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ON A GENERALIZATION OF THE RÉNYI–SRIVASTAVA CHARACTERIZATION OF THE POISSON LAW

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Abstract

We give a new method of proof for a result of D. Pierre-Loti-Viaud and P. Boulongne which can be seen as a generalization of a characterization of Poisson law due to Rényi and Srivastava. We also provide explicit formulas, in terms of Bell polynomials, for the moments of the compound distributions occurring in the extended collective model in non-life insurance.

Keywords: Poisson law; compound variables; Bell polynomials; non-life insurance

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Secondary 05A17; 60-08; 97M30

1. Introduction

The aim of this paper is to prove a generalization of the Rényi–Srivastava characterization of the Poisson law [10]; see Theorem 2 below. The generalization we deal with was stated by D. Pierre-Loti-Viaud and P. Boulongne in Theorem III.1 of [9] (in French), which is restated below as Theorem 1. In these two characterizations of Poisson law, only the necessary condition is non-trivial. We give two distinct, original proofs of Theorem 1, which are the first to be published in English; to date the only available proof of Theorem 1 has been in French [9]. A by-product of our new approach consists of explicit closed formulas for the moments and mixed moments of the so-called compound variables defined by (2)–(3). This type of variable provides a fundamental tool in a wide range of applications including probability theory, mathematical statistics, and their applications as reliability theory and non-life insurance. Therefore they appear under different names in the literature, as *stopped-sum distributions* in [6, Chapter 9], *compound random variables* in [5, Chapter 9], and *random sums* in [3, Example (d) and Exercise 24, Chapter 5].

Our paper is organized as follows. In Section 2 we introduce the random variables (RVs) $\mathbf{N}$, $\mathbf{S}$, and $(\mathbf{S}_r)_{1\leq r\leq R}$ defining the extended collective model in non-life insurance. We show that Theorem 1 can be seen as a generalization of a well-known characterization of Poisson law restated for reference in Theorem 2.

In Section 3 we introduce analytic tools as the characteristic and certain generating functions relevant to a general proof. The proof of Theorem 1 is given at the end of Section 3.

When the random variables under discussion are completely determined by their moments, a second proof of Theorem 1, essentially algebraic, is available and is the object of Section 4.

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The generating functions introduced in Section 3 admit the developments (20)–(21) involving moments and cumulants. Bell polynomials provide a convenient formalism to express the moments of our RVs. For the use of such formal tools in applied probability theory, the reader is referred to [13].

The analytical and algebraic methods we use are familiar to theoreticians but not to all practitioners. Thus the aim of this paper is to be self-contained in order to provide applied mathematicians with a survey of analytical and algebraic methods underlying the objects of their studies. Note that our first proof differs from that in [9] by the use of a wider variety of generating functions, whereas the second based on moments is a fundamentally new approach. All the RVs we shall consider in this paper take non-negative values and will be distinct from a degenerate RV constant equal to zero.

2. The collective model in non-life insurance

For a general introduction we refer to [9, Chapters I–III], [5, Chapter 9], and the numerous references therein. The *collective risk model*, or *aggregate loss model*, is an infinite set of RVs consisting of an RV $\mathbf{N}$ and a sequence $(\mathbf{Y}_k)_{k\geq 1}$. The RV $\mathbf{N}$ is the *claim count random variable*. It takes non-negative integer values and we set $\mathbb{E}\mathbf{N} = m \in (0, \infty]$, $\mathbb{P}(\mathbf{N} = k) = \nu_k$, $k \in \mathbb{N}$. Note the important particular case where the count distribution follows a Poisson law with expectation $m \in (0, \infty)$, in other words

$$\mathbb{P}(\mathbf{N} = k) = \mathrm{e}^{-m} m^k / k! \quad (k = 0, 1, \ldots).$$

For a positive integer k, $\mathbf{Y}_k$ is the kth *individual-loss random variable*. We assume that (i) the RVs $\mathbf{N}, \mathbf{Y}_1, \mathbf{Y}_2, \ldots$ are independent (hypothesis of independence of relative frequency and cost), and (ii) the RVs $\mathbf{Y}_1, \mathbf{Y}_2, \ldots$ are identically distributed (hypothesis of the stationary character of losses). They follow the distribution of an RV that will be denoted $\mathbf{Y}$.

The *aggregate losses* RV is the sum $\mathbf{S} = \sum_{k=1}^{\mathbf{N}} \mathbf{Y}_k$. A refinement of the collective model is provided by the *extended collective risk model*; see [9, Definition III.1]. In this model each loss $\mathbf{Y}_k$ is allotted to a specific class of risks denoted by an integer $r \in \{1, \ldots, R\}$. Henceforth r and r' will always denote two distinct elements from the set $\{1, \ldots, R\}$ with $R \geq 2$. Two classes denoted by distinct integers do not overlap. Mathematically this split of the sequence $(\mathbf{Y}_k)_{k\geq 1}$ can be modeled by a Bernoulli scheme, i.e. a finite set of random variables

$$\mathbf{B}_1, \ldots, \mathbf{B}_R \in \{0, 1\} \quad \text{with } \mathbf{B}_1 + \cdots + \mathbf{B}_R = 1 \tag{1}$$

and the probabilities $\mathbb{P}(\mathbf{B}_r = 1) = p_r > 0$ with $\sum_{r=1}^{R} p_r = 1$. We introduce a sequence

$$\{(\mathbf{B}_{r,k})_{1\leq r\leq R} \colon k = 1, 2, \ldots\}$$

of independent copies of $(\mathbf{B}_r)_{1\leq r\leq R}$, our last assumption in the model being that, for every triple (r, k, l), the RVs $\mathbf{N}, \mathbf{Y}_k, \mathbf{B}_{r,\ell}$ are independent.

Let us now define, for $k = 1, 2, \ldots$, the RVs

$$\hat{\mathbf{Y}}_r = \mathbf{B}_r \mathbf{Y} = \begin{cases} \mathbf{Y} & \text{if } \mathbf{B}_r = 1, \\ 0 & \text{if } \mathbf{B}_r = 0, \end{cases} \qquad \mathbf{Y}_{r,k} = \mathbf{B}_{r,k} \mathbf{Y}_k = \begin{cases} \mathbf{Y}_k & \text{if } \mathbf{B}_{r,k} = 1, \\ 0 & \text{if } \mathbf{B}_{r,k} = 0. \end{cases}$$

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Then the number of losses, say $\mathbf{N}_r$, and the aggregate losses, allotted to the rth class of risks, say $\mathbf{S}_r$, are given by

$$\mathbf{N}_r = \sum_{k=1}^{\mathbf{N}} \mathbf{B}_{r,k}, \quad \mathbf{S}_r = \sum_{k=1}^{\mathbf{N}} \mathbf{Y}_{r,k} \quad (r = 1, \ldots, R). \tag{2}$$

We are now in a position to state the following theorem.

Theorem 1. ([9].) *The random variables of the extended collective risk model satisfy the equality*

$$\mathbf{S} = \sum_{r=1}^{R} \mathbf{S}_r \tag{3}$$

and enjoy the following properties.

(i) *The RVs $\mathbf{S}_r$, $r = 1, \ldots, R$, are mutually independent if and only if $\mathbf{N}$ follows a Poisson law.*

(ii) *The RVs $\mathbf{S}_r$ and $\mathbf{S}_{r'}$ are identically distributed if and only if $p_r = p_{r'}$.*

Note that if $\mathbf{Y}$ is a deterministic variable, constant equal to 1, then $\mathbf{S}_r = \mathbf{N}_r$ corresponds to a binomial split of the counting variable $\mathbf{N}$, and assertion (i) in the above theorem can be rewritten as follows, in the case $R = 2$.

Theorem 2. ([10, 12].) *With the above notation we have $\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2$ and the variables $\mathbf{N}_1$ and $\mathbf{N}_2$ are mutually independent if and only if $\mathbf{N}$ follows a Poisson distribution.*

This remarkable characterization of the Poisson distribution was stated in the case $R = 2$ by Srivastava [12, Theorem 2], who presented this result as the corollary of a result proved by Rényi [10, Theorems 1 and 2]. For various characterizations of Poisson law, see [6, §4.8], particularly the reference to [12] on page 182.

Thus Theorem 1 can be seen as a generalization of the Rényi–Srivastava characterization of the Poisson law.

3. General proof

Let us introduce some of the functions characterizing a random vector $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ (with $n = 1, 2, \ldots,$ and without parentheses if $n = 1$). As well as the characteristic function (see [6, (1.264–265)], and [7, (34.8)]), defined for $t_1, \ldots, t_n \in \mathbb{R}$ by

$$\Phi_{(\mathbf{X}_1,\ldots,\mathbf{X}_n)}(t_1, \ldots, t_r) = \mathbb{E} \exp\left\{ i \sum_{r=1}^{n} t_r \mathbf{X}_r \right\},$$

we will make use of the (uncorrected) moments (see [6, (1.227)] and [7, (34.9)]), the (descending) factorial moments and factorial cumulants (see [6, (1.249), (1.256), and (1.274)]) generating functions denoted and defined by

$$M_{(\mathbf{X}_1,\ldots,\mathbf{X}_n)}(t_1, \ldots, t_n) = \mathbb{E}\left(e^{t_1 \mathbf{X}_1 + \cdots + t_n \mathbf{X}_n}\right)$$

for $t_1, \ldots, t_n \leq 0$, and

$$M_{[\mathbf{N}]}(t) = \mathbb{E}(1+t)^{\mathbf{N}} = \nu_0 + \sum_{k=1}^{\infty} \nu_k (1+t)^k,$$

$$K_{[\mathbf{N}]}(t) = \log M_{[\mathbf{N}]}(t)$$

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for $-1 < t \leq 0$. For the sake of notational simplicity we will also use the notations $\phi_{\mathbf{X}} = \Phi_{\mathbf{X}} - 1$ and $m_{\mathbf{X}} = M_{\mathbf{X}} - 1$.

These functions are easily seen to be well-defined and taking finite values on the given domain of definition. Furthermore $M_{\mathbf{X}}$ and $m_{\mathbf{X}}$ are increasing with respect to each variable, and for an integer-valued random variable $\mathbf{N}$ the following equalities hold: $M_{\mathbf{N}}(-\infty, 0] = (\nu_0, 1]$, $M_{[\mathbf{N}]}(\mathrm{e}^t - 1) = M_{\mathbf{N}}(t)$ and

$$M_{[\mathbf{N}]}(0) = 1, \quad K_{[\mathbf{N}]}(0) = 0, \tag{4}$$

$$M'_{[\mathbf{N}]}(0) = K'_{[\mathbf{N}]}(0) = \lim_{t \to 0^-} K_{[\mathbf{N}]}(t)/t = \mathbb{E}\mathbf{N}. \tag{5}$$

A first result is as follows.

Proposition 1. *The characteristic and moment generating functions of* $(\mathbf{S}_1, \ldots, \mathbf{S}_R)$*,* $\mathbf{S}$*, and* $\mathbf{S}_r$ *are given by*

$$\Phi_{(\mathbf{S}_1,\ldots,\mathbf{S}_R)}(t_1, \ldots, t_R) = M_{[\mathbf{N}]}\left\{\sum_{r=1}^{R} p_r \phi_{\mathbf{Y}}(t_r)\right\}, \tag{6}$$

$$M_{(\mathbf{S}_1,\ldots,\mathbf{S}_R)}(t_1, \ldots, t_R) = M_{[\mathbf{N}]}\left\{\sum_{r=1}^{R} p_r m_{\mathbf{Y}}(t_r)\right\}, \tag{7}$$

$$\Phi_{\mathbf{S}}(t) = M_{[\mathbf{N}]}\{\phi_{\mathbf{Y}}(t)\}, \tag{8}$$

$$M_{\mathbf{S}}(t) = M_{[\mathbf{N}]}\{m_{\mathbf{Y}}(t)\}, \tag{9}$$

$$\Phi_{\mathbf{S}_r}(t) = M_{[\mathbf{N}]}\{p_r \phi_{\mathbf{Y}}(t)\}, \tag{10}$$

$$M_{\mathbf{S}_r}(t) = M_{[\mathbf{N}]}\{p_r m_{\mathbf{Y}}(t)\}. \tag{11}$$

Proof. On one hand the formula for conditional expectation (see [3, (10.6)]) enables us to write

$$\begin{aligned}
\Phi_{(\mathbf{S}_1,\ldots,\mathbf{S}_R)}(t_1, \ldots, t_R) &= \nu_0 + \sum_{n=1}^{\infty} \nu_n \mathbb{E}\left(\exp\left\{\sum_{r=1}^{R} \mathrm{i}t_r \mathbf{S}_r\right\} \mid \mathbf{N} = n\right) \\
&= \nu_0 + \sum_{n=1}^{\infty} \nu_n \mathbb{E}\left(\exp\left\{\sum_{r=1}^{R} \mathrm{i}t_r \sum_{k=1}^{n} \mathbf{Y}_{r,k}\right\} \mid \mathbf{N} = n\right) \\
&= \nu_0 + \sum_{n=1}^{\infty} \nu_n \mathbb{E} \exp\left\{\sum_{k=1}^{n} \sum_{r=1}^{R} \mathrm{i}t_r \mathbf{Y}_{r,k}\right\} \\
&= \nu_0 + \sum_{n=1}^{\infty} \nu_n \prod_{k=1}^{n} \mathbb{E} \exp\left\{\sum_{r=1}^{R} \mathrm{i}t_r \mathbf{Y}_{r,k}\right\} \\
&= \nu_0 + \sum_{n=1}^{\infty} \nu_n \left\{\mathbb{E} \exp\left\{\sum_{r=1}^{R} \mathrm{i}t_r \hat{\mathbf{Y}}_r\right\}\right\}^n \\
&= M_{[N]}\left(\mathbb{E} \exp\left\{\sum_{r=1}^{R} \mathrm{i}t_r \hat{\mathbf{Y}}_r\right\} - 1\right),
\end{aligned}$$

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keeping in mind that $\mathbf{N}$ and $\mathbf{Y}_{r,k}$ are independent. On the other hand, keeping in mind the defining properties (1), it is easily checked that

$$\mathrm{e}^{\sum_{r=1}^{R} \mathrm{i}t_r \hat{\mathbf{Y}}_r} - 1 = \mathrm{e}^{\sum_{r=1}^{R} \mathrm{i}t_r \mathbf{B}_r \mathbf{Y}} - 1 = \sum_{r=1}^{R} \mathbf{B}_r \{\mathrm{e}^{\mathrm{i}t_r \mathbf{Y}} - 1\},$$

and by taking the expectation of both sides we obtain

$$\mathbb{E}\mathrm{e}^{\sum_{r=1}^{R} \mathrm{i}t_r \hat{\mathbf{Y}}_r} - 1 = \sum_{r=1}^{R} \mathbb{E}\mathbf{B}_r \mathbb{E}\{\mathrm{e}^{\mathrm{i}t_r \mathbf{Y}} - 1\} = \sum_{r=1}^{R} p_r m_{\mathbf{Y}}(t_r).$$

The proof for $M_{(\mathbf{S}_1,\ldots,\mathbf{S}_r)}$ is similar, replacing $\mathrm{i}t_r$ with t_r. The last four equalities are one-variable versions of the first two equalities, with $p_r = 1$ for $\mathbf{S}$. □

Let us see in which way $M_{\mathbf{X}}$ characterizes a non-negative random variable $\mathbf{X}$.

Lemma 1. *If $M_{\mathbf{X}}$ and $M_{\mathbf{X}'}$ coincide over a non-empty set $S \subseteq (-\infty, 0)$ having one accumulation point $s_0 \in S$, then the two non-negative random variables $\mathbf{X}$ and $\mathbf{X}'$ are identically distributed.*

Proof. Introduce the distribution function $F_{\mathbf{X}}(t) = \mathbb{P}(\mathbf{X} \le t)$, $t \ge 0$. The Laplace transform of $\mathbf{X}$ is given by the Stieljes integral

$$\Lambda_{\mathbf{X}}(z) = \int_0^{\infty} \mathrm{e}^{-uz} \, \mathrm{d}F_{\mathbf{X}}(u),$$

where $z = x + \mathrm{i}y$ is a complex variable with real and imaginary parts $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ respectively ([14, (1)]). It is clear that $\Lambda_{\mathbf{X}}(z)$ exists over the half-plane $H^+ = \{z\colon \mathrm{Re}(z) > 0\}$. In fact it defines an analytic function over H^+ ([14, Theorem 5a]). Now, by using similar notations for $\mathbf{X}'$, our assumption implies, for each $t \in S$,

$$\Lambda_{\mathbf{X}}(-t) = M_{\mathbf{X}}(t) = M_{\mathbf{X}'}(t) = \Lambda_{\mathbf{X}'}(-t).$$

We have shown that the two functions $\Lambda_{\mathbf{X}'}$ and $\Lambda_{\mathbf{X}}$ are analytic over H^+ and coincide over $-S$. This means that $\Lambda_{\mathbf{X}} - \Lambda_{\mathbf{X}'}$ is an analytic function for which $-s_0$ is a non-isolated zero. Thus $\Lambda_{\mathbf{X}} - \Lambda_{\mathbf{X}'}$ is the zero function over H^+ (see [2, Theorem 5.1]). By uniqueness of the Laplace transform ([14, Theorem 6.3]), we have $F_{\mathbf{X}} = F_{\mathbf{X}'}$. □

Let us now recall two characterizations of Poisson law, noting that (13) expresses the linearity of the factorial cumulant generating function.

Lemma 2. *Assume $m > 0$. The following propositions are equivalent.*

(i) *The RV $\mathbf{N}$ follows a Poisson distribution with expectation m.*

(ii) *For each $t \in \mathbb{R}$ we have*

$$M_{[\mathbf{N}]}(t) = \exp(mt), \tag{12}$$

$$K_{[\mathbf{N}]}(t) = mt. \tag{13}$$

(iii) *There exists a non-empty open interval $I \subset (-\infty, 0)$ such that*

$$M_{[\mathbf{N}]}(t) = \exp(mt) \quad \textit{for all } t \in I, \tag{14}$$

$$K_{[\mathbf{N}]}(t) = mt \quad \textit{for all } t \in I. \tag{15}$$

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Proof. From [6, §4.3] we know that (i) implies (ii), and perforce (iii). Next (iii) implies (i) by Lemma 1 with $S = I$. □

The following lemma is elementary but deserves to be stated explicitly as a key step in our general proof.

Lemma 3. *If the* $(S_r)_{1\le r\le R}$ *are mutually independent, then for any* $r \neq r'$ *we have*

$$K_{[\mathbf{N}]}\{p_r m_{\mathbf{Y}}(t) + p_{r'} m_{\mathbf{Y}}(t)\} = K_{[\mathbf{N}]}\{p_r m_{\mathbf{Y}}(t)\} + K_{[\mathbf{N}]}\{p_{r'} m_{\mathbf{Y}}(t)\} \quad \textit{for all } t \le 0. \tag{16}$$

Therefore, if we set

$$x_0 = -(p_r + p_{r'}) m_{\mathbf{Y}}(-\infty), \;\; p = p_r/(p_r + p_{r'}),$$

the function $f(x) = -K_{[\mathbf{N}]}(-x)$ *is non-negative, continuously differentiable, and satisfies relations*

$$f(0) = 0, \qquad \lim_{x\to 0+} f(x)/x = m, \tag{17}$$

$$f(x) = f(p\,x) + f(\{1-p\}x) \quad \textit{for all } x \in [0, x_0). \tag{18}$$

Proof. Independence implies

$$M_{(\mathbf{S}_1,\dots,\mathbf{S}_R)}(t_1, \dots, t_R) = \prod_{r=1}^{R} M_{\mathbf{S}_r}(t_r).$$

Then choose $(t_1, \dots, t_R)$ with all coordinates equal to zero except $t_r = t_{r'} = t$, write (7), and take the logarithm to obtain (16). Then (17) is simply a reformulation of the appropriate identities in (4)–(5), and (18) is (16) with $x = -(p_r + p_{r'}) m_{\mathbf{Y}}(t)$. The non-negativity and continuous differentiability of f follows from that of K. □

A converse of the above lemma is provided by the following one which, like the former, though elementary, is a key step in our proof of Theorem 1 (i).

Lemma 4. *Let* $x_0 > 0$, $m \in \mathbb{R}$ *and let* $f\colon [0, x_0) \to [0, \infty)$ *be a continuously differentiable function satisfying* (17)–(18)*. Then* f *is the linear function* $f(x) = mx$.

Proof. Without loss of generality we can assume $p \le 1/2$ so that $q = 1 - p \ge p$. From (18) and by a straightforward induction we see that for every positive integer n

$$f(s) = \sum_{k=0}^{n} \binom{n}{k} f\big(p^k q^{n-k} s\big). \tag{19}$$

By continuity of f', for every positive integer n, the numbers $a_n = \min_{x\in[p^n, q^n]} f'(x)$ and $b_n = \max_{x\in[p^n, q^n]} f'(x)$ are well-defined, finite, and satisfy $\lim a_n = \lim b_n = f'(0) = m$. Rolle's theorem implies, for every n, $a_n p^k q^{n-k} s \le f(p^k q^{n-k} s) \le b_n p^k q^{n-k} s$. By summing the latter inequalities over k and in view of (19), we obtain

$$a_n s = \sum_{k=0}^{n} \binom{n}{k} a_n p^k q^{n-k} s \le f(s) = \sum_{k=0}^{n} \binom{n}{k} f\big(p^k q^{n-k} s\big) \le \sum_{k=0}^{n} \binom{n}{k} b_n p^k q^{n-k} s = b_n s.$$

These inequalities lead, as n tends to infinity, to $f(s) = ms$, which proves the desired result. □

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Proof of Theorem 1(i). If the random variables $\mathbf{S}_r$ and $\mathbf{S}_{r'}$ are independent, then we know from Lemmas 3 and 4 that (15) holds with $I=(-x_0,0)$, so $\mathbf{N}$ is a Poisson random variable. Conversely, if $\mathbf{N}$ follows a Poisson law then (12) holds for $t\in\mathbb{R}$, so relations (6) and (10) in Proposition 1 imply $\Phi_{(\mathbf{S}_1,\ldots,\mathbf{S}_R)}(t_1,\ldots,t_R)=\prod_{r=1}^R \Phi_{\mathbf{S}_r}(t_r)$; in other words $\mathbf{S}_1,\ldots,\mathbf{S}_R$ are mutually independent. □

Proof of Theorem 1(ii). We infer from Lemma 1 that the random variables $\mathbf{S}_r$ and $\mathbf{S}_{r'}$ are identically distributed if and only if $M_{\mathbf{S}_r}=M_{\mathbf{S}_{r'}}$. In view of (9) and keeping in mind that $M_{[\mathbf{N}]}$ is injective, this is equivalent to $p_r m_{\mathbf{Y}}=p_{r'} m_{\mathbf{Y}}$; the latter is equivalent to $p_r=p_{r'}$ since $m_{\mathbf{Y}}$ is not the zero function. □

4. A proof based on moments and cumulants

In this section we shall prove Theorem 1 by means of moments and cumulants. For $d=1,2,\ldots$ we will let $\mu_d(\mathbf{X})=\mathbb{E}\mathbf{X}^d$ denote the moment of order d of the RV $\mathbf{X}$. Recall that assuming $\mu_d(\mathbf{X})<\infty$ is equivalent to assuming the finiteness of the factorial moments $\mu_{[d]}(\mathbf{X})=\mathbb{E}\mathbf{X}(\mathbf{X}-1)\cdots(\mathbf{X}-d+1)$, or of the cumulants $\kappa_d(\mathbf{X})$, or of the factorial cumulants $\kappa_{[d]}(\mathbf{X})$, these coefficients being given, provided relation

$$\limsup_{d\to\infty}\left\{d^{-1}\mu_d^{1/d}(\mathbf{X})\right\}<\infty$$

holds (see [3, XV.4, (4.15)]), by the developments

$$M_{\mathbf{X}}(t)=1+\sum_{k=1}^{\infty}\mu_k(\mathbf{X})\frac{t^k}{k!}=1+m_{\mathbf{X}}(t), \tag{20}$$

$$K_{\mathbf{X}}(t)=\sum_{k=1}^{\infty}\kappa_k(\mathbf{X})\frac{t^k}{k!}, \tag{21}$$

$$M_{[\mathbf{N}]}(t)=1+\sum_{k=1}^{\infty}\mu_{[k]}(\mathbf{N})\frac{t^k}{k!}, \tag{22}$$

$$K_{[\mathbf{N}]}(t)=\sum_{k=1}^{\infty}\kappa_{[k]}(\mathbf{N})\frac{t^k}{k!} \tag{23}$$

(see [6, §1.2.7], specifically formulas (1.227) for $M_{\mathbf{X}}$, (1.249) for $K_{\mathbf{X}}$, (1.256) for $K_{\mathbf{N}}$, and (1.274) for $M_{[\mathbf{N}]}$). The latter developments converge over a neighbourhood of the origin.

Let us now introduce basic facts and notation from combinatorics. The first tool we will need from this field consists of the partitions of integers and the associated Young diagrams. Consider two integers $d\geq k\geq 1$. The k-tuple of integers $L=(\ell_1,\ldots,\ell_k)$ is called a k-partition of d provided (see [1, 1a]) $\sum_{i=1}^k \ell_i=d$ and $\ell_1\geq\cdots\geq\ell_k\geq 1$. The set of k-partitions of d is denoted $\mathcal{L}_{d,k}$. Any element of the set $\mathcal{L}_d=\cup_{k=1}^d\mathcal{L}_{d,k}$ is called a partition of d.

A convenient and visual representation of partitions is provided by Young diagrams, equivalent to Ferrers diagrams discussed in [1, §2.4].

The Young diagram associated with $L=(\ell_1,\ldots,\ell_k)\in\mathcal{L}_{d,k}$ is a set of d cells arranged in k left justified rows, the ith row consisting of ℓ_i cells. For a given partition let

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$s = \text{Card}\{\ell_1, \ldots, \ell_k\}$ be the number of distinct summands. We will let $\lambda_1 = \ell_1 > \cdots > \lambda_s = \ell_k \geq 1$ and $\alpha_1, \ldots, \alpha_s \geq 1$ denote the two sequences such that

$$L = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\alpha_1 \text{ times}}, \ldots, \underbrace{\lambda_s, \ldots, \lambda_s}_{\alpha_s \text{ times}}).$$

These coefficients satisfy the equalities

$$\sum_{i=1}^{s} \alpha_i \lambda_i = d, \quad \sum_{i=1}^{s} \alpha_i = k.$$

Therefore an alternative notation for the partition $L = (\ell_1, \ldots, \ell_k)$ is $L = [\lambda_1^{\alpha_1}, \ldots, \lambda_s^{\alpha_s}]$, where an exponent can be omitted if it equals 1.

For $L = (\ell_1, \ldots, \ell_k) = [\lambda_1^{\alpha_1}, \ldots, \lambda_s^{\alpha_s}] \in \mathcal{L}_{d,k}$, let us introduce the coefficients

$$c_L = \frac{d!}{\ell_1! \cdots \ell_k! \, \alpha_1! \cdots \alpha_s!} = \frac{d!}{\lambda_1!^{\alpha_1} \cdots \lambda_s!^{\alpha_s} \, \alpha_1! \cdots \alpha_s!},$$

$$\mu_L(\mathbf{Y}) = \mu_{\ell_1}(\mathbf{Y}) \cdots \mu_{\ell_k}(\mathbf{Y}) = \mu_{\lambda_1}^{\alpha_1}(\mathbf{Y}) \cdots \mu_{\lambda_s}^{\alpha_s}(\mathbf{Y}).$$

A convenient tool we will borrow from combinatorics is the family of Bell polynomials. The *complete Bell polynomials* $\{B_n(X_1, \ldots, X_n)\}_{n \geq 1}$, the *partial Bell polynomials*

$$\{B_{n,k}(X_1, \ldots, X_{n-k+1})\}_{n \geq k \geq 1}$$

(see [1, §III.3, formulas [3a]–[3c]]), and the polynomials $\{b_n(X;.)\}_{n \geq 1}$ which we will call *associated Bell polynomials* (see [11, §4.1.8]), are defined via the following generating functions:

$$1 + \sum_{n=1}^{\infty} \frac{b_n(x; x_1, \ldots, x_n)}{n!} t^n = \exp\left(x \sum_{k=1}^{\infty} \frac{x_k}{k!} t^k \right), \tag{24}$$

$$\begin{aligned} B_n(x_1, \ldots, x_n) &= b_n(1; x_1, \ldots, x_n) \\ &= \sum_{k=1}^{n} B_{n,k}(x_1, \ldots, x_{n-k+1}), \end{aligned} \tag{25}$$

$$\begin{aligned} b_n(x; x_1, \ldots, x_n) &= \sum_{k=1}^{n} B_{n,k}(x_1, \ldots, x_{n-k+1}) x^k \\ &= \sum_{k=1}^{n} B_{n,k}(x x_1, \ldots, x x_{n-k+1}) \\ &= B_n(x x_1, \ldots, x x_n). \end{aligned} \tag{26}$$

All these polynomials have positive integer-valued coefficients and $B_{n,k}$ is k-homogeneous. By convention we set $b_0 = B_0 = B_{0,0} = \mathbf{1}$ and $B_{n,0} = \mathbf{0}$ for $n \geq 1$. Bell polynomials are related to Stirling numbers, of the first kind by formula $s(n, k) = (-1)^{n+k} B_{n,k}(0!, \ldots, (n-k+1)!)$, of the second kind by formula $S(n, k) = B_{n,k}(1, \ldots, 1)$ (see [1, eqs [3g], [3i], and [5d]]). From (20)–(21) and definitions (24)–(25) the moments and cumulants of an RV are related by the equality

$$B_d(\kappa_1, \ldots, \kappa_d) = \mu_d;$$

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this formula amounts to (3.33) of [4], a fundamental reference in which, however, Bell polynomials are not referred to. Henceforth, when the RV is omitted μ_d will always mean $\mu_d(\mathbf{Y})$, and $\mu_L = \mu_L(\mathbf{Y})$. A standard expression for partial Bell polynomials (see [1, Theorem A]) is given by

$$B_{d,k}(x_1, \ldots, x_{d-k+1}) = \sum \frac{d!}{c_1!c_2! \cdots (1!)^{c_1}(2!)^{c_2} \cdots} x_1^{c_1} x_2^{c_2} \cdots , \tag{27}$$

where the summation takes place over the set $C(d, k)$ of sequences $c\colon c_1, c_2, \ldots \geq 0$ of integers such that

$$c_1 + 2c_2 + 3c_3 + \cdots = d, \; c_1 + c_2 + c_3 + \cdots = k.$$

The following lemma gives, for partial Bell polynomials (27), a more explicit expression (28) in terms of Young diagrams.

Lemma 5. *We have*

$$B_{d,k}(\mu_1, \ldots, \mu_{d-k+1}) = \sum_{L \in \mathcal{L}_{d,k}} c_L \mu_L. \tag{28}$$

Proof. Each sequence in $C(d, k)$ has a finite number of non-zero terms whose list can be written in a unique way as $c_{\phi(1)}, \ldots, c_{\phi(s)}$ with $\phi(1) > \cdots > \phi(s)$. We build a bijection between $C(d, k)$ and $\mathcal{L}_{d,k}$ by associating c with the Young diagram

$$L(c) = \left[\lambda_1^{\alpha_1}, \ldots, \lambda_s^{\alpha_s}\right] = [\phi(1)^{c_{\phi(1)}}, \ldots, \phi(s)^{c_{\phi(s)}}]$$

for which

$$\begin{aligned} c_L &= \frac{d!}{\lambda_1!^{\alpha_1} \cdots \lambda_s!^{\alpha_s} \, \alpha_1! \cdots \alpha_s!} \\ &= \frac{d!}{\phi(1)!^{c_{\phi(1)}} \cdots \phi(s)!^{c_{\phi(s)}} \, c_{\phi(1)}! \cdots c_{\phi(s)}!} \\ &= \frac{d!}{(1!)^{c_1}(2!)^{c_2} \cdots c_1!c_2! \cdots}, \\ \mu_L &= \mu_{\lambda_1}^{\alpha_1} \cdots \mu_{\lambda_s}^{\alpha_s} = \mu_{\phi(1)}^{c_{\phi(1)}} \cdots \mu_{\phi(s)}^{c_{\phi(s)}} = \mu_1^{c_1} \mu_2^{c_2} \cdots , \end{aligned}$$

and this completes the proof of the claimed result. □

We are now in a position to state a first important result.

Theorem 3. *Consider an integer* $d \geq 1$.

(i) *The moment of order d of the random variable* $\mathbf{S}$ *admits the expression*

$$\begin{aligned} \mu_d(\mathbf{S}) &= \sum_{k=1}^{d} \mu_{[k]}(\mathbf{N}) B_{d,k}(\mu_1, \ldots, \mu_{d-k+1}) \\ &= \sum_{L \in \mathcal{L}_d} \mu_L(\mathbf{N}) \mu_L(\mathbf{Y}) \\ &= \sum_{k=1}^{d} \mu_{[k]}(\mathbf{N}) \sum_{L \in \mathcal{L}_{d,k}} c_L \mu_L(\mathbf{Y}), \end{aligned} \tag{29}$$

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and therefore satisfies the inequality

$$\mu_d(\mathbf{S}) \le \mu_d(\mathbf{N})\mu_d(\mathbf{Y}), \tag{30}$$

with equality if and only if $\mu_k(\mathbf{Y}) = \mu_1^k(\mathbf{Y}),\ k = 1, \ldots, d.$

(ii) *The cumulant of order d of the random variable* $\mathbf{S}$ *admits the expression*

$$\begin{aligned}\kappa_d(\mathbf{S}) &= \sum_{k=1}^{d} \kappa_{[k]}(\mathbf{N}) B_{d,k}(\mu_1, \ldots, \mu_{d-k+1}) \\ &= \sum_{k=1}^{d} \kappa_{[k]}(\mathbf{N}) \sum_{L\in\mathcal{L}_{d,k}} c_L \mu_L(\mathbf{Y}).\end{aligned} \tag{31}$$

Proof. Use Theorem A (Faà di Bruno Formula) of [1] with functions f, g and $h = f \circ g$ defined to be $f = M_{[\mathbf{N}]}$, $g = m_{\mathbf{Y}}$, so that from (9) we get $h = M_{\mathbf{S}}$. Then formula [4c] from [1] is exactly (29). The proof is similar for the cumulants.

For inequality (30), let us first use the well-known Lyapunov inequality (see [3, §V.8(c)]): $\mu_\lambda \le \mu_d^{\lambda/d}$ provided $\lambda \le d$, to get

$$\mu_L(\mathbf{Y}) = \prod_{i=1}^{s} \mu_{\lambda_i}(\mathbf{Y})^{\alpha_i} \le \prod_{i=1}^{s} \mu_d(\mathbf{Y})^{\alpha_i(\lambda_i/d)} = \mu_d(\mathbf{Y})^{(\sum \alpha_i\lambda_i)/d} = \mu_d(\mathbf{Y}).$$

Note in passing that equality holds if and only if $\mu_\lambda = \mu_d^{\lambda/d}$ for $\lambda = 1, \ldots, d$, which is equivalent to $\mu_\lambda = \mu_1^\lambda$ for $\lambda = 1, \ldots, d$.

Now, using this inequality we obtain

$$\mu_d(\mathbf{S}) \le \mu_d(\mathbf{Y}) \sum_{k=1}^{d} \mu_{[k]}(\mathbf{N}) \sum_{L\in\mathcal{L}_{d,k}} c_L. \tag{32}$$

Lemma 5 with $\mathbf{Y}$ chosen as a constant equal to 1, hence $\mu_L(\mathbf{Y}) = 1$ for every L, leads to

$$\sum_{k=1}^{d} \mu_{[k]}(\mathbf{N}) \sum_{L\in\mathcal{L}_{d,k}} c_L = \sum_{k=1}^{d} \mu_{[k]}(\mathbf{N}) B_{d,k}(1, \ldots, 1) = \sum_{k=1}^{d} \mu_{[k]}(\mathbf{N}) S(d, k) = \mu_d(\mathbf{N})$$

(see [6, (1.246)] and [1, [2.c]]). When combined with (32), this completes the proof of (30). □

A straightforward consequence of (30), justifying our approach based on moments, is as follows.

Corollary 1. *Let* $\alpha = 1 - \beta \in [0, 1]$. *If* $\mathbf{N}$ *and* $\mathbf{Y}$ *are completely determined by their moments due to*

$$\begin{aligned} &\limsup_{d\to\infty}\{d^{-\alpha}\mu_d^{1/d}(\mathbf{N})\} < \infty, \\ &\limsup_{d\to\infty}\{d^{-\beta}\mu_d^{1/d}(\mathbf{Y})\} < \infty, \\ \text{then}\quad &\limsup_{d\to\infty}\{d^{-1}\mu_d^{1/d}(\mathbf{S})\} < \infty, \\ &\limsup_{d\to\infty}\{d^{-1}\mu_d^{1/d}(\mathbf{S}_r)\} < \infty \quad (r = 1, \ldots, R), \end{aligned}$$

then $\mathbf{S}$ *and* $\mathbf{S}_r$, $(r = 1, \ldots, R)$ *are completely determined by their moments.*

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In the case when $\mathbf{N}$ is a Poisson random variable with expectation $\mu_{[1]}=m$, the other factorial moments are given by

$$\mu_{[k]}=m^k \quad (k=1, 2, \ldots), \tag{33}$$

so that in view of (26), another corollary of (29) is the following. Recall that the associated Bell polynomials b_n, $n\geq 1$, were defined above by formulas (24)–(26).

Corollary 2. *If* $\mathbf{N}$ *follows a Poisson law with expectation m, then the moments and cumulants of* $\mathbf{S}$ *are given by*

$$\mu_d(\mathbf{S})=b_d(m; \mu_1, \ldots, \mu_d), \qquad \kappa_d(\mathbf{S})=m\mu_d(\mathbf{Y}).$$

For the extended collective model the following result holds.

Theorem 4. *The moment and the cumulant of order d of the random variable* $\mathbf{S}_r$ *are given by*

$$\mu_d(\mathbf{S}_r)=\sum_{k=1}^{d} p_r^k\, \mu_{[k]}(\mathbf{N})B_{d,k}(\mu_1, \ldots, \mu_{\lambda_{d-k+1}}), \tag{34}$$

$$\kappa_d(\mathbf{S}_r)=\sum_{k=1}^{d} p_r^k\, \kappa_{[k]}(\mathbf{N})B_{d,k}(\mu_1, \ldots, \mu_{\lambda_{d-k+1}}). \tag{35}$$

Proof. Apply Theorem 3 to the RV $\hat{\mathbf{Y}}_r$, i.e. (29)–(31) with $\mu_d=\mu_d(\hat{\mathbf{Y}}_r)=p_r\mu_d(\mathbf{Y})$ and the k-homogeneity of $B_{n,k}$. □

For a reason similar to that given before Corollary 2, the following result is a straightforward consequence of Theorem 4.

Corollary 3. *If* $\mathbf{N}$ *follows a Poisson law with expectation m, then the moments and cumulants of* $\mathbf{S}_r$, $r=1, \ldots, R$, *are given by*

$$\mu_d(\mathbf{S}_r)=b_d(p_r m; \mu_1, \ldots, \mu_d),$$
$$\kappa_d(\mathbf{S}_r)=p_r m\mu_d \quad (d=1, 2, \ldots).$$

We are now in a position to prove the necessary condition in Theorem 1 (i).

Proposition 2. *If the random variables* $\mathbf{S}_r$ *and* $\mathbf{S}_{r'}$ *are independent then* $\mathbf{N}$ *follows a Poisson distribution.*

Proof. The cumulants satisfy $\kappa_d(\mathbf{S}_r+\mathbf{S}_{r'})=\kappa_d(\mathbf{S}_r)+\kappa_d(\mathbf{S}_{r'})$. In view of (34), this implies, for every d,

$$\sum_{k=1}^{d}(p_r+p_{r'})^k\, \kappa_{[k]}(\mathbf{N})B_{d,k}(\mu_1, \ldots, \mu_{\lambda_{d-k+1}})=\sum_{k=1}^{d}\left(p_r^k+p_{r'}^k\right)\kappa_{[k]}(\mathbf{N})B_{d,k}(\mu_1, \ldots, \mu_{\lambda_{d-k+1}}).$$

These equalities, due to

$$B_{d,k}(\mu_1, \ldots, \mu_{\lambda_{d-k+1}})>0,\ \ p_r^k+p_{r'}^k<(p_r+p_{r'})^k \quad (k\geq 2)$$

can hold if and only if $\kappa_{[k]}(\mathbf{N})=0$ for $k\geq 2$, in other words $K_{[\mathbf{N}]}(t)=\kappa_{[1]}t$, which is one of the characterizations of a Poisson law in Lemma 2. □

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Let us now complete the proof of Theorem 1 (i) by proving the result stated below as Corollary 4.

To this end it suffices to show that, for any $J \in \{2, \ldots, R\}$,

$$\prod_{j=1}^{J} \mu_{d_j}(\mathbf{S}_{r_j}) = \prod_{j=1}^{J} \mathbb{E}\mathbf{S}_{r_j}^{d_j} = \mathbb{E}\left(\prod_{j=1}^{J} \mathbf{S}_{r_j}^{d_j}\right), \tag{36}$$

for $r_1 < \cdots < r_J \in \{1, \ldots, R\}$ and $d_1, \ldots, d_J \geq 1$. The left-hand side of (36) can be obtained from (34). Let us introduce some notations that will enable us to compute the right-hand side of (36). Introduce, for $1 \leq k \leq n$, the index sets

$$\mathcal{I}_{n,k} = \{I = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k,\ \mathrm{Card}\{i_1, \ldots, i_k\} = k\}.$$

In this setting, for $n \geq d$, the multinomial formula can be written as

$$(x_1 + \cdots + x_n)^d = \sum_{k=1}^{d} \sum_{L \in \mathcal{L}_{d,k}} \sum_{I \in \mathcal{I}_{n,k}} \frac{d!}{\ell_1! \cdots \ell_k!\, \alpha_1! \cdots \alpha_r!} x_{i_1}^{\ell_1} \cdots x_{i_k}^{\ell_k}. \tag{37}$$

Consider the elementary and power symmetric polynomials

$$\pi_m(t_1, \ldots, t_n) = t_1^m + \cdots + t_n^m,$$

$$\sigma_{k,n}(t_1, \ldots, t_n) = \frac{1}{k!} \sum_{I \in \mathcal{I}_{n,k}} t_{i_1} \cdots t_{i_k} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} t_{i_1} \cdots t_{i_k}$$

for $1 \leq k \leq n$, and the random variables

$$\mathbf{Z}_k = \begin{cases} 1 \text{ if } k \leq \mathbf{N}, \\ 0 \text{ if } k > \mathbf{N}, \end{cases} \qquad \mathbf{N}^{(n)} = \sum_{k=1}^{n} \mathbf{Z}_k,$$

$$\mathbf{S}_{r,n} = \sum_{k=1}^{n} \mathbf{Y}_{r,k} \mathbf{Z}_k.$$

We will first need the following basic identities.

Lemma 6. *Assume* $1 \leq k \leq n$. *Then we have*

$$\pi_k(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \pi_1(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \mathbf{N}^{(n)}, \tag{38}$$

$$k!\, \sigma_{k,n}(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \mathbf{N}^{(n)}(\mathbf{N}^{(n)} - 1) \cdots (\mathbf{N}^{(n)} - k + 1). \tag{39}$$

Proof. Equality (38) is a straightforward consequence of $\mathbf{Z}_i^k = \mathbf{Z}_i$ if k and i are positive integers. For (39) we will use some of the well-known properties of Bell polynomials; see

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e.g. [1], specifically formula [3i], the last equation of §9, and [5d–e]. Let us fix n and use the abbreviation $\pi_k = \pi_k(\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$, so that $\pi_k = \mathbf{N}^{(n)}$ for every $k \geq 1$. Calculations give

$$\begin{aligned}
&k!\,\sigma_{k,n}(\mathbf{Z}_1, \ldots, \mathbf{Z}_n)\\
&\quad = (-1)^k k!\,\sigma_{k,n}(-\mathbf{Z}_1, \ldots, -\mathbf{Z}_n)\\
&\quad = (-1)^k B_k(-\pi_1, -\pi_2, \ldots, -(k-1)!\pi_k)\\
&\quad = (-1)^k \sum_{j=1}^{k} B_{k,j}(-\mathbf{N}^{(n)}, \ldots, -(k-j+1)!\,\mathbf{N}^{(n)})\\
&\quad = (-1)^k \sum_{j=1}^{k} (-1)^j B_{k,j}(\mathbf{N}^{(n)}, \ldots, (k-j+1)!\,\mathbf{N}^{(n)})\\
&\quad = \sum_{j=1}^{k} (-1)^{j+k} (\mathbf{N}^{(n)})^j\, B_{k,j}(0!, \ldots, (k-j+1)!)\\
&\quad = \sum_{j=1}^{k} s(k,j)(\mathbf{N}^{(n)})^j = \mathbf{N}^{(n)}(\mathbf{N}^{(n)}-1)\cdots(\mathbf{N}^{(n)}-k+1),
\end{aligned}$$

and the result is proved. □

We are now equipped to compute the product-mixed moments.

Theorem 5. *With the notation above we have*

$$\mathbb{E}\big(\mathbf{S}_r^d \mathbf{S}_{r'}^{d'}\big) = \sum_{k=1}^{d} \sum_{k'=1}^{d'} p_r^k\, p_{r'}^{k'}\, \mu_{[k+k']}(\mathbf{N}) B_{d,k}(\mu_1, \ldots, \mu_{d-k+1})\, B_{d',k'}(\mu_1, \ldots, \mu_{d'-k'+1}), \tag{40}$$

and more generally, for $J \in \{2, \ldots, R\}$,

$$\mathbb{E}\bigg(\prod_{j=1}^{J} \mathbf{S}_{r_j}^{d_j}\bigg) = \sum_{k_1=1}^{d_1} \cdots \sum_{k_J=1}^{d_J} \bigg\{ \mu_{\left[\sum_{j=1}^{J} k_j\right]}(\mathbf{N}) \prod_{j=1}^{J} p_{r_j}^{k_j}\, B_{d_j,k_j}(\mu_1, \ldots, \mu_{d_j-k_j+1}) \bigg\}. \tag{41}$$

Proof. We shall only prove (40), the mechanism being easily extended to prove (41). Let $\mathbf{X}_{r,i} = \mathbf{Y}_{r,i}\mathbf{Z}_i$, $i = 1, 2 \ldots$. First note that for $(i_1, \ldots, i_k) = I \in \mathcal{I}_{d,k}$,

$$\begin{aligned}
\mathbb{E}\mathbf{X}_{r,i_1}^{\ell_1} \cdots \mathbf{X}_{r,i_k}^{\ell_k} &= \mathbb{E}\big(\mathbf{Y}_{r,i_1}^{\ell_1} \cdots \mathbf{Y}_{r,i_k}^{\ell_k}\, \mathbf{Z}_{i_1}^{\ell_1} \cdots \mathbf{Z}_{i_k}^{\ell_k}\big)\\
&= \mathbb{E}\hat{\mathbf{Y}}_r^{\ell_1} \cdots \mathbb{E}\hat{\mathbf{Y}}_r^{\ell_k} \mathbb{E}(\mathbf{Z}_{i_1} \cdots \mathbf{Z}_{i_k})\\
&= p_r^k\, \mathbb{E}\mathbf{Y}^{\ell_1} \cdots \mathbb{E}\mathbf{Y}^{\ell_k} \mathbb{E}(\mathbf{Z}_{i_1} \cdots \mathbf{Z}_{i_k})\\
&= p_r^k\, \mu_L(\mathbf{Y})\, \mathbb{E}(\mathbf{Z}_{i_1} \cdots \mathbf{Z}_{i_k}).
\end{aligned}$$

Thus, by using (37), we obtain

$$\begin{aligned}
\mathbb{E}\mathbf{S}_{r,n}^d \mathbf{S}_{r',n}^{d'} &= \mathbb{E}(\mathbf{X}_{r,1} + \cdots + \mathbf{X}_{r,n})^d (\mathbf{X}_{r',1} + \cdots + \mathbf{X}_{r',n})^{d'}\\
&= \mathbb{E}\bigg(\sum_{k=1}^{d} \sum_{L \in \mathcal{L}_{d,k}} c_L \sum_{I \in \mathcal{I}_{n,k}} \mathbf{X}_{r,i_1}^{\ell_1} \cdots \mathbf{X}_{r,i_k}^{\ell_k}\bigg) \times \bigg(\sum_{k'=1}^{d'} \sum_{L' \in \mathcal{L}_{d',k'}} c_{L'} \sum_{I' \in \mathcal{I}_{n,k'}} \mathbf{X}_{r',i'_1}^{\ell'_1} \cdots \mathbf{X}_{r',i'_{k'}}^{\ell'_{k'}}\bigg).
\end{aligned}$$

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On one hand, if $I \cap I' \neq \emptyset$, then

$$\mathbf{X}_{r,i_1}^{\ell_1} \cdots \mathbf{X}_{r,i_k}^{\ell_k} \mathbf{X}_{r,i'_1}^{\ell'_1} \cdots \mathbf{X}_{r,i'_{k'}}^{\ell'_{k'}} = 0$$

because for each k, $\mathbf{Y}_{r,k}\mathbf{Y}_{r',k} = 0$. Thus

$$\begin{aligned}
&\mathbb{E}\mathbf{S}_{r,n}^{d}\mathbf{S}_{r',n}^{d'} \\
&= \sum_{k=1}^{d} \sum_{L \in \mathcal{L}_{d,k}} \sum_{k'=1}^{d'} \sum_{L' \in \mathcal{L}_{d',k'}} \sum_{I \in \mathcal{I}_{n,k}, I' \in \mathcal{I}_{n,k'}, I \cap I' = \emptyset} c_L c_{L'} \mathbb{E}\mathbf{X}_{r,i_1}^{\ell_1} \cdots \mathbf{X}_{r,i_k}^{\ell_k} \mathbf{X}_{r',i'_1}^{\ell'_1} \cdots \mathbf{X}_{r',i'_{k'}}^{\ell'_{k'}} \\
&= \sum_{k=1}^{d} \sum_{L \in \mathcal{L}_{d,k}} \sum_{k'=1}^{d'} \sum_{L' \in \mathcal{L}_{d',k'}} p_r^k c_L \mu_L(\mathbf{Y}) p_{r'}^{k'} c_{L'} \mu_{L'}(\mathbf{Y}) \mathbb{E} \sum_{I \in \mathcal{I}_{n,k}, I' \in \mathcal{I}_{n,k'}, I \cap I' = \emptyset} \mathbf{Z}_{i_1} \cdots \mathbf{Z}_{i_k} \mathbf{Z}_{i'_1} \cdots \mathbf{Z}_{i'_{k'}} \\
&= \sum_{k=1}^{d} \sum_{L \in \mathcal{L}_{d,k}} \sum_{k'=1}^{d'} \sum_{L' \in \mathcal{L}_{d',k'}} p_r^k c_L \mu_L(\mathbf{Y}) p_{r'}^{k'} c_{L'} \mu_{L'}(\mathbf{Y}) (k+k')! \mathbb{E}\sigma_{k+k',n}(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) \\
&= \sum_{k=1}^{d} \sum_{L \in \mathcal{L}_{d,k}} \sum_{k'=1}^{d'} \sum_{L' \in \mathcal{L}_{d',k'}} p_r^k p_{r'}^{k'} c_L c_{L'} \mu_L(\mathbf{Y}) \mu_{L'}(\mathbf{Y}) \mu_{[k+k']}(\mathbf{N}^{(n)}).
\end{aligned}$$

The claimed result is obtained as n tends to infinity by the dominated convergence theorem and the increasing convergence $0 \leq \mathbf{S}_{r,n}^{d}\mathbf{S}_{r',n}^{d'} \nearrow \mathbf{S}_r^d \mathbf{S}_{r'}^{d'}$. □

The sufficient condition for Theorem 1 (i) is now within our reach.

Corollary 4. *If* $\mathbf{N}$ *follows a Poisson distribution, then the random variables* $\mathbf{S}_r$, $r = 1, \ldots, R$ *are mutually independent.*

Proof. For the Poisson law, (33) holds. Write the product whose factors are given by formula (34) for the values $r_1, \ldots, r_J$, with $\mu_{[k_j]}(\mathbf{N}) = m^{k_j}$; then write (41) with

$$\mu_{[\sum_{j=1}^{J} k_j]}(\mathbf{N}) = m^{\sum_{j=1}^{J} k_j}.$$

It is clear that (36) is satisfied, which proves the claimed result. □

The proof of Theorem 1 (ii) is the object of the following result.

Corollary 5. *The random variables* $\mathbf{S}_r$ *and* $\mathbf{S}_{r'}$ *are identically distributed if and only if* $p_r = p_{r'}$.

Proof. The two random variables are determined by their moments, so they are identically distributed if and only if all their moments are equal. Let us use Theorem 3. If $p_r = p_{r'}$, then $\mu_d(\mathbf{S}_r) = \mu_d(\mathbf{S}_{r'})$ for every d. Conversely, if $\mu_d(\mathbf{S}_r) = \mu_d(\mathbf{S}_{r'})$ for every d, then $d = 1$ yields

$$\mu_1(\mathbf{S}_r) = p_r \mu_{[1]}(\mathbf{N}) B_{1,1}(\mu_1) = p_{r'} \mu_{[1]}(\mathbf{N}) B_{1,1}(\mu_1) = \mu_1(\mathbf{S}_{r'}),$$

so $p_r = p_{r'}$ because $\mu_{[1]}(\mathbf{N}) B_{1,1}(\mu_1) = \mu_{[1]}(\mathbf{N}) \mu_1(\mathbf{Y}) > 0$. This proves our result. □

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