/S_n,\ \text{such that } |v| > R \text{ and } \forall i \neq j,\ |v_i - v_j| > \varepsilon|v|\},$$

$$\check{\mathcal{C}}_{\mathfrak{p}} = \biggl\{(v_1, \dots, v_n) \in (\mathbb{C}^2)_0^n/A_{\mathfrak{p}},\ \text{such that } |v| > R,$$
$$\forall (i,j) \in \Delta_{\mathfrak{p}},\ |v_i - v_j| > \sqrt{\frac{2}{n(n-1)}}|v| \text{ and } \forall (i,j) \in \Delta_{\mathfrak{p}}^c, |v_i - v_j| < 2\epsilon|v|\biggr\}.$$

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It is clear that if ϵ is small enough then

$$(\mathbb{C}^2 \setminus R\mathbb{B})^n_0/S_n = \bigcup_{\mathfrak{p}} \phi_{\mathfrak{p}}(\check{\mathcal{C}}_{\mathfrak{p}}).$$

Moreover, on $\mathcal{C}_{\mathfrak{p}} := (\pi_{\mathfrak{p}} \times \mathrm{Id})^{-1}(\check{\mathcal{C}}_{\mathfrak{p}})$, the asymptotic (3.2) is

$$\nabla^l(\psi_{\mathfrak{p}}^* g - (g_{\mathfrak{p}} + \mathrm{eucl}_{V_{\mathfrak{p}}})) = O\biggl(\frac{1}{\sigma^{2+l}\rho^2}\biggr). \tag{3.3}$$

Remark 3.1. It can be shown that if all metrics $g_{\mathfrak{p}}$ are QALE and if the estimates (3.3) are satisfied, then g is also QALE.

3.3. Joyce's result

The result of Joyce concerning the Hilbert scheme of n points on $\mathbb{C}^2$ is the following.

Theorem 3.2. *Up to scaling,* $\mathrm{Hilb}^n_0(\mathbb{C}^2)$ *has a unique QALE hyperkähler metric asymptotic to* $(\mathbb{C}^2)^n_0/S_n$.

4. The asymptotic of Nakajima's metric

4.1. The induction hypothesis

In this section we will prove the following result by induction on n.

(i) On $\mathrm{Hilb}^n_0(\mathbb{C}^2)$, Nakajima's metric g_N satisfies the estimate (3.3) for $l = 0$; more precisely, if $g_{\mathfrak{p}}$ is the sum of Nakajima's metric on $\mathrm{Hilb}^{\mathfrak{p}}_0(\mathbb{C}^2)$, then there are an $\epsilon > 0$ small enough and an R large enough such that for all partitions $\mathfrak{p}$ we have, on $(\pi_{\mathfrak{p}} \times \mathrm{Id})^{-1}(\check{\mathcal{C}}_{\mathfrak{p}})$,

$$\psi_{\mathfrak{p}}^*(g_N) - g_{\mathfrak{p}} + \mathrm{eucl}_{V_{\mathfrak{p}}} = O\biggl(\frac{1}{\sigma^2\rho^2}\biggr).$$

(ii) There is a constant C such that if $z = (A, B, x, 0) \in \mathbb{L}_t$ then

$$\forall h \in \mathfrak{u}_n, \quad \|l_z(h)\|^2 = \|[h, A]\|^2 + \|[h, B]\|^2 + \|hx\|^2 \geqslant C\|h\|^2.$$

(iii) There is a constant M such that for all $z \in \mathbb{L}_t$ and $(\delta A, \delta B, 0, 0) \in T_z\mathbb{L}_t$ orthogonal to $\mathrm{Im}\, l_z$ we have

$$\|[\delta A, \delta A^*]\| + \|[\delta B, \delta B^*]\| \leqslant \frac{M}{\sigma^2}(\|\delta A\|^2 + \|\delta B\|^2).$$

It is easy to check these three conditions for $\mathrm{Hilb}^2_0(\mathbb{C}^2)$ thanks to the explicit description of $\mathbb{L}_t$ in this case (see (2.3)). So we now assume that these induction hypotheses are true for all $m < n$.

4.2. The case of well-separated points

We first prove the easiest case: that is, we prove the induction hypothesis for the partition $\mathfrak{p}_0$. More precisely, we consider $\boldsymbol{q} = (q_1, q_2, \dots, q_n) \in (\mathbb{C}^2)^n_0$ such that, for all $i \neq j$, $|q_i - q_j| > R$ (R will be chosen large enough), the set of such qs will be denoted by $\mathcal{O}_0$.

If $q_j = (\lambda_j, \mu_j)$, we search for a solution $z = (A, B, x, 0) \in \mathbb{M}$ of the equations

$$[A, A^*] + [B, B^*] + xx^* = t\,\mathrm{Id}, \qquad [A, B] = 0, \tag{4.1}$$

where A, B are upper triangular matrices with respective diagonals $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $(\mu_1, \mu_2, \dots, \mu_n)$ and upper diagonal coefficients $\boldsymbol{a} = (a_{i,j})$, $\boldsymbol{b} = (b_{i,j})$. We obtain the following equation* for the (i, j) coefficients of Equation (4.1):

$$\left.\begin{aligned} (\bar\lambda_i - \bar\lambda_j)a_{i,j} + (\bar\mu_i - \bar\mu_j)b_{i,j} + \sum_k [\bar a_{k,i}a_{k,j} + \bar b_{k,i}b_{k,j} - a_{i,k}\bar a_{j,k} - b_{i,k}\bar b_{j,k}] = x_i\bar x_j, \\ -(\bar\mu_i - \bar\mu_j)a_{i,j} + (\lambda_i - \lambda_j)b_{i,j} + \sum_k [a_{i,k}b_{k,j} - b_{i,k}a_{k,j}] = 0. \end{aligned}\right\} \tag{4.2}$$

And the equation for the diagonal coefficient (i, i) of (4.1) gives

$$\sum_k [|a_{i,k}|^2 - |a_{k,i}|^2 + |b_{i,k}|^2 - |b_{k,i}|^2] + |x_i|^2 = t. \tag{4.3}$$

We let $R_{i,j} = \sqrt{|\lambda_i - \lambda_j|^2 + |\mu_i - \mu_j|^2}$ and

$$x_i^0 = \sqrt{t}, \qquad a_{i,j}^0 = (\lambda_i - \lambda_j)\frac{t}{R_{i,j}^2}, \qquad b_{i,j}^0 = (\mu_i - \mu_j)\frac{t}{R_{i,j}^2}. \tag{4.4}$$

Then, if we write Equations (4.2) and (4.3) in the synthetic form

$$F(\boldsymbol{q}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}) = 0,$$

where $F : (\mathbb{C}^2)^n_0 \times \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^n \to \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^{n(n-1)/2} \times \mathbb{C}^n$, we have

$$F(\boldsymbol{q}, \boldsymbol{a}^0, \boldsymbol{b}^0, \boldsymbol{x}^0) = O(\sigma^{-2}).$$

Moreover, it is easy to check that when σ is large enough the partial derivative in the last three arguments $D_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{x})}F(\boldsymbol{q}, \boldsymbol{a}^0, \boldsymbol{b}^0, \boldsymbol{x}^0)$ is invertible and the norm of the inverse is uniformly bounded. Because the map F is polynomial of degree 2 in its arguments, the implicit function theorem implies that Equations (4.2) and (4.3) have a unique solution such that

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x})(q) = (\boldsymbol{a}^0, \boldsymbol{b}^0, \boldsymbol{x}^0) + O(\sigma^{-2}) \quad \text{and} \quad D_q(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x})(q) = O(\sigma^{-2}). \tag{4.5}$$

We have built a map

$$\begin{aligned} \hat\Psi_0 : \mathcal{O}_0 &\to \mathbb{L}_t, \\ q &\mapsto (A(\boldsymbol{q}), B(\boldsymbol{q}), x(\boldsymbol{q}), 0). \end{aligned}$$

* With the convention that $a_{i,j} = b_{i,j} = 0$ if $j \leqslant i$.

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Moreover, $\hat{\Psi}_0(\boldsymbol{q})$ and $\hat{\Psi}_0(\boldsymbol{q'})$ live in the same $\mathrm{U}(n)$-orbit if and only if $\boldsymbol{q}$ and $\boldsymbol{q'}$ live in the same S_n-orbit; hence $\hat{\Psi}_0$ induces a map

$$\Psi_0 : \mathcal{O}_0/S_n \to \mathrm{Hilb}_0^n(\mathbb{C}^2).$$

Moreover, this map Ψ_0 is holomorphic according to Lemma 2.2.

The diagonal entries of the matrices $A(q)$ and $B(q)$ are given by (4.4); hence the q-derivative of the diagonal entries of $A(q)$, $B(q)$ can be computed explicitly. We then have

$$x(q) = \sqrt{t}(1, \dots, 1) + O(\sigma^{-2});$$

hence with (4.5) we find that

$$D_q x(q) \cdot v = O(\sigma^{-2})v.$$

We have

$$|\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v|^2 = |D_q A(\boldsymbol{q}) \cdot v|^2 + |D_q B(\boldsymbol{q}) \cdot v|^2 + |D_q x(q)|^2,$$

and if $v = ((\delta\lambda_1, \delta\mu_1), \dots, (\delta\lambda_n, \delta\mu_n))$, then

$$|D_q A(\boldsymbol{q}) \cdot v|^2 = \sum_j |\delta\lambda_j|^2 + |D_q a(\boldsymbol{q}) \cdot v|^2,$$
$$|D_q B(\boldsymbol{q}) \cdot v|^2 = \sum_j |\delta\mu_j|^2 + |D_q b(\boldsymbol{q}) \cdot v|^2.$$

With (4.5), we obtain

$$|\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v|^2 = |v|^2 + O\left(\frac{|v|^2}{\sigma^4}\right). \tag{4.6}$$

In order to check point (i) of the induction hypothesis we must show that

$$|\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v|^2 - |\Pi_z(\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v)|^2 = |v|^2 + O\left(\frac{|v|^2}{\sigma^4}\right),$$

where, if $\hat{\Psi}_0(q) = z \in \mathbb{L}_t$, Π_z is the orthogonal projection onto the space $\operatorname{Im} l_z$. But, by construction, if $X \in \operatorname{Im} l_z$, then IX is normal to $T_z\mathbb{L}_t$, and hence $\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot (Iv) \perp IX$; in particular, $\Pi_z(I \cdot \mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot (Iv)) = 0$. Hence

$$\Pi_z(\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v) = \Pi_z(\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v + I\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot Iv) = 2\Pi_z(\bar{\partial}\hat{\Psi}_0(\boldsymbol{q}) \cdot v). \tag{4.7}$$

But, by construction,

$$|\bar{\partial}\hat{\Psi}_0(\boldsymbol{q}) \cdot v|^2 = |\bar{\partial}\boldsymbol{a}|^2 + |\bar{\partial}\boldsymbol{b}|^2 + |\bar{\partial}\boldsymbol{x}|^2 = O\left(\frac{|v|^2}{\sigma^4}\right). \tag{4.8}$$

Assertion (i) of the induction hypothesis follows from the estimates (4.6)–(4.8).

For point (ii) of the induction hypothesis, we have, for $z = \hat{\Psi}_0(\boldsymbol{q})$ and $h = (h_{i,j}) \in \mathfrak{u}_n$,

$$\|l_z(h)\|^2 \geqslant \frac{1}{2}\left[\sum_{i,j} R_{i,j}^2 |h_{i,j}|^2 + t\sum_i |h_{i,i}|^2\right] - C\sigma^{-2}\|h\|^2.$$

Hence, if R is chosen large enough, part (ii) of the induction hypothesis holds on $\mathcal{O}_0$.

Now we check part (iii) of the induction hypothesis. Let

$$(\delta A, \delta B, 0, 0) = \mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v - \Pi_z(\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v).$$

We have just said that

$$\|\Pi_z(\mathrm{d}\hat{\Psi}_0(\boldsymbol{q}) \cdot v)\| = O(\sigma^{-2})|v|,$$

so the off-diagonal parts of δA and δB are bounded by $O(\sigma^{-2})|v|$, which implies that

$$\|[\delta A, \delta A^*]\|^2 + \|[\delta B, \delta B^*]\|^2 \leqslant O(\sigma^{-4})|v|^4.$$

4.3. The general case

We now examine the region $\mathcal{C}_{\mathfrak{p}}$ associated with another partition $\mathfrak{p} \neq \mathfrak{p}_0$; we can always assume that

$$\mathfrak{p} = (\{m_0 = 1, \dots, m_1\}, \{m_1 + 1, \dots, m_2\}, \dots, \{m_{k-1} + 1, \dots, n = m_k\}).$$

Let $n_l = m_l - m_{l-1}$. We consider the set $\mathcal{O}_{\mathfrak{p}}$ of

$$(\boldsymbol{q}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{x}) \in (\mathbb{C}^2)^k_0 \times \bigoplus_{j=1}^{k} \mathcal{M}_{n_j}(\mathbb{C}) \times \bigoplus_{j=1}^{k} \mathcal{M}_{n_j}(\mathbb{C}) \times \bigoplus_{j=1}^{k} \mathbb{C}^{n_j}$$

such that if $\boldsymbol{q} = (q_1, q_2, \dots, q_k)$ then, for all $i \neq j$, $|q_i - q_j| > \sqrt{1/n(n-1)}|\boldsymbol{q}|$ and $|\boldsymbol{q}| \geqslant R$; and if $\boldsymbol{A} = (A_1, A_2, \dots, A_k)$, $\boldsymbol{B} = (B_1, B_2, \dots, B_k)$, $\boldsymbol{x} = (x_1, x_2, \dots, x_k)$, then each (A_j, B_j, x_j) satisfies $\operatorname{tr} A_j = \operatorname{tr} B_j = 0$ and the moment map equation

$$[A_j, A_j^*] + [B_j, B_j^*] + x_j x_j^* = t\,\mathrm{Id}_{n_j}, \qquad [A_j, B_j] = 0;$$

and, moreover,

$$\sup_j (\|A_j\|^2 + \|B_j\|^2) \leqslant \tau^2 |\boldsymbol{q}|^2.$$

We will search for a solution $z = (A, B, x, 0)$ of the moment map equation that is approximatively

$$A \simeq \begin{pmatrix} A_1 + \lambda_1 & 0 & \cdots & 0 \\ 0 & A_2 + \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & A_k + \lambda_k \end{pmatrix},$$

$$B \simeq \begin{pmatrix} B_1 + \mu_1 & 0 & \cdots & 0 \\ 0 & B_2 + \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & B_k + \mu_k \end{pmatrix},$$

$$x \simeq (x_1, x_2, \dots, x_k).$$

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First we fix some $\zeta = (\boldsymbol{q}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{x}) \in \mathcal{O}_{\mathfrak{p}}$ and we search for a $z_0 = (A^0, B^0, x^0, 0)$ such that if $q_j = (\lambda_j, \mu_j)$ then $x^0 = (x_1, x_2, \dots, x_k)$, A^0 (respectively B^0) is upper block triangular with diagonal $(A_1 + \lambda_1, A_2 + \lambda_2 \operatorname{Id}, \dots, A_k + \lambda_k)$ (respectively $(B_1 + \mu_1, B_2 + \mu_2, \dots, B_k + \mu_k)$) and $\mu(z_0)$, $\mu_{\mathbb{C}}(z_0)$ are block diagonal. Hence we search for matrices $A_{i,j}, B_{i,j} \in \mathcal{M}_{n_i,n_j}(\mathbb{C})$, $i < j$, such that for all $i < j$,

$$\left.\begin{aligned} &(A_i^* + \bar{\lambda}_i)A_{i,j} - A_{i,j}(A_j^* + \bar{\lambda}_j) + (B_i^* + \bar{\mu}_i)B_{i,j} \\ &\qquad\qquad -B_{i,j}(B_j^* + \bar{\mu}_j) + Q_1(i,j) + Q_2(i,j) = x_i x_j^*, \\ &-(B_i + \mu_i)A_{i,j} + A_{i,j}(B_j + \mu_j) + (A_i + \lambda_i)B_{i,j} - B_{i,j}(A_j + \lambda_j) + Q_3(i,j) = 0, \end{aligned}\right\} \tag{4.9}$$

where $Q_1(i,j)$ (respectively $Q_2(i,j)$) is a quadratic expression depending on the $A_{\alpha,\beta}$s (respectively on the $B_{\alpha,\beta}$s) and $Q_3(i,j)$ is bilinear in the $A_{\alpha,\beta}$s and the $B_{\alpha,\beta}$s. For $\tau > 0$ small enough, with the same arguments given in the preceding paragraph, the implicit function theorem implies the following lemma.

Lemma 4.1. *Equations (4.9) have a solution $A_{i,j}, B_{i,j} \in \mathcal{M}_{n_i,n_j}(\mathbb{C})$, $i < j$, which depends smoothly on $\zeta \in \mathcal{O}_{\mathfrak{p}}$; moreover, we have that*

$$\sum_{i<j} \|A_{i,j}\|^2 + \|B_{i,j}\|^2 = O\biggl(\frac{1}{|\boldsymbol{q}|^2}\biggr).$$

And the differential of the map $\zeta \mapsto (A_{i,j}, B_{i,j})$ is bounded by $O(1/|\boldsymbol{q}|^2)$.

We then obtain that $z_0(\zeta) = (A^0, B^0, x^0, 0) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \times \mathbb{C}^n \times (\mathbb{C}^n)^*$ satisfies

$$[A^0, B^0] = 0, \qquad 2\mathrm{i}\mu(z_0) - t = O\biggl(\frac{1}{|\boldsymbol{q}|^2}\biggr).$$

That is, $z_0(\zeta)$ is an almost solution of the moment map equation. More precisely, the off-block-diagonal terms of the moment map equations are zero. We will now use an argument that we learned from a paper of Donaldson [**8**, Proposition 17]: we will find a Hermitian matrix $h = \mathrm{i}k$ such that, if

$$z_h = \mathrm{e}^{\mathrm{i}k} \cdot z_0 = (\mathrm{e}^h A^0 \mathrm{e}^{-h}, \mathrm{e}^h B^0 \mathrm{e}^{-h}, \mathrm{e}^h \cdot x^0, 0),$$

then $2\mathrm{i}\mu(z_h) - t \operatorname{Id} = 0$ and $\mu_{\mathbb{C}}(z_h) = 0$ (this latter condition being obvious).

From part (ii) of the induction hypothesis, and if we take τ small enough and R large enough, we have

$$\forall \eta \in \mathfrak{u}_n, \quad \|l_{z_0}(\eta)\| \geqslant C\|\eta\|,$$

the constant C being uniform on $\mathcal{O}_{\mathfrak{p}}$. Hence, if $h = \mathrm{i}\eta$ with

$$\|k\| \leqslant \delta := \min\biggl\{1, \frac{C\mathrm{e}^{-2}}{4|z_0|}\biggr\},$$

then

$$\forall \eta \in \mathfrak{u}_n, \quad \|l_{z_h}(\eta)\| \geqslant \tfrac{1}{2}C\|\eta\|.$$

So as soon as we have

$$\left|\mu(z_0) - \frac{t}{2\mathrm{i}}\,\mathrm{Id}\right| < \frac{4}{C^2}\delta,$$

Proposition 17 in [**8**] furnishes an $h = \mathrm{i}k$ with $\mu(\mathrm{e}^h \cdot z_0) = (t/2\mathrm{i})\,\mathrm{Id}$ and with

$$\|h\| \leqslant \frac{4}{C^2}\left|\mu(z_0) - \frac{t}{2\mathrm{i}}\,\mathrm{Id}\right|.$$

But when R is large enough, the condition

$$\left|\mu(z_0) - \frac{t}{2\mathrm{i}}\,\mathrm{Id}\right| < \frac{4}{C^2}\delta$$

is satisfied; hence there is a Hermitian matrix $h = \mathrm{i}k$ such that $2\mathrm{i}\mu(z_h) - t\,\mathrm{Id} = 0$ and $\mu_{\mathbb{C}}(z_h) = 0$.

We need to recall how h is found. For $z \in \mathbb{M}$ we have a linear map $l_z : \mathfrak{u}_n \to T_z\mathbb{M} \simeq \mathbb{M}$ and l_z^* its adjoint; from the definition of the moment map we have $l_z^* = \mathrm{d}\mu(z) \circ I$. The endomorphism Q_z of $\mathfrak{u}_n$ is given by $Q_z = l_z^* l_z$. Then, for every $h = \mathrm{i}k$, with $|k| < \delta$, Q_{z_h} is invertible and $Q_{z_h}^{-1}$ has an operator norm bounded by $4C^{-2}$. Letting $a(z) = Q_z^{-1}(\mu(z) - (t/2\mathrm{i})\,\mathrm{Id})$, we follow the maximal solution of the equation

$$\frac{\mathrm{d}z}{\mathrm{d}s} = -\mathrm{i}l_z(a(z)), \quad z(0) = z_0. \tag{4.10}$$

By definition we have

$$\frac{\mathrm{d}\mu(z_s)}{\mathrm{d}s} = -\left(\mu(z_s) - \frac{t}{2\mathrm{i}}\,\mathrm{Id}\right);$$

hence

$$\mu(z_s) - \frac{t}{2\mathrm{i}}\,\mathrm{Id} = \mathrm{e}^{-s}\left(\mu(z_0) - \frac{t}{2\mathrm{i}}\,\mathrm{Id}\right).$$

In fact, $z_s = g_s \cdot z_0$, where

$$\frac{\mathrm{d}g_s}{\mathrm{d}s} = \mathrm{i}a(z_s) \cdot g_s, \quad g_s \in \mathrm{GL}_n(\mathbb{C}).$$

The arguments of [**8**] ensure that the maximal solution of (4.10) is defined on $[0, +\infty[$ and if $g_s = \mathrm{e}^{\eta_s}\mathrm{e}^{h_s}$, where $\eta_s \in \mathfrak{u}_n$ and h_s is Hermitian, then $|h_s| \leqslant \delta$. We therefore also obtain

$$\|\dot{g}_s\| \leqslant \frac{4}{C^2}\left|\mu(z_0) - \frac{t}{2\mathrm{i}}\,\mathrm{Id}\right|\mathrm{e}^{-s}\mathrm{e}^{\delta},$$

and hence $g_\infty = \lim_{s\to+\infty} g_s$ exists and

$$\|g_\infty - \mathrm{Id}\,\| \leqslant \frac{4\mathrm{e}^{\delta}}{C^2}\left|\mu(z_0) - \frac{t}{2\mathrm{i}}\,\mathrm{Id}\right| = O\left(\frac{1}{|\boldsymbol{q}|^2}\right). \tag{4.11}$$

We clearly have $2\mathrm{i}\mu(g_\infty \cdot z_0) = t\,\mathrm{Id}$ and $h = \mathrm{i}k$ is given by $\mathrm{e}^{2h} = g_\infty^* g_\infty$, i.e. the polar decomposition of g_∞ is $g_\infty = \mathrm{e}^{\eta_\infty}\mathrm{e}^h$. Moreover, if $s \geqslant 0$, then the operator norm of $l_{z_s} Q_{z_z}^{-1}$ remains less than $2/C$ and hence

$$\|g_\infty \cdot z_0 - z_0\| \leqslant \frac{2}{C}\left|\mu(z_0) - \frac{t}{2\mathrm{i}}\,\mathrm{Id}\right| = O\left(\frac{1}{|\boldsymbol{q}|^2}\right). \tag{4.12}$$

The implicit function theorem tells us that h depends smoothly on z_0 and hence on $\zeta \in \mathcal{O}_{\mathfrak{p}}$; indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mu(\mathrm{e}^{tik} \cdot z) = Q_z(k).$$

The following lemma gives an estimate of the size of the differential of $h : \zeta \in \mathcal{O}_{\mathfrak{p}} \to h(\zeta) \in \mathrm{i}\mathfrak{u}_n$.

Lemma 4.2. *Let $\boldsymbol{v} \in T_\zeta \mathcal{O}_{\mathfrak{p}}$ is a vector of unit length. Then*

$$\|\mathrm{d}h(\zeta) \cdot \boldsymbol{v}\| = O\bigg(\frac{1}{|\boldsymbol{q}|^2}\bigg).$$

Proof. Let $\dot{z}_0 = \mathrm{d}z_0(\zeta_0) \cdot \boldsymbol{v}$ and $\dot{h} = \mathrm{d}h(\zeta) \cdot \boldsymbol{v}$. If

$$\boldsymbol{v} = (((\delta\lambda_1, \delta\mu_1), \dots, (\delta\lambda_k, \delta\mu_k)), (\delta A_1, \dots, \delta A_k), (\delta B_1, \dots, \delta B_k), (\delta x_1, \dots, \delta x_k)),$$

then we define $v := (\delta A, \delta B, \delta x, 0) \in \mathbb{M}$, where δA and δB are block diagonal matrices with respective main diagonal block

$$\delta A_1 + \delta\lambda_1 \,\mathrm{Id}_{n_1}, \dots, \delta A_k + \delta\lambda_k \,\mathrm{Id}_{n_k} \quad \text{and} \quad \delta B_1 + \delta\mu_1 \,\mathrm{Id}_{n_1}, \dots, \delta B_k + \delta\mu_k \,\mathrm{Id}_{n_k}$$

and $\delta x = (\delta x_1, \dots, \delta x_k)$.

We have

$$\mathrm{d}\mu(z_h) \cdot (D\exp(h)\dot{h} \cdot z_0 + \mathrm{e}^h \cdot \dot{z}_0) = 0.$$

Recall that

$$D\exp(h)\dot{h} = \frac{\mathrm{e}^{\mathrm{ad}\, h} - \mathrm{Id}}{\mathrm{ad}\, h} \cdot \dot{h} \cdot \mathrm{e}^h.$$

Let $\mathrm{i}\dot{\eta}$ be the Hermitian part of $D\exp(h)\dot{h}$ and let $\dot{\xi}$ be its skew Hermitian part. Then

$$\mathrm{d}\mu(z_h) \cdot (D\exp(h)\dot{h} \cdot z_0) = \mathrm{d}\mu(z_h)(\mathrm{i}l_{z_h}\dot{\eta}) + \mathrm{d}\mu(z_h)(l_{z_h}\dot{\xi}) = Q_{z_h}(\dot{\eta}),$$

and

$$Q_{z_h}(\dot{\eta}) + \mathrm{d}\mu(z_h) \cdot (\mathrm{e}^h \cdot \dot{z}_0) = 0.$$

Moreover, from the construction of z_0 and Lemma 4.1, we easily obtain that

$$\mathrm{d}\mu(z_0)(\dot{z}_0) = O\bigg(\frac{1}{|\boldsymbol{q}|^2}\bigg)$$

and

$$\dot{z}_0 = v + O\bigg(\frac{1}{|\boldsymbol{q}|^2}\bigg).$$

So if $k \in \mathrm{U}(n)$ is such that $g_\infty = k\mathrm{e}^h$, then

$$\begin{aligned}
&\mathrm{Ad}(k)\, \mathrm{d}\mu(z_h) \cdot (\mathrm{e}^h \cdot \dot{z}_0) \\
&\qquad = \mathrm{d}\mu(g_\infty \cdot z_0) \cdot (g_\infty \cdot \dot{z}_0) \\
&\qquad = \mathrm{d}\mu(g_\infty \cdot z_0) \cdot ((g_\infty - \mathrm{Id}) \cdot \dot{z}_0) + \mathrm{d}\mu(g_\infty \cdot z_0 - z_0) \cdot \dot{z}_0 + \mathrm{d}\mu(z_0)(\dot{z}_0).
\end{aligned}$$

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Hence

$$Q_{z_h}(\dot\eta) + \mathrm{d}\mu(z_h)\cdot(k^{-1}\cdot(g_\infty - \mathrm{Id})\cdot \dot z_0) = O\biggl(\frac{1}{|\boldsymbol{q}|^2}\biggr).$$

We now take the scalar product of this quantity with $\dot\eta$ and we obtain

$$\begin{aligned}\|l_{z_h}(\dot\eta)\|^2 &\leqslant O\biggl(\frac{1}{|\boldsymbol{q}|^2}\biggr)\|\dot\eta\| - \langle l_{z_h}(\dot\eta), k^{-1}\cdot I(g_\infty - \mathrm{Id})\cdot \dot z_0\rangle \\ &\leqslant O\biggl(\frac{1}{|\boldsymbol{q}|^2}\biggr)(\|\dot\eta\| + \|l_{z_h}(\dot\eta)\|).\end{aligned}$$

But our construction gives that

$$\|\dot\eta\| \leqslant \frac{2}{C}\|l_{z_h}(\dot\eta)\|;$$

hence we obtain

$$\|\dot\eta\| \leqslant \frac{2}{C}\|l_{z_h}(\dot\eta)\| = O\biggl(\frac{1}{|\boldsymbol{q}|^2}\biggr).$$

Now $\dot h$ is a Hermitian matrix and $\|h\| = O(|\boldsymbol{q}|^{-2})$, hence, from the definition of $\dot\eta$ and $\dot\xi$, we have

$$\dot h - \mathrm{i}\dot\eta = O(|\boldsymbol{q}|^{-2}).$$

The lemma is therefore proved. □

We note that it is straightforward to verify point (ii) at z_h because, by construction,

$$\forall \eta\in\mathfrak{u}_n,\quad \|l_{z_h}(\eta)\| \geqslant \tfrac{1}{2}C\|\eta\|.$$

We have built a map $f_{\mathfrak{p}}$ from $\mathcal{O}_{\mathfrak{p}}$ to $\mathbb{L}_t$ whose value at a point $\zeta = (\boldsymbol{q}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{x}) \in \mathcal{O}_{\mathfrak{p}}$ is the z_h constructed before. This map is $\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\cdots\times\mathrm{U}(n_k)$-equivariant and it therefore induces a map

$$\psi_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}}/(\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\cdots\times\mathrm{U}(n_k)) \to \mathrm{Hilb}_0^n(\mathbb{C}^2).$$

We remark that by adjusting ϵ, R and τ we obtain

$$\mathcal{C}_{\mathfrak{p}} \subset \mathcal{O}_{\mathfrak{p}}/(\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\cdots\times\mathrm{U}(n_k)) \subset \mathrm{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2)\times(\mathbb{C}^2)_0^k,$$

where the last inclusion is an isometry if $\mathrm{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2)\times(\mathbb{C}^2)_0^k$ is endowed with the product metric. We now want to compare the metric $\psi_{\mathfrak{p}}^* g_N$ and the product metric on $\mathrm{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2)\times(\mathbb{C}^2)_0^k$. Let $\boldsymbol{v}$ be a vector of $T_\zeta\mathcal{O}_{\mathfrak{p}}$ that is orthogonal to the $\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\cdots\times\mathrm{U}(n_k)$ orbit of ζ. We will again follow the notation used in the proof of Lemma 4.2 ($f(\zeta) = z_h = \mathrm{e}^h\cdot z_0$, $\dot h$, v, $\dots$).

Recall that we have denoted by $\varPi_q$ the orthogonal projection onto $\operatorname{Im} l_q$. We therefore need to compare

$$\|\boldsymbol{v}\|^2 \quad\text{and}\quad \|(\mathrm{Id} - \varPi_{z_h})\cdot \mathrm{d}f_{\mathfrak{p}}(\zeta)\cdot\boldsymbol{v}\|^2 = \|\mathrm{d}f_{\mathfrak{p}}(\zeta)\cdot\boldsymbol{v}\|^2 - \|\varPi_{z_h}\cdot\mathrm{d}f_{\mathfrak{p}}(\zeta)\cdot\boldsymbol{v}\|^2.$$

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But

$$\mathrm{d}f_{\mathfrak{p}}(\zeta)\cdot \boldsymbol{v} = l_{z_h}(\dot{\xi}) + \mathrm{i}l_{z_h}(\dot{\eta}) + \mathrm{e}^h\cdot \dot{z}_0.$$

Hence

$$(\mathrm{Id} - \Pi_{z_h})\cdot \mathrm{d}f_{\mathfrak{p}}(\zeta)\cdot v = (\mathrm{Id} - \Pi_{z_h})\cdot(\mathrm{i}l_{z_h}(\dot{\eta}) + \mathrm{e}^h\cdot \dot{z}_0).$$

But we have already seen that

$$\|l_{z_h}(\dot{\eta})\|^2 = O\biggl(\frac{\|v\|^2}{|\boldsymbol{q}|^4}\biggr).$$

But $\mathrm{i}l_{z_h}(\dot{\eta})$ is orthogonal to $T_{z_h}\mathbb{L}_t$ and hence to the range of Π_{z_h}, so we also have

$$\begin{aligned}\langle(\mathrm{Id} - \Pi_{z_h})(\mathrm{i}l_{z_h}(\dot{\eta})), (\mathrm{Id} - \Pi_{z_h})\cdot(\mathrm{e}^h\cdot \dot{z}_0)\rangle &= \langle(\mathrm{Id} - \Pi_{z_h})\cdot \mathrm{i}l_{z_h}(\dot{\eta}), (\mathrm{e}^h\cdot \dot{z}_0)\rangle\\ &= \langle il_{z_h}(\dot{\eta}), (\mathrm{e}^h\cdot \dot{z}_0)\rangle\\ &= -\langle\dot{\eta}, \mathrm{d}\mu(z_h)(\mathrm{e}^h\cdot \dot{z}_0)\rangle.\end{aligned}$$

But

$$\mathrm{d}\mu(z_h)((\mathrm{d}f_{\mathfrak{p}}(\zeta)\cdot \boldsymbol{v}) = 0 = \mathrm{d}\mu(z_h)(\mathrm{i}l_{z_h}(\dot{\eta}) + \mathrm{e}^h\cdot \dot{z}_0),$$

so

$$\begin{aligned}\langle(\mathrm{Id} - \Pi_{z_h})(\mathrm{i}l_{z_h}(\dot{\eta})), (\mathrm{Id} - \Pi_{z_h})\cdot(\mathrm{e}^h\cdot \dot{z}_0)\rangle &= -\langle\dot{\eta}, \mathrm{d}\mu(z_h)(\mathrm{e}^h\cdot \dot{z}_0)\rangle\\ &= \langle\dot{\eta}, \mathrm{d}\mu(z_h)(\mathrm{i}l_{z_h}(\dot{\eta}))\rangle\\ &= \|l_{z_h}(\dot{\eta})\|^2\\ &= O\biggl(\frac{\|v\|^2}{|\boldsymbol{q}|^4}\biggr).\end{aligned}$$

It now remains to estimate

$$\|(\mathrm{Id} - \Pi_{z_h})(\mathrm{e}^h\cdot \dot{z}_0)\|^2 = \|\mathrm{e}^h\cdot \dot{z}_0\|^2 - \|\Pi_{z_h}(\mathrm{e}^h\cdot \dot{z}_0)\|^2.$$

Recall that

$$\Pi_{z_h} = l_{z_h}Q_{z_h}^{-1}l_{z_h}^* \quad \text{and} \quad l_{z_h}^*(\mathrm{e}^h\cdot \dot{z}_0) = \mathrm{d}\mu(z_h)(\mathrm{i}\mathrm{e}^h\cdot \dot{z}_0).$$

We have already noticed that

$$I\cdot \dot{z}_0 = \mathrm{d}z_0(\zeta)\cdot(I\cdot \boldsymbol{v}) + w,$$

where $w = O(|\boldsymbol{q}|^{-2})\|v\|$, so we get

$$l_{z_h}^*(\mathrm{e}^h\cdot \dot{z}_0) = \mathrm{d}\mu(z_h)(\mathrm{e}^h\,\mathrm{d}z_0(\zeta)\cdot(I\boldsymbol{v})) + l_{z_h}^*(\mathrm{e}^h\cdot w)$$

and the proof of Lemma 4.2 furnishes a $w' = O(|\boldsymbol{q}|^{-2})\|v\|$ such that

$$\mathrm{d}\mu(z_h)(\mathrm{e}^h\,\mathrm{d}z_0(\zeta)\cdot(I\boldsymbol{v})) = \mathrm{d}\mu(z_h)(w') + O(|\boldsymbol{q}|^{-2})\|v\|.$$

As the operator norm of $l_{z_h} Q_{z_h}^{-1}$ is bounded by $2/C$, we have obtained

$$\|\Pi_{z_h}(\mathrm{e}^h \cdot \dot{z}_0)\|^2 = O\left(\frac{\|v\|^2}{|\boldsymbol{q}|^4}\right).$$

Hence we have obtained

$$\begin{aligned}\psi_{\mathfrak{p}}^* g_N(\boldsymbol{v}, \boldsymbol{v}) &= \|\mathrm{e}^h \cdot \dot{z}_0\|^2 + O\left(\frac{1}{|\boldsymbol{q}|^4}\right)\|v\|^2 \\ &= \|\dot{z}_0\|^2 + 2\langle \dot{z}_0, h\dot{z}_0\rangle + O\left(\frac{1}{|\boldsymbol{q}|^4}\right)\|v\|^2.\end{aligned}$$

By construction,

$$|\dot{z}_0|^2 = \|v\|^2 + O\left(\frac{1}{|\boldsymbol{q}|^4}\right)\|v\|^2.$$

And if $\boldsymbol{v} = (\delta q, \delta A_1, \delta A_2, \dots, \delta A_k, \delta B_1, \delta B_2, \dots, \delta B_k, 0)$ and if $h_{i,j}$ are the blocks of h (of size $n_i \times n_j$), then

$$\begin{aligned}\langle \dot{z}_0, h\dot{z}_0\rangle &= \langle v, hv\rangle + O\left(\frac{1}{|\boldsymbol{q}|^4}\right)\|v\|^2 \\ &= \sum_j \langle \delta A_j, [h_{j,j}, \delta A_j]\rangle + \langle \delta B_j, [h_{j,j}, \delta B_j]\rangle + O\left(\frac{1}{|\boldsymbol{q}|^4}\right)\|v\|^2 \\ &= \sum_j \langle [\delta A_j, \delta A_j^*] + [\delta B_j, \delta B_j^*], h_{j,j}\rangle + O\left(\frac{1}{|\boldsymbol{q}|^4}\right)\|v\|^2 \\ &= O\left(\frac{1}{\sigma^2|\boldsymbol{q}|^2}\right)\|v\|^2\end{aligned}$$

according to hypothesis (iii).

In order to finish the proof we need to check property (iii) at the point z_h. With what has been proved in the preceding paragraph, we only need to check that if $(\delta A, \delta B, 0, 0)$ is a unitary vector in the tangent space of $\mathbb{L}_t$ at z_h and orthogonal to the $\mathrm{U}(n)$ orbit of z_h, then $[\delta A_j, \delta A_j^*] + [\delta B_j, \delta B_j^*]$ is bounded. This is evident.

5. Conclusion

With the previous asymptotic description of Nakajima's metric, we will show that Nakajima's metric coincides with Joyce's one. One way to prove such a result would be to verify the estimates (3.1) for the orders $l \geqslant 1$; this is probably possible with some extra work but we will give a different proof here that follows the classical proof of the uniqueness of the solution of the Calabi–Yau problem. Moreover, our argument gives a new analytical result on the mapping property of the Laplace operator on QALE space. For new results that extended those of Joyce and go further than our result, there is the forthcoming work of Degeratu and Mazzeo [**7**].

We have already seen that Kronheimer's classification of hyperkähler ALE four-dimensional manifolds implies that Nakajima's metric is the Eguchi–Hansen metric on

$\mathrm{Hilb}_0^2(\mathbb{C}^2) \simeq T^*\mathbb{P}^1(\mathbb{C})$. We are going to prove our result by induction on n. Hence we now assume that up to a scaled factor, Joyce's and Nakajima's metrics coincide on $\mathrm{Hilb}_0^l(\mathbb{C}^2)$ for all integers $l < n$. We consider g to be the Joyce metric on $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ and ω to be the Kähler form associated with g (for the complex structure I), and for simplicity of the forthcoming notation we denote by g' Nakajima's metric on $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ with associated Kähler form ω'.

5.1. The (co)homology of $\mathrm{Hilb}_0^n(\mathbb{C}^2)$

The homology of $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ is well known; the odd Betti numbers of $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ are zero and the even ones can be computed explicitly (see [**10**] and [**25**, Corollary 5.10]).

Recall that $\nu = (\nu_1, \dots, \nu_k) \in \mathbb{N}^k$ is a partition of n if

$$\nu_1 \geqslant \nu_2 \geqslant \dots \geqslant \nu_k \quad \text{and} \quad \sum_{j=1}^{k} \nu_j = n.$$

The length of such a ν is

$$l(\nu) = k.$$

We then have that

$$b_{2k}(\mathrm{Hilb}_0^n(\mathbb{C}^2)) = \mathrm{Card}\{\nu,\ \nu \text{ a partition of } n,\ l(\nu) = 2n - 2k\}.$$

Moreover, there is a geometric description of a basis of $H_*(\mathrm{Hilb}_0^n(\mathbb{C}^2))$ (see [**25**, Example 8.18.2]). For instance, $H_2(\mathrm{Hilb}_0^n(\mathbb{C}^2))$ has dimension 1 and the corresponding partition of n is

$$(2, 1, \dots, 1).$$

We consider the top-dimensional stratum $S_0(n)$ of the singular set of $(\mathbb{C}^2)_0^n/S_n$; we have

$$S_0(n) = \hat{S}_0(n)/S_n,$$

where

$$q \in \hat{S}_0(n) \Leftrightarrow \mathrm{Card}\{\{i, j\},\ i < j,\ q_i = q_j\} = 1.$$

Then, for any $\bar{q} \in S_0(n)$, the homology class of $\pi^{-1}\{\bar{q}\} \simeq \mathbb{P}^1(\mathbb{C})$ generates $H_2(\mathrm{Hilb}_0^n(\mathbb{C}^2))$. Moreover, $S_0(n)$ being path connected, this homology class does not depend on the choice of $\bar{q} \in S_0(n)$: we will call it Σ_n.

5.2. Comparison of the two metrics

We can assume that, for any $\bar{q} \in S_0(n)$,

$$\int_{\pi^{-1}\{\bar{q}\}} \omega = \int_{\pi^{-1}\{\bar{q}\}} \omega'.$$

We will note that $R(n) = \hat{R}(n)/S_n$, which is the smooth part of $(\mathbb{C}^2)_0^n/S_n$.

Moreover, for each partition $\mathfrak{p}$ of $\{1, 2, \dots, n\}$, we have

$$\psi_{\mathfrak{p}}^* g = g_{\mathfrak{p}} + \mathrm{eucl} + O\biggl(\frac{1}{\sigma^2 \rho^2}\biggr)$$

and

$$\psi_{\mathfrak{p}}^* g' = g'_{\mathfrak{p}} + \mathrm{eucl} + O\biggl(\frac{1}{\sigma^2 \rho^2}\biggr)$$

on $\mathcal{C}_{\mathfrak{p}} \subset \mathrm{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2) \times V_{\mathfrak{p}}$, where $g_{\mathfrak{p}}$ (respectively $g'_{\mathfrak{p}}$) is the sum of the Joyce (respectively Nakajima) metric on $\mathrm{Hilb}_0^{\mathfrak{p}}(\mathbb{C}^2) \simeq \mathrm{Hilb}_0^{n_1}(\mathbb{C}^2) \times \mathrm{Hilb}_0^{n_2}(\mathbb{C}^2) \times \cdots \times \mathrm{Hilb}_0^{n_k}(\mathbb{C}^2)$. We identity $V_{\mathfrak{p}} \simeq (\mathbb{C}^2)_0^k / S_n$ and $W_{\mathfrak{p}} \simeq \bigoplus_{j=1}^k (\mathbb{C}^2)_0^{n_j}$.

We fix $j \in \{1, \dots, k\}$ and $(q_1, \dots, q_k) \in W_{\mathfrak{p}} \simeq \bigoplus_{j=1}^k (\mathbb{C}^2)_0^{n_j}$ such that if $i \neq j$, then $q_i \in \hat{R}(n_i)$, whereas $q_j \in \hat{S}_0(n_j)$ and we take $v \in \hat{R}(k) \subset V_{\mathfrak{p}} \simeq (\mathbb{C}^2)_0^k$. We also consider

$$C_v = \pi^{-1}(S_n \cdot (q, v)).$$

We have, by assumption,

$$\int_{C_v} \omega = \int_{C_v} \omega'.$$

Let g_j (respectively g'_j) be the Joyce (respectively Nakajima) metric on $\mathrm{Hilb}_0^{n_j}(\mathbb{C}^2)$ and let ω_j (respectively ω'_j) be its Kähler form; that is to say, $g_{\mathfrak{p}} = g_1 + g_2 + \cdots + g_k$ and $g'_{\mathfrak{p}} = g'_1 + g'_2 + \cdots + g'_k$. We have

$$\begin{aligned}\int_{C_v} \omega &= \int_{\Sigma_{n_j}} \omega_j + O\biggl(\frac{1}{\sigma^2 \rho^2}\biggr) \\ &= \int_{C_v} \omega' \\ &= \int_{\Sigma_{n_j}} \omega'_j + O\biggl(\frac{1}{\sigma^2 \rho^2}\biggr).\end{aligned}$$

When $\|v\|$ tends to ∞, we obtain

$$\int_{\Sigma_{n_j}} \omega'_j = \int_{\Sigma_{n_j}} \omega_j.$$

Our induction hypothesis yields that $g_j = g'_j$ for all j and, eventually, we have proven that

$$g - g' = O\biggl(\frac{1}{\sigma^2 \rho^2}\biggr).$$

5.3. Coincidence of the Joyce and Nakajima metrics

Following the classical proof of the uniqueness of the solution of the Calabi–Yau problem, we would like to find a function ϕ such that

$$\omega - \omega' = \mathrm{i}\partial\bar{\partial}\phi.$$

However, this is not easy because the weight $\sigma^{-2}\rho^{-2}$ is critical in Joyce's analysis of QALE manifolds. To circumvent this difficulty, we remark that both metrics g and g' have an $\mathbb{S}^1$ invariance property coming from the diagonal action of $\mathbb{S}^1$ on $(\mathbb{C}^2)^n_0/S_n$. For Joyce's metric it comes from the uniqueness result of the QALE Kähler–Einstein metric, which is asymptotic to $(\mathbb{C}^2)^n_0/S_n$. For Nakajima's metric, the action of $\mathbb{S}^1$ on $\mathbb{M}$ is as follows. If $\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{S}^1$ and if $z = (A, B, x, 0) \in \mathbb{L}_t$, then $\mathrm{e}^{\mathrm{i}\theta} \cdot z := (\mathrm{e}^{\mathrm{i}\theta}A, \mathrm{e}^{\mathrm{i}\theta}B, \mathrm{e}^{\mathrm{i}\theta}x, 0) \in \mathbb{L}_t$. And this action is isometric. This $\mathbb{S}^1$ action is holomorphic for the complex structure I but not for the complex structures J and K. Let X be the g or g' Killing field associated with the infinitesimal action of $\eta = \frac{1}{2}\mathrm{i}$. Then X has linear growth on $\mathrm{Hilb}^n_0(\mathbb{C}^2)$: that is to say, there is a constant c such that

$$X(z) \leqslant c(\rho(z) + 1).$$

Moreover, if ω_1 is the Kähler form of (g, J), ω_2 is the Kähler form of (g, K) and ω_1' and ω_2' are the corresponding 2-forms associated with the metric g', then

$$\omega_1 = \mathrm{d}(i_X\omega_2) \quad \text{and} \quad \omega_1' = \mathrm{d}(i_X\omega_2').$$

Hence if we let

$$\beta = i_X\omega_2 - i_X\omega_2',$$

then we have

$$\omega_1 - \omega_1' = \mathrm{d}\beta \quad \text{and} \quad \beta = O\left(\frac{1}{\sigma^2\rho}\right).$$

We now consider the Kähler manifold $(\mathrm{Hilb}^n_0(\mathbb{C}^2), g, J)$. The following analytical result is the key point of our proof.

Proposition 5.1. *There is a* $(0,1)$*-form* α *on* $\mathrm{Hilb}^n_0(\mathbb{C}^2)$ *such that*

$$\alpha = O\left(\frac{\log(\rho + 2)}{\rho}\right)$$

and

$$\beta^{0,1} = \Delta_{\bar\partial}\alpha = \bar\partial\bar\partial^*\alpha + \bar\partial^*\bar\partial\alpha.$$

First we explain why this proposition implies that $\omega_1 = \omega_1'$. This proposition will be proven in the next subsection.

The 1-form $\Phi = \bar\partial^*\bar\partial\alpha$ satisfies $\bar\partial\beta^{0,1} = 0 = \bar\partial\Phi$ and $\bar\partial^*\Phi = 0$. Moreover, the metric g has, by definition, bounded geometry, and we therefore have the following uniform local elliptic estimate:

$$\begin{aligned} \|\Phi\|_{L^2(B(x,1))} &= \|\bar\partial^*\bar\partial\alpha\|_{L^2(B(x,1))} \\ &\leqslant c\|\Delta_{\bar\partial}\alpha\|_{L^2(B(x,2))} + c'\|\alpha\|_{L^2(B(x,2))} \\ &\leqslant O\left(\frac{\log(\rho + 2)}{\rho}\right). \end{aligned}$$

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But because Φ is harmonic we also have a uniform estimate

$$|\Phi(x)| \leqslant c\|\Phi\|_{L^2(B(x,1))}.$$

Hence we obtain that

$$\Phi = O\biggl(\frac{\log(\rho+2)}{\rho}\biggr).$$

But the Ricci curvature of g is zero, so the Bochner formula and the Kato inequality imply that $|\Phi|$ is a subharmonic function and hence Φ is zero by the maximum principle. We also obtain $\beta^{0,1} = \bar{\partial}\bar{\partial}^*\alpha$, and the same argument shows that we can find a $(1,0)$-form $\tilde{\alpha}$ such that $\beta^{1,0} = \partial\partial^*\tilde{\alpha}$. Hence if we let

$$\mathrm{i}\phi = \bar{\partial}^*\alpha - \partial^*\tilde{\alpha},$$

then we have

$$\mathrm{d}\beta = \mathrm{i}\partial\bar{\partial}\phi.$$

Again the same argument as before, using the fact that g has bounded geometry, implies that

$$\phi = O\biggl(\frac{\log(\rho+2)}{\rho}\biggr).$$

Both ω_1 and ω_1' are Kähler–Einstein with zero scalar curvature so there is a pluriharmonic function f such that

$$\omega_1^m = \mathrm{e}^f(\omega_1')^m;$$

the function e^f is computed with the determinant of $(g'^{-1})g$, so we have

$$f = O\biggl(\frac{1}{\sigma^2\rho^2}\biggr).$$

By the maximum principle we deduce that $f = 0$. We finish the proof with a classical argument: the function ϕ is subharmonic for the metric g [**4**, Exposé VI, Lemma 1.6] and decays at infinity, hence by the maximum principle ϕ is negative; but, reversing the role of g and g', $-\phi$ is also subharmonic for the metric g' and $-\phi$ is positive and decays at infinity; hence ϕ is zero.

5.4. Proof of the analytical result

We first remark that because $(\mathrm{Hilb}_0^n(\mathbb{C}^2), g)$ is asymptotic to the Euclidean metric on $(\mathbb{C}^2)_0^n/S_n$, we have

$$\lim_{r\to\infty} \frac{\mathrm{vol}\, B(x,r)}{r^d} = \frac{w_d}{n!},$$

where $d = 4(n-1)$ is the real dimension of $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ and w_d is the volume of the unit ball in $\mathbb{R}^d$. The Bishop–Gromov inequality tells us that for any point $x \in \mathrm{Hilb}_0^n(\mathbb{C}^2)$

$$\frac{w_d r^d}{n!} \leqslant \mathrm{vol}\, B(x,r) \leqslant w_d r^d.$$

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The result of Li and Yau [**21**] implies that the Green kernel G of the metric g (that is to say, the Schwartz kernel of the operator Δ^{-1}) satisfies

$$G(x,y) \leqslant \frac{c}{d(x,y)^{d-2}}.$$

Moreover, because g is Ricci flat, the Hodge–de Rham operator acting on 1-forms is the rough Laplacian

$$\forall v \in C_0^\infty(T^*\mathrm{Hilb}_0^n(\mathbb{C}^2)), \quad \Delta v = \mathrm{d}\mathrm{d}^* v + \mathrm{d}^*\mathrm{d} v = \nabla^*\nabla v.$$

Hence the Kato inequality implies that if $\boldsymbol{G}(x,y)$ is the Schwartz kernel of the operator Δ^{-1}, then it satisfies

$$|\vec{G}(x,y)| \leqslant G(x,y) \leqslant \frac{c}{d(x,y)^{d-2}}.$$

Proposition 5.1 is a consequence of the following lemma.

Lemma 5.2. *If $f \in L^\infty_{\mathrm{loc}}(\mathrm{Hilb}_0^n(\mathbb{C}^2))$ is a non-negative function that satisfies*

$$f = O\biggl(\frac{1}{\sigma^2\rho}\biggr),$$

then

$$u(x) = \int_{\mathrm{Hilb}_0^n(\mathbb{C}^2)} \frac{f(y)}{d(x,y)^{d-2}}\,\mathrm{d}y$$

is well defined and satisfies

$$u = O\biggl(\frac{\log(\rho+2)}{\rho}\biggr).$$

Proof. Let $o \in \mathrm{Hilb}_0^n(\mathbb{C}^2)$ be a fixed point. We can assume that $\rho(x) = d(o,x)$. We remark that u is well defined because there is a constant c such that, for $R > 1$,

$$\int_{B(o,R)} f \leqslant cR^{d-3}.$$

As a matter of fact, the function $1/\sigma^2\rho$ is asymptotic to a homogeneous function h of degree -3 on $(\mathbb{C}^2)_0^n/S_n$: $h(r\theta) = r^{-3}\bar{h}(\theta)$, where $\bar{h}$ is a positive function on $\mathbb{S}^{2d-1}/S_n$; this function $\bar{h}$ is singular on the singular locus of $\mathbb{S}^{2d-1}/S_n$. If we call this singular locus Σ, then $\bar{h}$ behaves like $d(\cdot\,,\Sigma)^{-2}$ near Σ but the real codimension of Σ is 4 so $\bar{h}$ is integrable on $\mathbb{S}^{2d-1}/S_n$ and we have

$$\lim_{R\to\infty} R^{3-d}\int_{B(o,R)} \frac{1}{\rho\sigma^2} = \frac{1}{d-3}\int_{\mathbb{S}^{2d-1}/S_n} \bar{h}.$$

In order to finish our estimate, we must find a constant c such that if $\rho(x) \geqslant 10$ then

$$F(x) = \int_{\mathrm{Hilb}_0^n(\mathbb{C}^2)} \frac{1}{d(x,y)^{d-2}}\frac{1}{\rho(y)\sigma(y)^2}\,\mathrm{d}y \leqslant c\frac{\log\rho(x)}{\rho(x)}.$$

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We decompose

$$\mathrm{Hilb}_0^n(\mathbb{C}^2) = (B(o,2\rho(x))\setminus B(x,\rho(x)/2))\cup B(x,\rho(x)/2)\cup(\mathrm{Hilb}_0^n(\mathbb{C}^2)\setminus B(o,2\rho(x))). \quad (5.1)$$

Then we have $F = F_1 + F_2 + F_3$, where F_i is the integral of $d(x,y)^{2-d}\rho^{-1}\sigma^{-2}$ on the ith region of the decomposition (5.1). The first and last integrals are easy to estimate:

$$F_1(x) \leqslant \left(\frac{2}{\rho(x)}\right)^{d-2} \int_{B(o,2\rho(x))} \frac{1}{\rho\sigma^2}\,\mathrm{d}y \leqslant C\frac{1}{\rho(x)}.$$

Concerning F_3 we have

$$\begin{aligned} F_3(x) &= \int_{\mathrm{Hilb}_0^n(\mathbb{C}^2)\setminus B(o,2\rho(x))} \frac{1}{d(x,y)^{d-2}}\frac{1}{\rho(y)\sigma(y)^2}\,\mathrm{d}y \\ &\leqslant \int_{\mathrm{Hilb}_0^n(\mathbb{C}^2)\setminus B(o,2\rho(x))} \frac{2^{d-2}}{\rho(y)^{d-1}}\frac{1}{\sigma(y)^2}\,\mathrm{d}y \\ &\leqslant \sum_{k=1}^{\infty}\int_{B(o,2^{k+1}\rho(x))\setminus B(o,2^k\rho(x))} \frac{2^{d-2}}{\rho(y)^{d-1}}\frac{1}{\sigma(y)^2}\,\mathrm{d}y \\ &\leqslant \sum_{k=1}^{\infty}\frac{1}{(2^k\rho(x))^{d-2}}\int_{B(o,2^{k+1}\rho(x))} \frac{1}{\rho(y)\sigma(y)^2}\,\mathrm{d}y \\ &\leqslant C\sum_{k=1}^{\infty}\frac{1}{(2^k\rho(x))^{d-2}}(2^{k+1}\rho(x))^{d-3} \\ &\leqslant C'\frac{1}{\rho(x)}. \end{aligned}$$

It remains to estimate F_2. We have

$$F_2(x) \leqslant \frac{2}{\rho(x)}\int_{B(x,\rho(x)/2)} \frac{1}{d(x,y)^{d-2}}\frac{1}{\sigma(y)^2}\,\mathrm{d}y.$$

Let

$$V(\tau) = \int_{B(x,\tau)} \frac{1}{\sigma(y)^2}\,\mathrm{d}y$$

and note $\mathrm{d}V$, the Riemann–Stieljes measure associated with the increasing function V. We have

$$\begin{aligned} \int_{B(x,\rho(x)/2)} \frac{1}{d(x,y)^{d-2}}\frac{1}{\sigma(y)^2}\,\mathrm{d}y &= \int_0^{\rho(x)/2}\frac{1}{\tau^{d-2}}\,\mathrm{d}V(\tau) \\ &= \frac{V(\rho(x)/2)}{(\rho(x)/2)^{d-2}} + (d-2)\int_0^{\rho(x)/2}\frac{V(\tau)}{\tau^{d-1}}\,\mathrm{d}\tau. \end{aligned} \quad (5.2)$$

We will estimate V: if we let S be the pullback to $\mathrm{Hilb}_0^n(\mathbb{C}^2)$ of the singular locus of $(\mathbb{C}^2)_0^n/S_n$ and $\mathcal{O} = \{y \in \mathrm{Hilb}_0^n(\mathbb{C}^2),\ \text{such that } \sigma(y) \leqslant 2\}$, then we have $V(\tau) = V_1(\tau) + V_2(\tau)$, where V_1 is the integral over $B(x,\tau)\cap\mathcal{O}$ and V_2 is the integral over $B(x,\tau)\setminus\mathcal{O}$.

V_1 is easy to estimate because on this region σ^{-2} is bounded and hence

$$V_1(\tau) \leqslant C \operatorname{vol}(B(x,\tau) \cap \mathcal{O}) \leqslant C \min\{\tau^d, \tau^{d-4}\}. \tag{5.3}$$

Outside $\mathcal{O}$ the metric is quasi-isometric to the Euclidean metric and we can estimate V_2 by a similar integral on $(\mathbb{C}^2)^n_0/S_n$. Let

$$D = \{q \in (\mathbb{C}^2)^n_0,\ \text{such that } \forall i \neq j,\ |q_i - q_j| \geqslant |q_1 - q_2|\}$$

and let $D' = \{q \in D,\ |q_1 - q_2| \geqslant 1\}$. D is a fundamental domain for the action of S_n on $(\mathbb{C}^2)^n_0$ and if $\bar{x} \in D$ is such that $S_n \cdot \bar{x} = \pi(x)$, then

$$V_2(\tau) \leqslant \frac{C}{n!} \sum_{\gamma \in S_n} \int_{D' \cap B(\gamma\bar{x},\tau)} \frac{1}{|q_1 - q_2|^2}\, \mathrm{d}q.$$

We give three different estimates for V_2 according to the relative size of $\sigma(x)$ and τ:

(1) if $\sigma(x) \leqslant \frac{3}{2}$, then for $\tau \in [0, \frac{1}{2}]$ we have

$$V_2(\tau) = 0;$$

(2) if $\sigma(x) \geqslant \frac{3}{2}$, then for $\tau \leqslant \sigma(x)/2$ we have

$$V_2(\tau) \leqslant \frac{C}{\sigma(x)^2}\tau^d;$$

and finally

(3) if $\tau \geqslant \frac{1}{2}\sigma(x)$, then there is a point $z \in S$ such that $d(x,z) = \sigma(x) - 1$, and if $\bar{z} \in D$ such that $S_n\bar{z} = \pi(z)$, then

$$\int_{D' \cap B(\gamma\bar{x},\tau)} \frac{1}{|q_1 - q_2|^2}\, \mathrm{d}q \leqslant \int_{D' \cap B(\gamma\bar{z},3\tau)} \frac{1}{|q_1 - q_2|^2}\, \mathrm{d}q \leqslant C\tau^{d-2}.$$

Now, with the estimate (5.3), it is easy to show that in (5.2) the part coming from V_1 is bounded; concerning the part coming from V_2, when $\sigma(x) \leqslant \frac{3}{2}$ we get

$$\int_0^{\rho(x)/2} \frac{1}{\tau^{d-2}}\, \mathrm{d}V_2(\tau) \leqslant C + (d-2) \int_{1/2}^{\rho(x)/2} \frac{C\tau^{d-2}}{\tau^{d-1}}\, \mathrm{d}\tau = C' + C \log \rho(x),$$

and when $\sigma(x) \geqslant \frac{3}{2}$, we obtain

$$\begin{aligned}\int_0^{\rho(x)/2} \frac{1}{\tau^{d-2}}\, \mathrm{d}V_2(\tau) &\leqslant C + (d-2) \int_0^{\sigma(x)/2} \frac{C\tau^d}{\tau^{d-1}\sigma(x)^2}\, \mathrm{d}\tau + (d-2) \int_{\sigma(x)/2}^{\rho(x)/2} \frac{C\tau^{d-2}}{\tau^{d-1}}\, \mathrm{d}\tau \\ &= C' + C \log\left(\frac{\rho(x)}{\sigma(x)}\right).\end{aligned}$$

And from this we obtain the result. □

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References

1. M. Anderson, L^2 harmonic forms on complete Riemannian manifolds, in *Geometry and Analysis on Manifolds, Katata/Kyoto, 1987*, Lecture Notes in Mathematics, No. 1339, pp. 1–19 (Springer, 1988).
2. S. Bando and R. Kobayashi, Ricci-flat Kähler metrics on affine algebraic manifolds, II, *Math. Annalen* **287**(1) (1990), 175–180.
3. A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, *J. Diff. Geom.* **18**(4) (1983), 755–782.
4. J.-P. Bourguignon *et al.*, *Premiére classe de Chern et courbure de Ricci: preuve de la conjecture de Calabi*, Astérisque, Volume 58 (Société Mathématique de France, Paris, 1978).
5. G. Carron, Cohomologie L^2 des variétés QALE, *J. Reine Angew. Math.*, in press.
6. G. Carron, L^2 harmonics forms on non-compact manifolds, preprint (arXiv:math.DG/0704.3194; 2007).
7. A. Degeratu and R. Mazzeo, Fredholm results on QALE manifolds, in preparation (see *Oberwolfach Rep.* **4**(3) (2007), 2398–2400).
8. S. K. Donaldson, Scalar curvature and projective embeddings, I, *J. Diff. Geom.* **59** (2001), 479–522.
9. M. Gromov, Kähler hyperbolicity and L_2-Hodge theory, *J. Diff. Geom.* **33** (1991), 263–292.
10. G. Ellingsrud and S. A. Strømme, On a cell decomposition of the Hilbert scheme of points in the plane, *Invent. Math.* **91** (1988), 365–370.
11. T. Hausel, S-duality in hyperkähler Hodge theory, in *The many facets of geometry: a tribute to Nigel Hitchin* (ed. O. Garcia-Prada, J. P. Bourguignon and S. Salamon), in press (Oxford University Press).
12. T. Hausel, E. Hunsicker and R. Mazzeo, Hodge cohomology of gravitational instantons, *Duke Math. J.* **122**(3) (2004), 485–548.
13. N. Hitchin, L^2-cohomology of hyperkähler quotient, *Commun. Math. Phys.* **211** (2000), 153–165.
14. N. Hitchin, A. Karlhede, U. Lindström and M. Rocek, Hyperkähler metrics and supersymmetry, *Commun. Math. Phys.* **108** (1987), 535–589.
15. J. Jost and K. Zuo, Vanishing theorems for L^2-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry, *Commun. Analysis Geom.* **8**(1) (2000), 1–30.
16. D. Joyce, Quasi-ALE metrics with holonomy SU(m) and Sp(m), *Annals Global Analysis Geom.* **19**(2) (2001), 103–132.
17. D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs (Oxford University Press, 2000).
18. P. Kronheimer, The construction of ALE spaces as hyper-Kähler quotients, *J. Diff. Geom.* **29** (1989), 665–683.

19. P. Kronheimer, A Torelli-type theorem for gravitational instantons, *J. Diff. Geom.* **29** (1989) 685–697.
20. C. LeBrun, Complete Ricci-flat Kähler metrics on $\mathbb{C}^n$ need not be flat, *Proc. Symp. Pure Math.* **52**(2) (1991), 297–304.
21. P. Li and S.-T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156**(3–4) (1986), 153–201.
22. J. McNeal, L^2 harmonic forms on some complete Kähler manifolds, *Math. Annalen* **323**(2) (2002), 319–349.
23. R. Mazzeo, Resolution blowups, spectral convergence and quasiasymptotically conic spaces, *Actes du Colloque EDP, Évian 2006*.
24. H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras, *Duke Math. J* **76**(2) (1994), 365–416.
25. H. Nakajima, *Lectures on Hilbert schemes of points on surfaces* (American Mathematical Society, Providence, RI, 1999).
26. N. Nekrasov and A. Schwarz, Instantons on non-commutative $\mathbb{R}^4$, and $(2,0)$ superconformal six-dimensional theory, *Commun. Math. Phys.* **198**(3) (1998), 689–703.
27. G. Tian and S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature, I, *J. Am. Math. Soc.* **3**(3) (1990), 579–609.
28. G. Tian and S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature, II, *Invent. Math.* **106**(1) (1991), 27–60.
29. C. Vafa and E. Witten, A strong coupling test of S-duality, *Nuclear Phys.* B **431**(1–2) (1994), 3–77.
30. S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I, *Commun. Pure Appl. Math.* **31**(3) (1978), 339–411.