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Dynamics of waves and multidimensional solitons of the Zakharov–Kuznetsov equation: II. Higher-order calculation; nonlinear development

E. INFELD and A. A. SKORUPSKI
Soltan Institute for Nuclear Studies, Hoża 69, 00–681 Warsaw, Poland

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Abstract. The Zakharov–Kuznetsov equation describes the propagation of weak ion acoustic waves in a strongly magnetized plasma. Their dynamics have been studied in a series of papers, one of which gives growth rates of instabilities found numerically, as well as pictures of soliton collisions [*J. Plasma Phys.* **64**, 397 (2000) – Part I]. In the present paper, we find good approximate formulas for growth rates of the dominant instability, vastly improving those of Part I. This is done by proceeding to higher order in the expansion, combined with an incorporation of exact values for the boundaries of the unstable region in the formulas. The result is better than we had any right to expect. We next depart from linear stability analysis and look at nonlinear dynamics to obtain a pulse in time. The maximum amplitude of this pulse is seen to be proportional to the linear growth rate, a result that was so far suspected from numerics but not derived theoretically. (This paper can be read independently of Part I.)

Dedication

This work is dedicated to John Dougherty on the occasion of his official retirement. Knowing John, this formality should not limit his versatile activities too much!

1. Introduction

The Zakharov–Kuznetsov equation, describing the propagation of weak ion acoustic waves in a strongly magnetized plasma, is (Zakharov and Kuznetsov 1974)

$$n_t + nn_x + (\Delta n)_x = 0, \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{1.1}$$

This equation can be solved in one space dimension, yielding nonlinear travelling waves, functions of $x - vt$. In the coordinate system of the wave, $n \to n - v$, $x \to x - vt$, the solution is

$$n_0(x, m) = 4(m + 1) - 12\, m\, \mathrm{sn}^2(x|m), \quad 0 \leqslant m \leqslant 1. \tag{1.2}$$

When $m > 1$, we find a solution by rescaling so as to obtain $-12\,\mathrm{sn}^2(m^{1/2}x|m^{-1})$ as the second term. Here sn is the Jacobi elliptic function. Its period is $4K(m)$. As

m approaches one, $K \to \infty$ and the nonlinear wave becomes a soliton train. For $m = 1$, we in fact obtain a soliton

$$n_0 = -4 + 12\,\mathrm{sech}^2 x, \tag{1.3}$$

the -4 term being introduced by the motion of the system. We recover the more familiar form by returning to the original frame, where $n_0 = 12\,\mathrm{sech}^2(x - 4t)$.

2. Small-m theory

We now perturb (1.2) and linearize:

$$n = n_0 + \tilde{\delta} n\, e^{(\gamma t + iK_x x + iK_y y)}. \tag{2.1}$$

The strongest instability was shown in Infeld et al. (2000; henceforth referred to as Part I) to be associated with wavelength doubling, $K_x = K_0/2$, where K_0 is the wavelength of the basic structure (1.2). We begin the expansion in m by taking

$$\tilde{\delta} n_1 = a\cos(\xi + \alpha), \quad \xi = x(1 + \tfrac{1}{4}m + \tfrac{9}{64}m^2)^{-1} \simeq \frac{\pi x}{2K(m)}. \tag{2.2}$$

The expansion in m, taken to one order higher than in Part I, then yields (Appendix)

$$\gamma = (1 - \tfrac{1}{4}m)\sqrt{(3 + \tfrac{3}{2}m + \tfrac{15}{32}m^2 - K_y^2)(K_y^2 - 3 + \tfrac{9}{2}m - \tfrac{15}{32}m^2)}. \tag{2.3}$$

The unstable region is bounded by

$$K_{y1}^2 = 3 - \tfrac{9}{2}m + \tfrac{15}{32}m^3, \quad K_{y2}^2 = 3 + \tfrac{3}{2}m + \tfrac{15}{32}m^2. \tag{2.4}$$

Obviously, this calculation, though improved, is still only useful for small m, as K_{y1}^2 becomes negative before $m = 1$.

3. Exact limits

We have been able to solve (1.1) and (2.1) for $\gamma = 0$ at four values of K_y^2. The two that interest us, corresponding to $K_x = K_0/2$, are

$$\left.\begin{aligned} K_{y1\,\mathrm{ex}}^2 &= -(m+1) + \sqrt{(m+1)^2 + 15(m-1)^2}, \\ K_{y2\,\mathrm{ex}}^2 &= 2m - 1 + \sqrt{(2m-1)^2 + 15}, \end{aligned}\right\} \tag{3.1}$$

yielding K_{y1}^2 and K_{y2}^2, respectively, for small m, as they should. No contradiction with the above small-m calculation will therefore be introduced if we replace (2.3) by

$$\gamma = (1 - \tfrac{1}{4}m)\sqrt{(K_{y2\,\mathrm{ex}}^2 - K_y^2)(K_y^2 - K_{y1\,\mathrm{ex}}^2)}. \tag{3.2}$$

We have therefore been able to improve on the calculation of Part I by combining two very different results:

(i) following the m expansion to next order;

(ii) incorporating the exact limits of the unstable region.

4. Comparison with numerics

We calculated γ numerically in Part I, by using iterations similar to those described in Infeld et al. (2002). Comparisons of our formula (3.2) with γ obtained from simulations are presented for six values of m in Fig. 1. (The first three also reproduce results for the $K_x = 0$, Benjamin–Feir instability from Part I. Here no progress has been made.) The agreement is excellent up to $m = 0.5$, and good even up to $m = 0.99$. For the soliton, $m = 1$, the solid line represents the numerical values taken from Infeld and Rowlands (2000). Here, since the wavelength is infinite, the two instabilities merge. Comparison with the results of Part I reveals just how improved (3.2) is.

5. Nonlinear behaviour of the perturbation

We will now use our result to help us look at nonlinear dynamics. Equation (1.1) yields, assuming $n = n_0 + \delta n$,

$$\frac{\partial \delta n}{\partial t} + \frac{\partial^3}{\partial x^3}\delta n + \frac{\partial}{\partial x}\{[4(m+1) - 12m\,\mathrm{sn}^2(x|m)]\delta n + \tfrac{1}{2}(\delta n)^2\} + \frac{\partial^3}{\partial x \partial y^2}\delta n = 0, \quad (5.1)$$

with the understanding that $K_x = K_0/2$. The above linear analysis follows when $(\delta n)^2$ is neglected. However, we now look for solutions in the form (m is still small, and ξ is defined in (2.2))

$$\delta n = a\cos(\xi + \alpha_1)\cos(K_y y) + b\cos(2\xi + \alpha_2) + c, \quad (5.2)$$

and search for solutions such that a, b, c and the αs are functions of time and

$$\lim_{a\to 0} \frac{a_t}{a} = \gamma. \quad (5.3)$$

Here γ is taken to be given by (3.2). We find from straightforward calculations that no generality is lost by taking $\alpha_1 = \alpha_2 \equiv \alpha$, and then

$$\begin{aligned} 3am\sin 2\alpha + a_t &= 0, \\ [3m(\cos 2\alpha - 1) - \delta K_y^2 + \alpha_t + c + \tfrac{1}{2}b]a &= 0, \\ b_t + 12mc\sin 2\alpha &= 0, \\ b(\alpha_t + c) + \tfrac{1}{8}a^2 + 6mc\cos 2\alpha &= 0, \end{aligned}$$

where

$$\delta K_y^2 = K_y^2 - 3 - \tfrac{3}{2}m,$$

yielding (2.3) to lower order (without the $-\frac{1}{4}m$ and $\frac{15}{32}m^2$ terms) when $b = c = \alpha_t = 0$ ((5.1) of Part I). One equation for a can be derived from the four in a, b, c, α, when condition (5.3) is satisfied:

$$\left(\frac{da}{dt}\right)^2 = \gamma^2 a^2 - \tfrac{1}{8}a^4, \quad (5.4)$$

where here γ is a constant of integration. The solution is

$$a = \pm\sqrt{8}\gamma\,\mathrm{sech}[\gamma(t - t_0)]. \quad (5.5)$$

where we take γ to be given by (3.2), for reasons explained in Sect. 3. Equation (5.5) describes a pulse driven by an initially linear instability. Importantly, $a_{\max} \propto \gamma$, a result suggested in Casali et al. (1998) as following from numerics. This is now confirmed theoretically. More generally, this argues for recognition of a subclass of Landau-type models in which there is one parameter instead of the usual two (5.4) (a second parameter usually multiplies the higher-order term – Infeld and Rowlands 2000).

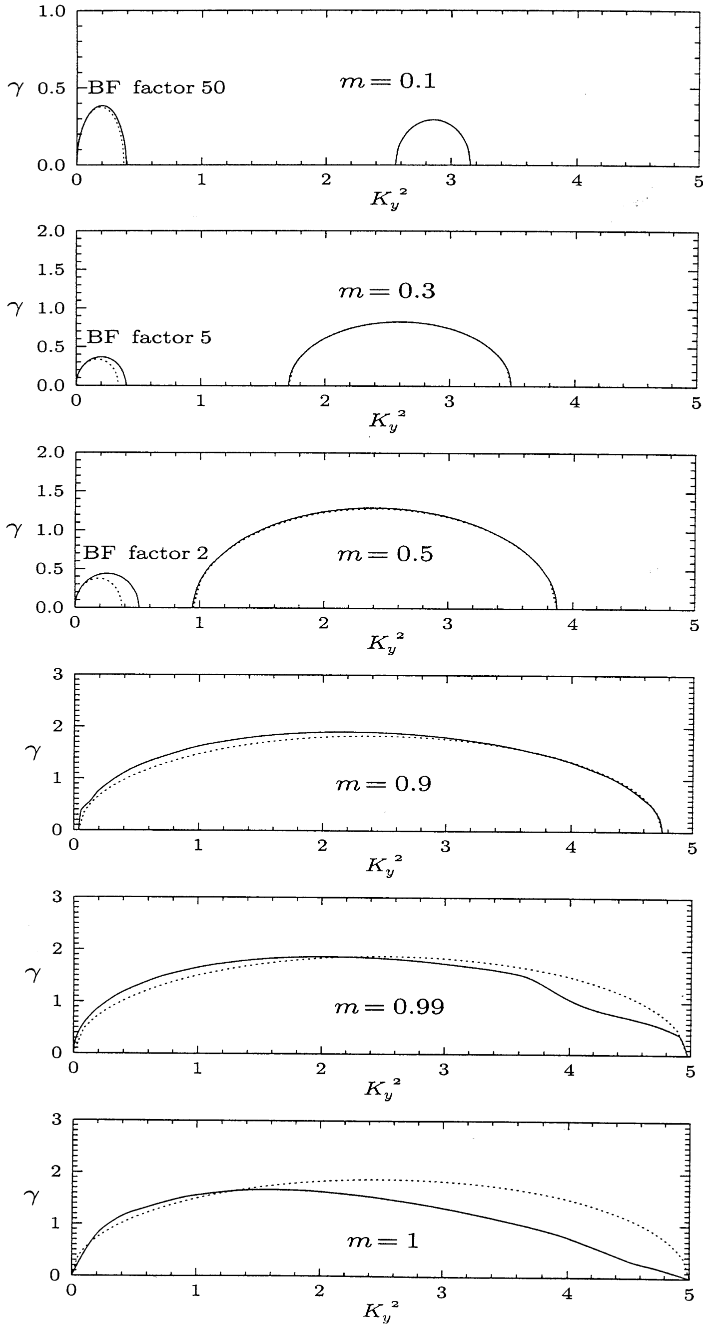

Figure 1. Growth rate γ versus K_y^2 for both the Benjamin–Feir (BF) and wavelength-doubling instabilities. Numerical results are designated by solid lines, our models by broken lines. The graphs corresponding to the BF instability are blown up as indicated (by a factor of 50 for $m = 0.1$, etc.).

The appearance of a doubly space-periodic pulse (5.2), (5.5) that disappears after a while does not tell the whole story. A 2D lump will meanwhile detach itself from each crest of this time-dependent wave at $\xi+\alpha = 2n\pi$, $y = 2m\pi K_y^{-1}$. It will proceed forward at a greater speed than its parent structure (5.2), (5.5) (Frycz et al. 1992; Pelinovsky and Stepanyants 1993; Infeld et al. 1995; Senatorski and Infeld 1998; Infeld and Skorupski 2000). The dynamics of one such lump have been investigated by Frycz et al. (1992). Collisions of two or more lumps were followed by Feng et al. (1999), Infeld et al. (2000) and Infeld and Skorupski (2000). A similar treatment of the Kadomtsev–Petviashvili equation will be presented elsewhere (Infeld et al. 2002). Once again, (5.4) is obtained, pointing at its generic character.

6. Conclusions

Taken jointly with Part I, this work yields a wide panorama of linear instabilities of wave solutions to the Zakharov–Kuznetsov equation. Models for growth rates are found. Corresponding numerical values are calculated and compared with these models. Agreement is surprisingly good for the dominant instability. Nonlinear development is followed by using a different, extended calculation. A refinement of Landau saturation is indicated.

Note added in proof

Since $a_{\max} \propto \gamma$, see (5.5), δn given by (5.2) is approximately proportional to γ (since b is of higher order than a). This result rather than $a_{\max} \propto \gamma$ was suggested by Casali et al. (1998) for KPI, and is now demonstrated by us for ZK by a simple theoretical argument.

Appendix

We assume m to be small, and expand in this quantity, essentially the amplitude of the nonlinear wave. Thus

$$\gamma = \gamma_1 + \gamma_2 + \ldots, \tag{A 1}$$

$$K_y^2 = 3 + K_{y1}^2 + K_{y2}^2 + \ldots, \tag{A 2}$$

$$\delta n = \delta n_1 + \delta n_2 + \ldots. \tag{A 3}$$

From (5.1), using (2.2), we obtain, up to and including third order,

$$\gamma(1 + \tfrac{1}{4}m)\partial_\xi^{-1}\delta n + (1 - \tfrac{1}{2} - \tfrac{3}{32}m^2)\partial_\xi^2\delta n + [4(1+m) - 12m(\sin^2\xi + \tfrac{1}{8}m\sin^2 2\xi)]\delta n - K_y^2\delta n = 0. \tag{A 4}$$

Here we have expanded $\mathrm{sn}^2(x|m)$ up to *first* order.

First order yields

$$L\delta n_1 = 0, \quad L = \partial_\xi^2 + 1, \tag{A 5}$$

$$\delta n_1 = \cos(\xi + \alpha). \tag{A 6}$$

At second order, we obtain two conditions for avoidance of secular terms:

$$3m\sin 2\alpha = \gamma_1, \quad \text{avoids } \sin(\xi+\alpha), \tag{A 7}$$

$$3m\cos 2\alpha = 3m + [K_y^2 - (3 + \tfrac{3}{2}m)], \quad \text{avoids } \cos(\xi+\alpha). \tag{A 8}$$

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Squaring these two equations and adding them yields γ_1, equation (5.1) of Part I. When there are no secular terms, all that is left is

$$L\delta n_2 = -3m\cos(3\xi + \alpha), \qquad \text{(A 9)}$$

$$\delta n_3 = \tfrac{3}{8}m\cos(3\xi + \alpha). \qquad \text{(A 10)}$$

We use this at third order, (A 4). Removal of secular terms now demands

$$\gamma_2 + \tfrac{1}{4}m\gamma_1 = 0, \qquad \text{(A 11)}$$

$$K_{y2}^2 - \tfrac{15}{32}m^2 = 0. \qquad \text{(A 12)}$$

When these are added to the left-hand sides of (A 7) and (A 8), we obtain

$$3m\sin 2\alpha = (1 + \tfrac{1}{4}m)\gamma, \qquad \text{(A 13)}$$

$$3m\cos 2\alpha = 3m + K_y^2 - (3 + \tfrac{3}{2}m + \tfrac{15}{32}m^2), \qquad \text{(A 14)}$$

where γ and K_y^2 are taken up to second order. If we drop the subscripts, square and add as before, we obtain (2.3). Similar improvement for the Benjamin–Feir instability would demand a formidable, fourth-order calculation. Furthermore, the exact value of K_{y2}^2 is not known (a glance at Fig. 1 shows that, without this value, little progress can be made).

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