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Complete Disorder is Impossible: The Mathematical Work of Walter Deuber

Complete disorder is impossible – this theme of Ramsey Theory, as stated by Theodore S. Motzkin, was a guiding theme throughout Walter Deuber's scientific life.

1. Ramsey's theorem and Deuber's diploma thesis

In 1928 the young British mathematician Frank Plumpton Ramsey had written a paper 'On a problem in formal logic', which was published in 1930 in the *Proceedings of the London Mathematical Society*. It is the paper for which he became eponymous for the field of Discrete Mathematics nowadays known as *Ramsey Theory*. The objective was to give a decision procedure for the sentences of propositional logic. The need for such procedures – in present day terminology we would call them algorithms – arose with the crisis of the foundations of mathematics around 1900. It is somewhat ironic that a purely mathematical result (which is now called Ramsey's theorem) from Ramsey's paper has proved to be of greater consequence than the metamathematical investigations for which it was made a tool.

Theorem (Ramsey). *Let G be an infinite set and let k and r be positive integers. Then, for every colouring of the k-element subsets of G with r colours, there exists an infinite subset F of G which is monochromatic.*

An easy compactness argument now shows that in case G is finite, say $|G| = m$, then there exists a smallest positive integer $n =: R(m,k,r)$ such that every n-element set F fulfils the requirement of Ramsey's theorem with respect to G, k, and r. Determining the exact values for $R(m,k,r)$ is a very difficult problem in combinatorics and even the so-called graphical case, *i.e.*, determining $R(m,2,2)$, is wide open. In 1966, when Walter wrote his thesis [7], only the trivial cases $R(3,2,2) = 6$ and $R(4,2,2) = 18$ were known. He re-proved these results and studied some other small cases but could not make any real progress. This turns out to be not such a surprise: the number $R(5,2,2)$ is still not known and the best bounds are $43 \leqslant R(5,2,2) \leqslant 49$ (see [39]). Paul Erdős has given us a metaphorical impression of how difficult determining these Ramsey numbers might be.

He claims:

Suppose an evil spirit would tell us, ‘Unless you tell me the value of $R(5,2,2)$ I will exterminate the human race.’ Our best strategy would perhaps be to get all the computers and computer scientists to work on it. If he would ask for $R(6,2,2)$ our best bet would perhaps be to try to destroy him before he destroys us.

2. Issai Schur and Richard Rado

Quite a while before Ramsey proved his partition theorem for finite sets, some results were established which form an even earlier part of Ramsey Theory than Ramsey's theorem itself. One of these results is due to Issai Schur [42].

Theorem (Schur). *Let r be a positive integer. Then there exists a least positive integer $n = S(r)$ such that for every colouring, say Δ, of $[1,n]$, the first n positive integers, with r colours, there exist positive integers $x, y \in [1,n]$ such that $\Delta(x) = \Delta(y) = \Delta(x+y)$.*

Issai Schur (1875–1941) was a student of Frobenius and profoundly influenced by him. With the exception of three years in Bonn (1913–1916) as the successor of Felix Hausdorff, Schur spent his entire academic life at the University of Berlin. In September 1935 Schur was dismissed by the Nazis. At this time, Schur was the last Jewish professor to lose his job at the University of Berlin. Four years later, he emigrated to Palestine, where he died in 1941.

Schur was a mathematician with widely spread interests, most famous for his results in the representation theory of groups. He studied the above question on partitions in order to give a new and elegant proof of a modular version of Fermat's conjecture: the fact that the congruence $x^m + y^m \equiv z^m (\mathrm{mod}\ p)$ has a solution for every m and for all sufficiently large primes p follows immediately from Schur's theorem.

The distribution of quadratic residues and non-residues is an old problem in number theory. Schur also worked on this problem. He conjectured that for every positive integer k and every sufficiently large prime p there exist k consecutive integers which are quadratic residues and there exist k consecutive integers which are quadratic non-residues (modulo p). Schur first tried to show that for every k there exists an n such that, for every colouring of $[1,n]$ with two colours, one of the two colour classes contains an arithmetic progression of length k. Schur did not succeed in any of these questions and both remained open for several years (compare Brauer [6]).

Very likely, the conjecture on arithmetic progressions was transmitted to Göttingen by the gossip of the Dutch student Baudet. This was how Bartel Leendert van der Waerden and other mathematicians in Göttingen heard about the problem. See Alexander Soifer's paper ‘The Baudet–Schur conjecture on monochromatic arithmetic progressions: An historical investigation’ [43] for some historical background. After some discussions with Artin and Schreier, van der Waerden resolved the problem. The answer to this conjecture is entitled ‘Beweis einer Baudetschen Vermutung’ [45].

Theorem (van der Waerden). *Let k and r be positive integers. Then there exists a positive integer $n = W(k, r)$ such that for every colouring of $[1, n]$ with r colours there exists a monochromatic k-term arithmetic progression $\{a + id \mid i < k\}$.*

According to Brauer [6], 'a few days' after van der Waerden answered Schur's question on arithmetic progressions, he himself (a doctoral student of Schur at this time) was then able to use van der Waerden's result to solve the conjecture of Schur on quadratic residues and non-residues. Brauer's paper [5] contains a common generalization of van der Waerden's theorem and of Schur's theorem which Brauer attributes to Schur.

Richard Rado (1906–1989) became one of the most famous doctoral students of Schur. His thesis, entitled 'Studien zur Kombinatorik' [37], earned him a doctorate at the University of Berlin in 1933. The second examiner of Rado's thesis was Erhard Schmidt. Both of them graded the thesis (in November 1931) with 'valde laudabile'. Rado's family was Jewish, so when the Nazis came to power he emigrated to England in August 1933 with the support of Schur. There he entered the University of Cambridge and in 1935 completed a second PhD under Hardy's supervision on 'Linear transformations of sequences'.

In his 'Studien zur Kombinatorik', Rado investigates partition regular systems of equations, in particular, of linear equations. Let $A \cdot x = 0$ be a system of linear equations with integral coefficients. This system is called *partition regular* (in $\mathbb{N}$) if for every colouring of $\mathbb{N}$ with finitely many colours there exists a monochromatic solution of $A \cdot x = 0$. The single equation $x_0 + x_1 - x_2 = 0$ is partition regular, which is nothing else than a restatement of Schur's theorem. The system

$$\begin{aligned} x_0 + x_1 - x_2 &= 0 \\ x_0 + x_2 - x_3 &= 0 \\ &\vdots \\ x_0 + x_{k-2} - x_{k-1} &= 0 \end{aligned}$$

is partition regular, which follows from van der Waerden's theorem on arithmetic progressions. In fact, a monochromatic solution of the system above is a monochromatic arithmetic progression of length k with the additional property that the difference has the same colour, too. This is just the common generalization of van der Waerden's theorem and of Schur's theorem mentioned by Brauer. On the other hand, the system $x_0 - 2x_1 = 0$ is not partition regular. Just consider the 2-colouring of the natural numbers n by the parity of the number of twos in the prime factorization of n, *i.e.*, colour n red if $\max\{i \mid 2^i \text{ divides } n\}$ is odd and blue if it is even. One of the main problems Rado addressed in his thesis was to characterize those systems of linear equations which are partition regular.

The crucial notion in his characterization of linear partition regular systems $A \cdot x = 0$ is the notion of the *columns property* of the matrix A.

Definition (columns property). Let $A = (a^1, \ldots, a^n)$ be a matrix with columns $a^j \in \mathbb{Z}^m$. Then A has the columns property if there exists a partition $[1, n] = I_0 \cup \cdots \cup I_l$

such that

(1) $\sum_{i \in I_0} a^i = 0$, and

(2) for every $j < l$,

$\sum_{i \in I_{j+1}} a^i$ is a rational linear combination of the columns in $I_0 \cup \cdots \cup I_j$.

Clearly, the Schur matrix $(1, 1, -1)$ has the columns property and also the matrix arising from van der Waerden's system of linear equations is easily seen to have the columns property (just view all columns besides the first one as belonging to I_0). On the other hand, the matrix $(1, -2)$ obviously does not have the columns property. In his thesis, Rado [37] shows that the columns property of a matrix uniquely describes the partition regular system of equations.

Theorem (Rado). *Let A be a matrix with integral coefficients. The linear system $A \cdot x = 0$ is partition regular (in $\mathbb{N}$) if and only if the matrix A has the columns property.*

Let us go one step further and, following Rado, call a set $S \subseteq \mathbb{N}$ *partition regular* if every partition regular system of linear equations can be solved in S. Note that for matrices A and B having the columns property, their direct sum

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

also has the columns property (and hence is also partition regular). This leads to the following corollary, which was observed by Rado.

Corollary. *For every colouring of the positive integers $\mathbb{N}$ with finitely many colours, one of the colour classes is partition regular.*

Rado conjectured an analogous result, if one colours an arbitrary partition regular set S, but was unable to prove his conjecture. This conjecture was the starting point of Deuber's doctoral dissertation.

3. Walter Deuber's doctoral dissertation

The main result of Walter Deuber's doctoral dissertation is a new characterization of partition regular linear systems $A \cdot x = 0$. The nature of this characterization is somewhat different from Rado's columns property. Deuber's approach is to describe the arithmetic structure of the sets of solutions of partition regular linear systems $A \cdot x = 0$. The crucial notion is the (m, p, c)-set.

Definition ((m, p, c)-set). Let m, p, c be positive integers. A set $D \subseteq \mathbb{N}$ is an (m, p, c)-set if there exist positive integers $d_0, \ldots, d_m$ such that $D = D_{p,c}(d_0, \ldots, d_m)$ consists of all numbers

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in the following list:

$$\begin{array}{r} cd_0 + l_1 d_1 + l_2 d_2 + \cdots + l_m d_m \\ cd_1 + l_2 d_2 + \cdots + l_m d_m \\ cd_2 + \cdots + l_m d_m \\ \vdots \\ cd_m, \end{array}$$

where l_i is an integer in $[-p, p]$. In other words, $D = D_{p,c}(d_0, \ldots, d_m)$ is the following set of positive integers:

$$D = D_{p,c}(d_0, \ldots, d_m) = \left\{ cd_i + \sum_{j=i+1}^{m} l_j d_j \mid i \leqslant m, l_j \in [-p, p] \right\}.$$

Hence, a $(1, 1, 1)$-set, *i.e.*, $\{d_0 - d_1, d_0, d_0 + d_1\} \cup \{d_1\}$ contains a solution of a Schur system and a $(1, p, 1)$-set is an arithmetic progression of length $2p + 1$ together with its difference. The somewhat technical definition of (m, p, c)-sets proves its value in Deuber's [8] characterization of partition regular linear systems. The two steps to obtain this characterization are the following lemmas.

Lemma. *Let m, p and c be positive integers. Then there exists a partition regular system with rational coefficients such that each of its solutions in the positive integers contains an (m, p, c)-set.*

Lemma. *Let $A \cdot x = 0$ be a partition regular system of linear equations with rational coefficients. Then there exist positive integers m, p, c such that every (m, p, c)-set contains a solution of $A \cdot x = 0$.*

Theorem (Deuber). *A linear system $A \cdot x = 0$ with rational coefficient is partition regular in $\mathbb{N}$ if and only if there exist positive integers m, p, c such that every (m, p, c)-set contains a solution of $A \cdot x = 0$.*

Proof. The 'if' part of the theorem is given by the second lemma. In order to prove the 'only if' part, assume that m, p, c are such that every (m, p, c)-set contains a solution of $A \cdot x = 0$. Then by the first lemma there exists a partition regular system with rational coefficients, say $B \cdot y = 0$, such that each of its solutions in the positive integers contains an (m, p, c)-set. If we now assume that $A \cdot x = 0$ is not partition regular, it follows that $B \cdot y = 0$ is not partition regular, a contradiction. □

This result moves (m, p, c)-sets into a central position within the partition theory for arithmetic structures. But the original motivation of Deuber for studying (m, p, c)-sets was to resolve Rado's conjecture. For this, he proved the following partition theorem for (m, p, c)-sets.

Theorem. *Let m, p, c and r be positive integers. Then there exist positive integers n, q and d such that for every (n, q, d)-set $D \subseteq \mathbb{N}$ and every r-colouring $\Delta : D \longrightarrow [1, r]$ there exists a monochromatic (m, p, c)-set $D' \subseteq D$.*

Deuber's original proof of this theorem relies on van der Waerden's theorem on arithmetic progressions. Later on, Klaus Leeb [28] observed a more elegant proof using the Hales–Jewett theorem on partitioning parameter words. Using the partition theorem for (m, p, c)-sets, Deuber easily derived the proof of Rado's conjecture.

Theorem (Deuber). *For every colouring of a partition regular set with finitely many colours one of the colour classes again is partition regular.*

Proof. Let $S \subseteq \mathbb{N}$ be a partition regular set, *i.e.*, every partition regular system of linear equations can be solved in S. Since, by the first lemma, for every n, q and d there exists a partition regular system of equations such that each of its solutions in the positive integers contains an (n, q, d)-set, S contains an (n, q, d)-set for every choice of parameters n, q and d.

Let r be a positive integer and let an arbitrary colouring $\Delta : S \longrightarrow [1, r]$ be given. Then, by the partition theorem for (m, p, c)-sets, for every m, p and c at least one colour class contains an (m, p, c)-set. Assume that none of the colour classes is partition regular. Then, by the second lemma, for every $i \in [1, r]$ there exist positive integers m_i, p_i, c_i such that the ith colour class does not contain an (m_i, p_i, c_i)-set. Choose

$$n_0 = \max_i m_i, \qquad q_0 = \left(\prod_{i=1}^r c_i\right) \cdot \max p_i, \qquad d_0 = \prod_{i=1}^r c_i.$$

Then, an elementary calculation shows that each (n_0, q_0, d_0)-set contains an (m_i, p_i, c_i)-set for every $i \in [1, r]$. But then one of the colour classes (the jth, say) must contain an (n_0, q_0, d_0)-set and thus an (m_j, p_j, c_j)-set, which is a contradiction. $\square$

4. Developments based on Rado's dissertation

Partition regular systems of equations and (m, p, c)-sets became a constant companion throughout Walter Deuber's entire mathematical life.

It is an easy observation that a system $A \cdot x = 0$ with integral coefficients is partition regular in $\mathbb{N}$ if and only if it is partition regular in $\mathbb{Z}$ and this is the case if and only if it is partition regular in $\mathbb{Q}$. Rado proved, see [38], an extension of his original results to the fields of reals and complex numbers. Soon after having finished his doctoral thesis, Walter Deuber also started studying systems of linear equations in more general domains and established (one year after he obtained his PhD) a q-analogue of partition regular systems of linear equations for Abelian groups.

Let G be an Abelian group, considered as a $\mathbb{Z}$-module, and let A be a matrix with integral coefficients. He called A *partition regular over* G if, for every colouring of $G \setminus \{0\}$ with finitely many colours, at least one class contains a solution of the homogeneous system $A \cdot x = 0$. Let p be a prime. Then a matrix A is said to have the *p-columns property*

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if it has the columns property with all linear combinations taken modulo p. In 1975, Deuber [9] published the following q-analogue to Rado's theorem.

Theorem (Deuber). *Let G be an Abelian group and A be a finite matrix with integral coefficients. The linear system $A \cdot x = 0$ is partition regular in G if and only if one of the following conditions is true.*

(1) *$A \cdot x = 0$ has a solution in $G \backslash \{0\}$ with $x_1 = x_2 = \cdots = x_n$.*
(2) *For some prime p, the group G contains the infinite direct sum of the cyclic groups $\mathbb{Z}_p$, and A satisfies the p-columns property.*
(3) *G contains elements of arbitrarily high order or an element of infinite order, and A satisfies the columns property.*

Later on, in the paper 'Rado's theorem for finite fields' [3], with Vitaly Bergelson and Neil Hindman, he established a finitist version of Rado's theorem for reals, and in the paper 'Rado's theorem for commutative rings' [4] these authors together with Hanno Lefmann prove that an analogue of Rado's columns condition is sufficient for the partition regularity of a homogeneous system of equations in any commutative ring. Moreover, they show that for a wide class of commutative rings (the so-called Rado rings), this condition is also necessary.

Thanks to the work of Rado and 'on his shoulders' of Deuber (to cite Ron Graham, Bruce Rothschild and Joel Spencer [21]) the situation of finite systems of linear equations is quite well understood – though there are still a few outstanding open questions. Deuber gave in [13] an excellent account on the 'Developments based on Rado's dissertation "Studien zur Kombinatorik"' up to the end of the 1980s.

But beyond finite systems of linear equations, there are questions on partition regularity in which Deuber was very interested throughout his scientific life and which he frequently addressed. One is about a characterization of non-linear partition regular systems of equations. Here, still not much is known. Some partial results can be found in [30].

A second rich area is that of the questions about infinite partition regular systems. Walter Deuber contributed the papers [12] and [17] to this interesting field of research. For a recent survey on 'Open problems in partition regularity', which contains many further results, see [26] (or the survey by Imre Leader [27]).

The third problem which Deuber wanted to see solved was a characterization of all partition regular systems of linear inequalities. Rado started to consider this problem in his thesis, but did not get far. To Walter Deuber's great satisfaction, his last doctoral student, Meike Schäffler (née Schröder) finally took care of this problem and was able to solve it in her doctoral thesis in 1996 (see [41]). The crucial notion in her characterization of partition regular systems of inequalities $A \cdot x \leqslant 0$ is the notion of the *columns property for inequalities* of a matrix A.

Definition (columns property for inequalities). Let $A = (a^1, \ldots, a^n)$ be a matrix with columns $a^j \in \mathbb{Z}^m$. Then A has the columns property for inequalities if there exists a partition $[1,n] = I_0 \cup \cdots \cup I_l$ such that

(1) $\sum_{i \in I_0} a^i \leqslant 0$, and

(2) for every $j < l$,

$\sum_{i \in I_{j+1}} a^i$ is bounded from above by a rational linear combination of the columns in $I_0 \cup \cdots \cup I_j$.

Clearly, if a matrix A has the columns property (in the original sense of Rado), the system $A \cdot x \leqslant \mathbf{0}$ is partition regular. But there are many other systems of inequalities which are partition regular without having the columns property. For example, the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

has the columns property for inequalities, but not the columns property (in the sense of Rado).

Theorem (Schäffler). *Let A be a matrix with integral coefficients. The linear system $A \cdot x \leqslant \mathbf{0}$ is partition regular in $\mathbb{N}$ if and only if the matrix A has the columns property for inequalities.*

An alternative characterization was given independently by Neil Hindman and Imre Leader [25].

5. Partition theorems for graphs

Let K_m be the complete graph on m vertices. Then the finite version of Ramsey's theorem can be rephrased by saying that for every r-colouring of the K_k-subgraphs of some K_n, where $n = R(m,k,r)$, there exists a K_m subgraph of this K_n such that all its K_k-subgraphs have the same colour. Using the Ramsey arrow notation, this assertion is abbreviated by

$$K_n \longrightarrow (K_m)_r^{K_k}.$$

A natural question which arose around 1970 is to what extent the complete graphs K_k, K_m, and K_n might be replaced by arbitrary graphs H, G, and F, *i.e.*, when do we have that

$$F \longrightarrow (G)_r^H \ ?$$

Notice that this question is only sensible if the notion of subgraphs always means *induced subgraphs*. In this case the question can be rephrased by asking: Does there exist a finite graph F such that, for every r-colouring of the induced H-subgraphs of F, there always exists an induced G-subgraph of F such that all its induced H have the same colour?

A straightforward product construction shows that for every finite graph G and every positive integer r there does exist a finite graph F such that $F \longrightarrow (G)_r^{K_1}$. One simply defines F as the rth power of G. The answer becomes more difficult if the K_1, a single

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vertex, is replaced by a K_2, an edge. A positive answer to this question was conjectured by Henson [24] and given independently by Deuber [10], Erdős, Hajnal and Pósa [20] and Rödl [40]. It is worth noticing that the first two of these papers were presented on the occasion of the 60th birthday of Paul Erdős.

This result was then generalized by Deuber [11] and by Nešetřil and Rödl [32] to a partition with respect to colouring complete subgraphs of finite graphs.

Theorem (Deuber, Nešetřil and Rödl). *Let k and r be positive integers and let G be a finite graph. Then there exists a finite graph F such that*

$$F \longrightarrow (G)_r^{K_k}.$$

This result was part of Walter Deuber's Habilitationsschrift which he defended in 1974 at the University of Hanover. The original proofs of this theorem were rather complicated, relying on *ad hoc* methods which were invented to construct such Ramsey graphs. Later on, Nešetřil and Rödl [34] found an elegant way to deduce this result from a particular case of the Graham–Rothschild theorem for parameter sets. A complete analogue to Ramsey's theorem for graphs was then established independently by Abramson and Harrington [1] and Nešetřil and Rödl [33].

Theorem (Abramson and Harrington, and Nešetřil and Rödl). *Let r be a positive integer and let G,H be a finite graphs. Then there exists a finite graph F such that*

$$F \longrightarrow (G)_r^{H}.$$

It is worth noting that Abramson and Harrington, as well as Nešetřil and Rödl, proved a much more powerful result, *viz.* a partition theorem for general set systems.

Apart from arithmetic structures, graphs and hypergraphs also played a central rôle in Walter Deuber's research throughout his life. So he immediately became interested when he learnt about Hajnal's question of graphs on arithmetic structures. Hajnal asked the following question (see [19]).

Question. *If G is a triangle-free graph on $\mathbb{N}$, does there always exist a Hindman set independent in G?*

Here a *Hindman set* is an infinite set of integers $(x_1, x_2, \dots)$ together with all its sums $\sum_{i \in I} x_i$ where $I \subset \mathbb{N}$ is finite. The name of the set is motivated by Hindman's theorem (see, *e.g.*, [21]) which states that for every two-colouring of the natural numbers there is an infinite set of numbers so that all of its finite sums have the same colour. Note that if the above question had an affirmative answer then this would imply that for every two-colouring of the pairs of natural numbers one would either find a monochromatic triangle in the first colour or a Hindman set H where all of the pairs of numbers in H have the second colour. However, Deuber, Gunderson, Hindman and Strauss [16] showed that the answer to the above question is negative. On the other hand, there are variants of this question which have been shown to have a positive answer, for example, if the condition ‘triangle-free’ is replaced by ‘$K_{m,m}$-free’; see [16, 31, 22].

For finite systems of linear equations an analogue to Hajnal’s question is true in a rather general sense, as shown in the paper ‘Independent Deuber sets in graphs on the natural numbers’ [23].

Theorem (Gunderson, Leader, Prömel and Rödl). *Let k, m, p, c be positive integers. Then there exist positive integers n, q, d so that any K_k-free graph on an (n, q, d)-set contains an independent (m, p, c)-set.*

Observe that this result is a common generalization of Deuber’s partition theorem for (m, p, c)-sets and Ramsey’s theorem (for colouring pairs). As an immediate consequence one obtains the following corollary on partition regular systems of linear equations.

Corollary. *Let $k \geqslant 2$ be an integer and G be any K_k-free graph on $\mathbb{N}$. Then one can solve any partition regular system of linear equations in an independent set of G.*

6. Canonization

The popularization of Ramsey’s theorem is inherently linked to Richard Rado and Paul Erdős. There are numerous results, derived independently and jointly, which are fundamental in Ramsey Theory. An excellent example is their paper ‘A combinatorial theorem’ [18]. In this paper they show what kind of patterns occur for complete graphs and hypergraphs if an unbounded number of colours is allowed to colour the edges. This paper gave rise for a new branch of Ramsey Theory, the so-called canonical Ramsey Theory.

Forty years later, Walter Deuber [14] published a survey entitled ‘Canonization’, which he dedicated to the 80th birthday of Paul Erdős. In this paper he describes the development which has emerged from the Erdős–Rado theorem and where he discusses his favourite results in this area.

In 1980, Paul Erdős and Ron Graham (unpublished) generalized van der Waerden’s theorem on arithmetic progressions to the extent that colourings with an unlimited number of colours are admitted. Their proof makes use of Szemerédi’s density result for arithmetic progressions, saying that, for every positive integer k and every $\epsilon > 0$, there exists a least positive integer $n = Sz(k, \epsilon)$ such that every subset $S \subseteq [1, n]$ satisfying $|S| > \epsilon\, n$ contains an k-term arithmetic progression.

Theorem (Erdős, Graham). *Let k be a positive integer. Then there exists n such that for every unbounded colouring $\Delta : [1, n] \longrightarrow \omega$ there exists a k-term arithmetic progression $\{a + id \mid i < k\}$ which is either monochromatic or coloured injectively (‘rainbow colouring’).*

Proof. Let $n = Sz(k, \frac{1}{k^4})$ according to Szemerédi’s theorem and consider an arbitrary colouring $\Delta : [1, n] \longrightarrow \omega$. We assume that every k-term arithmetic progression is coloured with at most $k - 1$ colours. Since the number of k-term arithmetic progressions in $[1, n]$ is at least $\frac{n^2}{k^2}$, there exist at least $\frac{n^2}{k^4}$ monochromatic two-element subsets in $[1, n]$. But then,

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there is one colour-class containing at least $\frac{n}{k^4}$ elements (with room to spare). By choice of n, this colour-class contains a k-term arithmetic progression. $\square$

In summer 1981, when he visited Walter Deuber, Ron Graham brought this result to Bielefeld. An immediate question which then arose in joint discussions was about the canonical patterns for the multidimensional version of van der Waerden's theorem.

Such a multidimensional version was proved by Tibor Gallai (see Rado [38]) and later by Ernst Witt [46]. The Gallai–Witt version focuses on the geometric nature of van der Waerden's result. A *homothetic mapping or homothety* is a mapping $h : \mathbb{R}^t \longrightarrow \mathbb{R}^t$ of the Euclidean space into itself which is of the form $h(\mathbf{s}) = \mathbf{a} + d \cdot \mathbf{s}$, where $\mathbf{a} \in \mathbb{R}^t$ is a translation vector and $d \in \mathbb{R}\backslash\{0\}$ describes a dilation. For a set $S \subseteq \mathbb{R}^t$, its image $h(S)$ is a *homothetic copy* of S. Thus, homothetic means similar without rotation. From a purely geometric point of view the group of homotheties is certainly not that exciting compared, for example, to the group of similarities. In spite of that, it still is an interesting group as this rather small group of geometric mappings admits the following Ramsey theorem.

Theorem (Gallai, Witt). *Let r and t be positive integers and let $S \subseteq \mathbb{N}^t$ be a finite set. Then there exists a finite set $T \subseteq \mathbb{N}^t$ such that for every colouring $\Delta : T \longrightarrow r$ there exists a homothetic copy of S in T which is monochromatic.*

Of course, when considering unbounded colourings of $\mathbb{N}^t$ one can never expect a homothetic copy of, say, $[1, n]^t$ which is either monochromatic or coloured injectively. Consider, for example, the colouring which assigns to each t-tuple its first coordinate as the colour. So the canonical patterns for the multidimensional version of van der Waerden's theorem must be more complex than just the monochromatic and injective colouring.

Let U be a linear subspace of $\mathbb{R}^t$. Then a *coset colouring modulo U* on $\mathbb{N}^t$ is defined by $\Delta(x) = \Delta(y)$ if and only if $x - y \in U$. Observe that every injective colouring is a coset colouring modulo $U = \{0\}$ and every monochromatic colouring is a coset colouring modulo $U = \mathbb{R}^t$. At the end of Ron Graham's stay in Bielefeld the following result [15] was proved.

Theorem (Deuber, Graham, Prömel, Voigt). *Let t be a positive integer and let $S \subseteq \mathbb{N}^t$ be a finite set. Then there exists a finite set $T \subseteq \mathbb{N}^t$ such that for every colouring $\Delta : T \longrightarrow \omega$ there exists a homothetic copy S^* of S in T and there exists a linear subspace $U \subseteq \mathbb{R}^t$ such that the restriction of Δ to S^* is a coset colouring modulo U.*

It is natural to ask about canonical partitions with respect to other groups as well. Evidently, the number of canonical partitions decreases if the groups are getting richer. Recall that the group of homotheties $h : \mathbb{R}^t \longrightarrow \mathbb{R}^t$ is a subgroup of the group of the group of similarities (a similarity is a composition of translations, dilatations, rotations and reflections; more precisely, a similarity is the composition of a dilatation and a special orthogonal mapping). Spencer [44] shows that the group of similarities is already rich enough to admit a so-called selectivity result.

Theorem (Spencer). *Let t be a positive integer and let $S \subseteq \mathbb{R}^t$ be a finite set. Then there exists a finite set $T \subseteq \mathbb{R}^t$ such that for every colouring $\Delta : T \longrightarrow \omega$ there exists a similar copy S^* of S in T which is either monochromatic or coloured injectively.*

Several results along these lines were proved in subsequent years, many of them with Walter Deuber's advice. In order to complete the picture of this survey, we will just mention one more result, *viz.* a canonical version of the partition theorem for (m, p, c)-sets which was proved by Deuber's former PhD student Hanno Lefmann [29]. Recall the representation of an (m, p, c)-set as a collection of $m + 1$ rows where each element of this (m, p, c)-set belongs (without loss of generality) to exactly one of these rows.

Theorem (Lefmann). *Let m, p and c be positive integers. Then there exist positive integers n, q and d such that for every (n, q, d)-set $D \subseteq \mathbb{N}$ and every colouring $\Delta : D \longrightarrow \omega$ there exists a (m, p, c)-set $D^* \subseteq D$ such that either the restriction of Δ to D^* is monochromatic, or injective, or it is a row-colouring, i.e., each row is monochromatic in a different colour.*

7. Biographical remarks

Walter Deuber was born on 6th October 1942 in Bern, Switzerland. He studied mathematics and physics at the Eidgenössische Technische Hochschule (ETH) Zürich, originally to become a high school teacher. Both his diploma thesis in 1966 as well as his PhD thesis at the beginning of the 1970s were written under the guidance of Ernst Specker in Zürich. In 1972 (one year before he formally obtained his PhD from Zürich in 1973) Walter Deuber went to Hanover to become Oberassistent at the Technische Universität. He spent the academic year 1974/75 as a visiting assistant professor at the University of California at Los Angeles. In 1976, he became Professor of Mathematics in Bielefeld where he remained until his untimely death on 16th July 1999. For further remarks on his life and his interests, see also [35].

A final remark concerning Walter Deuber's genealogical tree: Ernst Specker, Deuber's thesis advisor, obtained his PhD at the ETH Zürich in 1949. Specker's adviser was Heinz Hopf. Hopf was awarded his PhD at the University of Berlin in 1929. His main examiner was Erhard Schmidt (who was the second examiner of Rado two years later). Hopf, who had been fascinated by Schmidt's lectures on set theory in Breslau in 1917, had followed his teacher to the University of Berlin in 1920. There he also took classes with Issai Schur and became a friend of his; see [2] for historical details. Erhard Schmidt himself, who was professor at the University of Berlin from 1917 to 1950, was a doctoral student of David Hilbert and obtained his PhD at Göttingen in 1905 under Hilbert's supervision.

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