J. Appl. Prob. **56,** 1006–1019 (2019)
*doi:*10.1017/jpr.2019.58

PIECEWISE DETERMINISTIC PROCESSES FOLLOWING TWO ALTERNATING PATTERNS

NIKITA RATANOV,* ** *Universidad del Rosario*
ANTONIO DI CRESCENZO,*** **** AND
BARBARA MARTINUCCI,*** ***** *Università di Salerno*

Abstract

We propose a wide generalization of known results related to the telegraph process. Functionals of the simple telegraph process on a straight line and their generalizations on an arbitrary state space are studied.

Keywords: Telegraph process; rectifying diffeomorphism

2010 Mathematics Subject Classification: Primary 60J25
Secondary 60K15; 60K99

1. Motivation and problem settings

The aim of this paper is to study some examples of a continuous-time stochastic process with deterministic behaviour between random switching times, the so-called piecewise deterministic process with continuous paths.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and let $\varepsilon = (\varepsilon(t))_{t\geq 0}$ be an arbitrary measurable and adapted process defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$ with values in a finite space $\{1, \ldots, N\}$. Let $\phi_1, \ldots, \phi_N$ be N deterministic flows in a phase space $(G, \mathcal{G})$, where we assume that G is a topological space and $\mathcal{G}$ is the Borel σ-algebra. Let $\{\tau_n\}_{n\geq 1}$ be the sequence of switching times of ε. The piecewise deterministic process $\mathbb{X}$ is defined as

$$\mathbb{X}(t) = \phi_{\varepsilon(\tau_n)}(t), \qquad \tau_n \leq t < \tau_{n+1}.$$

The family of piecewise deterministic processes was introduced in [4], and a subclass of piecewise linear processes was first studied in [10]. This important class of random processes was then thoroughly studied in [5]; see [11] for a modern presentation. Piecewise deterministic processes are intensively exploited in biology [18], insurance [8], storage models [3], financial market modelling [16], and in many other fields.

To simplify our presentation we restrict ourselves to switchings driven by a Markov process with only two values (states). The simplest example of such a process is a piecewise linear (telegraph) process based on the two-state Markov process $\varepsilon = \varepsilon(t) \in \{0, 1\}$:

$$T(t) = V(0) \int_0^t (-1)^{N(\tau)}\, \mathrm{d}\tau, \qquad t > 0, \tag{1.1}$$

Received 5 November 2018; revision received 23 June 2019.
* Postal address: Facultad de Economía, Universidad del Rosario, Calle 12c, No. 4-69, Bogotá, D. C. Cundinamarca, Colombia.
** Email address: nratanov@urosario.edu.co
*** Postal address: Dipartimento di Matematica, Università di Salerno, 84084 Fisciano (SA), Italia.
**** Email address: adicrescenzo@unisa.it
***** Email address: bmartinucci@unisa.it

driven by a homogeneous Poisson process $N = N(t)$. The value $T(t)$ corresponds to the position of a particle moving on the line with velocities -1 and $+1$ alternating at Poisson times. The random starting velocity $V(0) \in \{-1, +1\}$ is independent of N.

The theory of telegraph processes is well developed, beginning from [12]. Over the past few decades, many generalizations of the telegraph process have been proposed in the literature including motions characterized by arbitrary numbers of possible velocities [13], by random velocities [24, 6], with velocity changes governed by an alternating renewal process (for instance [7] or perturbed by jumps [23, 19]). See also the monograph [16] and the references therein for full details on the telegraph process.

The classic telegraph model (1.1) can be easily generalized to the process $T(t)$ of inhomogeneous structure with velocities c_0 and c_1, $c_0 > c_1$, alternating with intensities λ_0 and λ_1 respectively. The distribution of the random variable $T(t)$ is given hereafter.

Let

$$f_i(x, t; n) = \mathrm{P}\{T(t) \in \mathrm{d}x, N(t) = n \mid \varepsilon(0) = i\}/\mathrm{d}x, \qquad n \geq 1,\ i \in \{0, 1\},$$

be the density function of $T(t)\mathbf{1}_{\{N(t)=n\}}$. Note that

$$\mathrm{P}\{T(t) \in \mathrm{d}x, N(t) = 0 \mid \varepsilon(0) = i\} = \mathrm{e}^{-\lambda_i t}\delta_{c_i t}(\mathrm{d}x), \qquad i \in \{0, 1\},$$

where $\delta_z(\cdot)$ denotes Dirac's delta-measure on a line throughout the paper.

Proposition 1.1. *The distribution of $T(t)$, $t > 0$, is described by*

$$\begin{gathered} f_i(x, t; n) = q_i(\xi, t-\xi; n)\theta(\xi, t-\xi), \\ \xi = \xi(x) = \frac{x - c_1 t}{c_0 - c_1}, \quad t - \xi = \frac{c_0 t - x}{c_0 - c_1}. \end{gathered} \tag{1.2}$$

Here, $q_i(\xi, \eta; n)$, $i \in \{0, 1\}$, $n \geq 1$, are separately defined for even and odd n by the equalities

$$\begin{gathered} q_0(\xi, \eta; 2k) = \frac{\lambda_0^k \lambda_1^k}{(k-1)!k!}\xi^k \eta^{k-1}, \qquad q_1(\xi, \eta; 2k) = \frac{\lambda_0^k \lambda_1^k}{(k-1)!k!}\xi^{k-1}\eta^k, \\ q_0(\xi, \eta; 2k+1) = \frac{\lambda_0^{k+1}\lambda_1^k}{k!^2}\xi^k\eta^k, \qquad q_1(\xi, \eta; 2k+1) = \frac{\lambda_0^k\lambda_1^{k+1}}{k!^2}\xi^k\eta^k, \\ \xi, \eta > 0, \end{gathered} \tag{1.3}$$

and

$$\theta(\xi, \eta) := \frac{\exp(-\lambda_0\xi - \lambda_1\eta)}{c_0 - c_1}\mathbf{1}_{\{\xi>0,\ \eta>0\}}. \tag{1.4}$$

For the proof, see, e.g., [16, Proposition 4.1]. In the following, Proposition 1.1 will be generalized to the case of a piecewise linear process in an arbitrary linear normed space; see Section 2.1.

The paper is structured as follows. In Section 2 piecewise deterministic flows are studied. After recalling some elementary properties of basic deterministic flows, Section 2 is divided into two main parts: Section 2.1 regarding the distribution of the telegraph process $\mathbb{T}(t)$, $t \geq 0$, in a normed vector space, and Section 2.2 where we study the time-homogeneous process $\mathbb{X}$ defined as $\mathbb{X}(t) = \Phi^{-1}(\Phi(x) + \mathbb{T}(t))$, $t \geq 0$ (with Φ a continuous injection defined on the state space of the process $\mathbb{X}$). In Section 3 we present two examples: a one-dimensional (1D) squared telegraph process and a two-dimensional process with alternating radial and circular movements. In Section 4 some observations concerning self-similarity are presented.

2. Piecewise deterministic flows

Consider the phase space $(G, \mathcal{G})$ where G is a topological space with the Borel σ-algebra $\mathcal{G}$. For any fixed $x \in G$ consider a *continuous flow* on G,

$$t \to \phi(t \mid x, s) \in G, \qquad t, s \in (-\infty, \infty),\ t > s,$$

starting at time s from position $x \in G$: $\phi(t \mid x, s)|_{t\downarrow s} = x = \phi(t \mid x, s)|_{s\uparrow t}$. Assume that for any $s, t, s < t$, the mapping $x \to \phi^{ts}(x) = \phi(t \mid x, s)$, $t > s$, is a homeomorphism.

Assume that ϕ^{ts} as well as the inverse mapping (the reverse flow) form a two-parameter semigroup under composition; see, e.g., [11].

In the following we will study piecewise deterministic flows consistently switching between two alternating patterns $\phi_0(t \mid \cdot\,)$ and $\phi_1(t \mid \cdot\,)$ at random times.

Let x denote the state of the process at initial time s, and let $t > s$. Consider two continuous functions $\tau \to g_0(\tau)$, $\tau \to g_1(\tau)$, $\tau \in [s, t]$, which are defined by iterated superposition of these two flows:

$$g_0(\tau) = \phi_1(t \mid \phi_0(\tau \mid x, s), \tau), \qquad g_1(\tau) = \phi_0(t \mid \phi_1(\tau \mid x, s), \tau), \qquad s \le \tau \le t. \tag{2.1}$$

These functions determine the pieces of continuous curves $\ell_0 = \ell_0(x)$ and $\ell_1 = \ell_1(x)$ on the space G,

$$\ell_0 = \{y \in G \mid y = g_0(\tau),\ \tau \in [s, t]\}, \qquad \ell_1 = \{y \in G \mid y = g_1(\tau),\ \tau \in [s, t]\}. \tag{2.2}$$

For any target point $y \in \ell_0(x)$, the time $\tau_0^*(y; x)$ when the flow is switched from ϕ_0 to ϕ_1 exists and is unique. Indeed, the equation $g_0(\tau) = y$ has the unique solution $\tau = \tau_0^*(y; x) \in [s, t]$. Similarly, $\tau_1^*(y; x) \in [s, t]$, $y \in \ell_1(x)$, is defined as the root of the equation $y = g_1(\tau)$.

Further, the stochastic switching mechanism between two deterministic flows ϕ_0 and ϕ_1 is defined by a two-state random process $\varepsilon = \varepsilon(t) \in \{0, 1\}$, $t \in (-\infty, \infty)$, with independent inter-switching times.

Let $s \in (-\infty, \infty)$ be the (fixed) starting time, and let τ^s be the first switching time after s, $\tau^s > s$. Denote by $F_i^s(t) = \mathrm{P}_i\{\tau^s < t\} = \mathrm{P}\{\tau^s < t \mid \varepsilon(s) = i\}$ the (conditional) distribution function of τ^s under the given initial state $\varepsilon(s) = i$, $i \in \{0, 1\}$. That is,

$$\mathrm{P}\{\varepsilon(t') = i \text{ for all } t' \in (s, t) \mid \varepsilon(s) = i\} = 1 - F_i^s(t), \qquad t > s.$$

We study the marginal distributions of the piecewise deterministic continuous random walk $\mathbb{X} = \mathbb{X}(t)$ on the topological space G which follows two patterns ϕ_0 and ϕ_1 alternating at switching times of ε. Let $N = N(s, t)$ count the number of switches of $\varepsilon(\cdot)$ during the time interval $[s, t)$.

By conditioning on the first pattern's switching, one can observe that the transition probabilities $P_i(A, t; n \mid x, s) := \mathrm{P}\{\mathbb{X}(t) \in A, N(s, t) = n \mid \mathbb{X}(s) = x, \varepsilon(s) = i\}$, $n \ge 0$, $i \in \{0, 1\}$, of $\mathbb{X}(t)$, $t > s$, satisfy the following coupled integral Chapman–Kolmogorov equations for $t > s$:

$$\begin{cases} P_0(\,\cdot\,, t; n \mid x, s) = \displaystyle\int_s^t P_1(\,\cdot\,, t; n-1 \mid \phi_0(\tau \mid x, s), \tau)\, \mathrm{d}F_0^s(\tau), \\ P_1(\,\cdot\,, t; n \mid x, s) = \displaystyle\int_s^t P_0(\,\cdot\,, t; n-1 \mid \phi_1(\tau \mid x, s), \tau)\, \mathrm{d}F_1^s(\tau), \end{cases} \quad n \ge 1. \tag{2.3}$$

The distribution of $\mathbb{X}(t)$ with no switchings till time t is given by

$$\begin{aligned} P_0(A, t; 0 \mid x, s) &= (1 - F_0^s(t))\, \delta_{\phi_0(t|x,s)}(A), \\ P_1(A, t; 0 \mid x, s) &= (1 - F_1^s(t))\, \delta_{\phi_1(t|x,s)}(A). \end{aligned} \tag{2.4}$$

In the following we consider in detail the *Markovian* case, that is,

$$F_i^s(t) = \mathrm{P}_i\{\tau^s < t\} = 1 - \mathrm{e}^{-\lambda_i(t-s)}, \qquad t \geq s, \ i \in \{0, 1\},$$

with $\lambda_0, \lambda_1 > 0$.

We begin with the example of a random walk $\mathbb{T}(t)$ that follows a *linear flow in a linear normed space*.

2.1. Piecewise linear processes in a linear normed space

Let V be a linear normed vector space and $c_0, c_1 \in V$, $c_0 \neq c_1$. We consider the linear time-homogeneous case, where $\mathbb{T} = \mathbb{T}(t)$, $t \geq 0$, is the piecewise linear process (the integrated telegraph process) on the space V, switching between two linear flows

$$\phi_0(t \mid x, s) = x + tc_0, \qquad \phi_1(t \mid x, s) = x + tc_1.$$

The current position $\mathbb{T}(t)$ is given by

$$\mathbb{T}(t) := \int_0^t c_{\varepsilon(\tau)}\, \mathrm{d}\tau = \sum_{n=0}^{N(t)-1} c_{\varepsilon_n}(\tau_{n+1} - \tau_n) + c_{\varepsilon_{N(t)}}(t - \tau_{N(t)}), \qquad t \geq 0, \tag{2.5}$$

where τ_n, $n \geq 0$, are the switching times, $\tau_0 = 0$, $\varepsilon_n = \varepsilon(\tau_n)$, $n \geq 0$, and $N(t)$ is the number of switchings occurring till time t, $t > 0$, $N(0) = 0$.

The distribution of $\mathbb{T}(t)$, $t > 0$, is supported on the straight segment $I_t \subset V$,

$$I_t = \{z \in V \mid z = \tau c_0 + (t - \tau)c_1, \ 0 \leq \tau \leq t\}. \tag{2.6}$$

Indeed, for any $z \in I_t$, we have $\mathbb{T}(t) = z = \tau c_0 + (t - \tau)c_1$, where $\tau \in [0, t]$ is the time spent by the underlying Markov process $\varepsilon(u)$, $0 \leq u \leq t$, in state 0.

Due to (2.3), the distribution densities

$$p_0^{\mathbb{T}}(z, t; n) := \mathrm{P}\{\mathbb{T}(t) \in \mathrm{d}z, N(t) = n \mid \varepsilon(0) = 0\}/\mathrm{d}z,$$
$$p_1^{\mathbb{T}}(z, t; n) := \mathrm{P}\{\mathbb{T}(t) \in \mathrm{d}z, N(t) = n \mid \varepsilon(0) = 1\}/\mathrm{d}z$$

follow the coupled integral equations

$$\begin{cases} p_0^{\mathbb{T}}(z, t; n) = \displaystyle\int_0^t \lambda_0 \mathrm{e}^{-\lambda_0 \tau} p_1^{\mathbb{T}}(z - \tau c_0, t - \tau; n - 1)\, \mathrm{d}\tau, \\ p_1^{\mathbb{T}}(z, t; n) = \displaystyle\int_0^t \lambda_1 \mathrm{e}^{-\lambda_1 \tau} p_0^{\mathbb{T}}(z - \tau c_1, t - \tau; n - 1)\, \mathrm{d}\tau, \end{cases} \qquad n \geq 1. \tag{2.7}$$

The case of no switchings, corresponding to $\mathbb{T}(t)\mathbf{1}_{N(t)=0}$, is given by

$$\begin{aligned} \mathrm{P}\{\mathbb{T}(t) \in \mathrm{d}z, N(t) = 0 \mid \varepsilon(0) = 0\} &= \exp(-\lambda_0 t)\delta_{tc_0}(\mathrm{d}z) \\ &= \exp(-\lambda_0 t)\delta(z - tc_0)\mathrm{d}z, \\ \mathrm{P}\{\mathbb{T}(t) \in \mathrm{d}z, N(t) = 0 \mid \varepsilon(0) = 1\} &= \exp(-\lambda_1 t)\delta_{tc_1}(\mathrm{d}z) \\ &= \exp(-\lambda_1 t)\delta(z - tc_1)\mathrm{d}z. \end{aligned} \tag{2.8}$$

In the particular case of linearly dependent vectors $c_0, c_1 \in V$, $c_0, c_1 \neq 0$, the random process $\mathbb{T} = \mathbb{T}(t)$ is one-dimensional and the distribution of $\mathbb{T}(t)$ is supported on the segment

I_t of the straight line L with direction vector c_0 (or c_1), $I_t \subset L \subset V$. Moreover, the density functions $p_0^{\mathbb{T}}(\,\cdot\,, t; n)$ and $p_1^{\mathbb{T}}(\,\cdot\,, t; n)$, $n \geq 1$, coincide with functions $f_0(\,\cdot\,, t; n)$ and $f_1(\,\cdot\,, t; n)$; see the formulae in (1.2) with ξ, $0 \leq \xi \leq t$, defined by the equation $z - tc_1 = \xi(c_0 - c_1)$, $z \in L$.

In general, the segment I_t given in (2.6) floats in V (with constant velocity $\frac{1}{2}(c_0 + c_1)$). By solving the equations in (2.7), the density functions $p_0^{\mathbb{T}}(z, t; n)$ and $p_1^{\mathbb{T}}(z, t; n)$, $n \geq 1$, can be shown to satisfy formulae similar to (1.2) with $\xi \in [0, t]$, which is defined as the (unique) solution $\xi = \varphi(z, t)$ of the equation

$$z - tc_1 = \xi(c_0 - c_1), \qquad z \in I_t. \tag{2.9}$$

Proposition 2.1. *The density functions* $p_0^{\mathbb{T}}(z, t; n)$ *and* $p_1^{\mathbb{T}}(z, t; n)$, $n \geq 1$, *are given by* $p_i^{\mathbb{T}}(z, t; n) = q_i(\xi, t - \xi; n)\theta(\xi, t - \xi)$, *where* $q_i(\xi, \eta; n)$ *are defined by* (1.3), *and the function* θ *is*

$$\theta(\xi, \eta) := \frac{1}{\|c_0 - c_1\|} \exp\left(-\lambda_0\xi - \lambda_1\eta\right)\mathbf{1}_{\{\xi>0,\eta>0\}}. \tag{2.10}$$

Here, $\xi = \varphi(z, t) \in [0, t]$, $z \in I_t$ *is the solution of* (2.9) *and* $\eta = t - \xi$.

See the proof in Appendix B.

2.2. Time-homogeneous piecewise deterministic process $\mathbb{X}$

Consider the time-homogeneous case, so that the deterministic pattern $\phi(t \mid x, s)$ depends on s, t through $t - s$ only. Assume that the flow ϕ is defined by

$$t \to \phi(t \mid x, s) = \Phi^{-1}(\Phi(x) + c(t - s)), \qquad t \geq s, \tag{2.11}$$

where $\Phi : G \to V$ is a continuous injective map from G to a topological vector space V and $c \in V$ is a constant. The reverse flow is defined by $s \to \Phi^{-1}(\Phi(y) - c(t - s))$, $s \leq t$.

In the following we will use the shortened notation

$$\phi(t; x) := \phi(t \mid x, 0).$$

Remark 2.1. Let $G = \mathbb{R}^d$, $V = \mathbb{R}^d$, and $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism. Therefore, the trajectory of ϕ defined by (2.11) is differentiable, $\Phi(\phi(t; x)) = \Phi(x) + ct$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}[\Phi(\phi(t; x))] \equiv c, \qquad t > 0.$$

This means that ϕ follows the differential equation

$$\frac{\mathrm{d}\phi(t; x)}{\mathrm{d}t} = a(\phi(t; x)), \qquad t > 0, \tag{2.12}$$

with the initial condition $\phi(t; x)|_{t\downarrow 0} = x$, where $a(y) = [\Phi'(y)]^{-1}c$.

The mapping Φ acts as a rectifying diffeomorphism for equation (2.12); see [1].

In the case when the time-homogeneous flows ϕ_0 and ϕ_1 are defined by (2.11) with $c_0, c_1 \in V$, $c_0 \neq c_1$, and are characterized by a *common rectifying mapping* $\Phi : G \to V$, that is,

$$\phi_0(t \mid x, s) = \Phi^{-1}(\Phi(x) + c_0(t - s)), \qquad \phi_1(t \mid x, s) = \Phi^{-1}(\Phi(x) + c_1(t - s)), \qquad t \geq s,$$

https://doi.org/10.1017/jpr.2019.58 Published online by Cambridge University Press

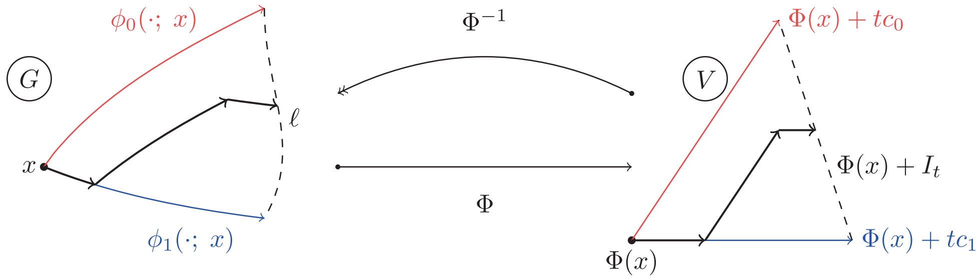

FIGURE 1: Flows $\phi_0(\,\cdot\,; x)$ and $\phi_1(\,\cdot\,; x)$ with common mapping $\Phi : G \to V$; a sample path of $\mathbb{X}^x(t)$.

the mappings g_0 and g_1 defined by (2.1) become

$$g_0(\tau) = \Phi^{-1}(\Phi(x) + c_0\tau + c_1(t-\tau)), \qquad \tau \in [0, t],$$
$$g_1(\tau) = \Phi^{-1}(\Phi(x) + c_1\tau + c_0(t-\tau)), \qquad \tau \in [0, t].$$

Hence, the curves ℓ_0 and ℓ_1 defined in (2.2) identify

$$\ell := \ell_0 = \ell_1 = \Phi^{-1}(\Phi(x) + I_t),$$

where I_t is the straight segment (2.6).

Let the time-homogeneous flows $\phi_0 = \phi_0(t; x)$ and $\phi_1 = \phi_1(t; x)$, $0 \le t < \infty$, be defined by (2.11) with a *common diffeomorphism* $\Phi : G \to V$ from the *open subset* G of a *linear normed space* into a linear normed space V, and with constant 'velocities' $c_0, c_1 \in V$, $c_0 \ne c_1$. Therefore, the corresponding piecewise deterministic time-homogeneous continuous process $\mathbb{X}^x = \mathbb{X}^x(t) \in G$ starting from point x is defined by

$$\mathbb{X}^x(t) = \Phi^{-1}(\Phi(x) + \mathbb{T}(t)), \quad 0 \le t < \infty; \qquad \mathbb{X}^x(0) = x. \tag{2.13}$$

Here, $\mathbb{T} = \mathbb{T}(t)$, $t \ge 0$, is the telegraph process defined by (2.5) with the two velocities $c_0, c_1 \in V$ alternating with switching intensities $\lambda_0, \lambda_1 > 0$.

For any fixed $t > 0$, the distribution of $\mathbb{T}(t)$ is supported on the straight segment $I_t \subset V$; see Proposition 2.1. Hence, the distribution of $\mathbb{X}^x(t)$ is supported on the segment of the continuous curve $\ell = \ell_{t,x}$, $\ell \subset G$, $\ell = \Phi^{-1}(\Phi(x) + I_t)$; see Figure 1.

Let $p_0^{\mathbb{X}}(y, t; n \mid x)$ and $p_1^{\mathbb{X}}(y, t; n \mid x)$ be the transition densities of $\mathbb{X}(t)$, $t > s$:

$$p_i^{\mathbb{X}}(y, t; n \mid x)\,\mathrm{d}y := \mathrm{P}\{\mathbb{X}^x(t) \in \mathrm{d}y, N(t) = n \mid \varepsilon(0) = i\}, \qquad i \in \{0, 1\},\ n = 0, 1, 2, \ldots$$

In the case of no switchings, $n = 0$, by (2.4) we have

$$p_0^{\mathbb{X}}(y, t; 0 \mid x) = \mathrm{e}^{-\lambda_0 t}\delta(y - \phi_0(t; x)), \qquad p_1^{\mathbb{X}}(y, t; 0 \mid x) = \mathrm{e}^{-\lambda_1 t}\delta(y - \phi_1(t; x)).$$

Theorem 2.1. *The transition densities $p_i^{\mathbb{X}}(y, t; n \mid x)$, $n \ge 1$, for each positive t are given by Proposition 2.1 with $\xi = \varphi(\Phi(y) - \Phi(x), t)$, see (2.9), and with θ given by*

$$\begin{aligned}\theta &= k(y)\exp\{-\lambda_0\xi - \lambda_1(t-\xi)\}\\ &= k(y)\exp\{-\lambda_0\varphi(\Phi(y) - \Phi(x), t) - \lambda_1(t - \varphi(\Phi(y) - \Phi(x), t))\},\end{aligned}$$

where $k(y) = \dfrac{\|\Phi'(y)\|}{\|c_0 - c_1\|}\mathbf{1}_{\{y\in\ell\}}$.

https://doi.org/10.1017/jpr.2019.58 Published online by Cambridge University Press

Further,

$$p_0^{\mathbb{X}}(y, t \mid x) = \mathrm{e}^{-\lambda_0 t}\delta(y - \phi_0(t; x)) + \theta\mathcal{P}_0(\xi, t - \xi; t),$$
$$p_1^{\mathbb{X}}(y, t \mid x) = \mathrm{e}^{-\lambda_1 t}\delta(y - \phi_1(t; x)) + \theta\mathcal{P}_1(\xi, t - \xi; t), \tag{2.14}$$

where

$$\mathcal{P}_0(\xi, \eta; t) = \lambda_0 I_0(2\sqrt{\lambda_0\lambda_1\xi\cdot\eta}) + \sqrt{\lambda_0\lambda_1\xi/\eta}\, I_1(2\sqrt{\lambda_0\lambda_1\xi\cdot\eta}),$$
$$\mathcal{P}_1(\xi, \eta; t) = \lambda_1 I_0(2\sqrt{\lambda_0\lambda_1\xi\cdot\eta}) + \sqrt{\lambda_0\lambda_1\eta/\xi}\, I_1(2\sqrt{\lambda_0\lambda_1\xi\cdot\eta}). \tag{2.15}$$

Proof. By (2.13),

$$\mathrm{P}_i\{\mathbb{X}(t) \in \mathrm{d}y \mid \mathbb{X}(0) = x\} = \mathrm{P}_i\{\Phi(x) + \mathbb{T}(t) \in \Phi(\mathrm{d}y)\}, \qquad i \in \{0, 1\}.$$

The proof follows from the result of Proposition 2.1. Summing over n one can obtain (2.14). □

The next section is related to other examples.

3. Examples

3.1. Squared telegraph process

First, we present the important example of the squared telegraph process,

$$\mathbb{X}(t) = \mathbb{X}^x(t) = (\sqrt{x} + T(t))^2, \qquad t > 0,$$

$\mathbb{X}^x(0) = x$, where the underlying telegraph process $T = T(t)$ is determined by velocities c_0, c_1, $c_0 > c_1$, and switching intensities λ_0, λ_1 (see (1.1)). Such a process can be obtained by (2.13), with $\Phi(x) = \sqrt{x}$, $x \geq 0$.

Although $x \to \Phi^{-1}(x) = x^2$, $x \in (-\infty, \infty)$, is not a diffeomorphism, Theorem 2.1 can be applied.

The density functions $p_i(\,\cdot\,, t; n \mid x)$, $n \geq 1$, of $\mathbb{X}^x(t)$ can be expressed using $f_0(x, t; n)$ and $f_1(x, t; n)$ defined in (1.2)–(1.4). The explicit expressions for $p_i(\,\cdot\,, t; n \mid x)$, $n \geq 1$, are different for the following four cases, defined by the four possible relationships between the parameters and the time value t, $t > 0$.

(A) $0 \leq \sqrt{x} + c_1 t < \sqrt{x} + c_0 t$:

The distribution of $\mathbb{X}^x(t)$ is supported on the segment

$$\Delta_\mathrm{A} := [(\sqrt{x} + c_1 t)^2, (\sqrt{x} + c_0 t)^2] \subset \mathbb{R}^1_+,$$

the equation $(\sqrt{x} + z)^2 = y$, $y \in \Delta_\mathrm{A}$, has the unique solution $z = \sqrt{y} - \sqrt{x}$, and

$$p_i(y, t; n \mid x) = \frac{1}{2\sqrt{y}} f_i(\sqrt{y} - \sqrt{x}, t; n), \qquad n \geq 1,\ i \in \{0, 1\},\ y \in \Delta_\mathrm{A}. \tag{3.1}$$

(B) $\sqrt{x} + c_1 t < 0 < -\sqrt{x} - c_1 t \leq \sqrt{x} + c_0 t$:

The distribution of $\mathbb{X}^x(t)$ is supported on

$$\Delta_\mathrm{B} := [0, (\sqrt{x} + c_0 t)^2] \subset \mathbb{R}^1_+.$$

https://doi.org/10.1017/jpr.2019.58 Published online by Cambridge University Press

For all y, $0 < y \le (\sqrt{x} + c_1 t)^2$, the equation $(\sqrt{x} + z)^2 = y$ has two roots $z = \pm\sqrt{y} - \sqrt{x}$; if $(\sqrt{x} + c_1 t)^2 < y \le (\sqrt{x} + c_0 t)^2$ this equation has the unique solution $z = \sqrt{y} - \sqrt{x}$ between $c_1 t$ and $c_0 t$. Hence, for $n \ge 1$, $i \in \{0, 1\}$, the density function $p_i(y, t; n \mid x)$ is given by

$$\frac{1}{2\sqrt{y}} \begin{cases} f_i(-\sqrt{y} - \sqrt{x}, t; n) + f_i(\sqrt{y} - \sqrt{x}, t; n), & 0 < y < (\sqrt{x} + c_1 t)^2, \\ f_i(\sqrt{y} - \sqrt{x}, t; n), & (\sqrt{x} + c_1 t)^2 < y \le (\sqrt{x} + c_0 t)^2. \end{cases} \tag{3.2}$$

(C) $\sqrt{x} + c_1 t \le -\sqrt{x} - c_0 t < 0 < \sqrt{x} + c_0 t$:

The distribution of $\mathbb{X}^x(t)$ is supported on

$$\Delta_{\mathrm{C}} := [0, (\sqrt{x} + c_1 t)^2] \subset \mathbb{R}^1_+.$$

For all y, $0 < y \le (\sqrt{x} + c_0 t)^2$, the equation $(\sqrt{x} + z)^2 = y$ has two roots $z = \pm\sqrt{y} - \sqrt{x}$; if $(\sqrt{x} + c_0 t)^2 < y \le (\sqrt{x} + c_1 t)^2$, this equation has the unique solution $z = -\sqrt{y} - \sqrt{x}$ between $c_1 t$ and $c_0 t$. Hence, for $n \ge 1$, $i \in \{0, 1\}$, the density function $p_i(y, t; n \mid x)$ is given by

$$\frac{1}{2\sqrt{y}} \begin{cases} f_i(-\sqrt{y} - \sqrt{x}, t; n) + f_i(\sqrt{y} - \sqrt{x}, t; n), & y < (\sqrt{x} + c_0 t)^2, \\ f_i(-\sqrt{y} - \sqrt{x}, t; n), & (\sqrt{x} + c_0 t)^2 < y \le (\sqrt{x} + c_1 t)^2, \end{cases} \tag{3.3}$$

$$n \ge 1, \quad i \in \{0, 1\}.$$

(D) $\sqrt{x} + c_1 t < \sqrt{x} + c_0 t \le 0$:

The distribution of $\mathbb{X}^x(t)$ is supported on the segment

$$\Delta_{\mathrm{D}} := [(\sqrt{x} + c_0 t)^2, (\sqrt{x} + c_1 t)^2] \subset \mathbb{R}^1_+,$$

the equation $(\sqrt{x} + z)^2 = y$, $y \in \Delta_{\mathrm{D}}$, has the unique root $z = -\sqrt{y} - \sqrt{x}$. Thus

$$p_i(y, t; n \mid x) = \frac{1}{2\sqrt{y}} f_i(-\sqrt{y} - \sqrt{x}, t; n), \qquad n \ge 1, \ i \in \{0, 1\}, \ y \in \Delta_{\mathrm{D}}. \tag{3.4}$$

As a result, the distribution of $\mathbb{X}(t)$ depends on the signs of velocities.

First, if both velocities are positive, $c_0 > c_1 \ge 0$, then $T(t)$ is a subordinator and the distribution of $\mathbb{X}^x(t) = (\sqrt{x} + T(t))^2$ fits case (A).

Second, let $c_0 \ge 0 > c_1$. For sufficiently small times, $0 < t \le \sqrt{x}/(-c_1)$, the value $\sqrt{x} + T(t)$ remains positive. Hence the density functions $p_i(y, t; n \mid x)$, $i \in \{0, 1\}$, are again given by (3.1) (case (A)).

For large t the solution depends on the relation between c_0 and $|c_1|$.

If $c_0 + c_1 < 0$ and $\sqrt{x}/(-c_1) < t \le 2\sqrt{x}/(-c_0 - c_1)$ or $c_0 + c_1 \ge 0$ and $t > \sqrt{x}/(-c_1)$, then $\sqrt{x} + c_1 t < 0 < -\sqrt{x} - c_1 t < \sqrt{x} + c_0 t$, which corresponds to case (B). Hence, the formula (3.2) holds.

If $c_0 + c_1 < 0$ and $t \ge 2\sqrt{x}/(-c_0 - c_1)$, then $\sqrt{x} + c_1 t < -\sqrt{x} - c_0 t < 0 < \sqrt{x} + c_0 t$, which is case (C), and (3.3) holds.

https://doi.org/10.1017/jpr.2019.58 Published online by Cambridge University Press

Third, let both velocities be negative, $0 > c_0 > c_1$. The distribution of $\mathbb{X}^x(t)$ is given separately for the different time intervals:

$$\begin{aligned} 0 < t \le \sqrt{x}/(-c_1) &\Longrightarrow \text{case (A) and formula (3.1);} \\ \sqrt{x}/(-c_1) < t \le 2\sqrt{x}/(-c_0-c_1) &\Longrightarrow \text{case (B) and formula (3.2);} \\ 2\sqrt{x}/(-c_0-c_1) \le t < \sqrt{x}/(-c_0) &\Longrightarrow \text{case (C) and formula (3.3);} \\ t > \sqrt{x}/(-c_0) &\Longrightarrow \text{case (D) and formula (3.4).} \end{aligned}$$

If $t = 2\sqrt{x}/(-c_0-c_1)$ (with $c_0+c_1<0$), case (B) coincides with case (C) and $p_i(y, t; n \mid x) = \frac{1}{2\sqrt{y}}[f_i(-\sqrt{y}-\sqrt{x}, t; n) + f_i(\sqrt{y}-\sqrt{x}, t; n)]$, $0 < y < (\sqrt{x}+c_1t)^2$.

A slightly different approach is given in [20].

3.2. Process in the plane and polar coordinates

The piecewise deterministic process in the plane has been studied in the past in various contexts [9, 14, 15, 21, 22]. Here we present an example of planar motion in the spirit of our construction (2.13).

Let $\Phi(\boldsymbol{x}) = (r(\boldsymbol{x}), \alpha(\boldsymbol{x}))$, $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, $\boldsymbol{x} \ne \boldsymbol{0}$, be the operator setting the polar coordinates $r(\boldsymbol{x}) = |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2} > 0$ and $\alpha(\boldsymbol{x}) \in S^1$ for any $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, $\boldsymbol{x} \ne \boldsymbol{0}$. The mapping Φ is the (local) diffeomorphism from $\mathbb{R}^2 \setminus \{\boldsymbol{0}\}$ to the semi-cylinder $(0, +\infty) \times S^1$.

Let $\mathcal{J} : \mathbb{C} \to \mathbb{R}^2$ be defined by

$$\mathcal{J}(z) = (r\cos(\alpha), r\sin(\alpha))^\top, \qquad z = re^{i\alpha} \in \mathbb{C}.$$

Consider the two basic deterministic flows $\phi_0(t; \boldsymbol{x})$ and $\phi_1(t; \boldsymbol{x})$ defined by (2.13) with $\boldsymbol{c} = \boldsymbol{c}_0 = (c_0, 0)^\top$ and $\boldsymbol{c} = \boldsymbol{c}_1 = (0, c_1)^\top$ respectively. Here, $c_0 > 0$ is the velocity of a radial flight and $c_1 > 0$ is the constant angular velocity.

The flow

$$\phi_0(t; \boldsymbol{x}) = \widehat{r}_{c_0t}(\boldsymbol{x}) = \boldsymbol{x} + c_0t\boldsymbol{x}/|\boldsymbol{x}| = (1 + c_0t/|\boldsymbol{x}|)\boldsymbol{x}$$

is the *radial* movement starting from point $\boldsymbol{x} \in \mathbb{R}^2$, $\boldsymbol{x} \ne \boldsymbol{0}$, and the flow $\phi_1(t; \boldsymbol{x})$ is the *circular* motion defined by rotation of $\boldsymbol{x}$:

$$\phi_1(t; \boldsymbol{x}) = \widehat{\omega}_{c_1t}(\boldsymbol{x}) = \mathcal{J}(r(\boldsymbol{x})e^{i(\alpha + c_1t)}), \qquad t \ge 0.$$

The process $\mathbb{X}^{\boldsymbol{x}}$ is defined by the radial-circular motion, switching from radial to circular motion with intensity λ_0 and vice versa with intensity λ_1.

The distribution of $\mathbb{X}^{\boldsymbol{x}}(t)$ is supported on the segment $\ell = \ell(t, \boldsymbol{x})$ of the Archimedean spiral, $\boldsymbol{y} \in \ell(t, \boldsymbol{x})$ (Figure 2),

$$\begin{cases} y_1 = (r(\boldsymbol{x}) + c_0\tau)\cos(\alpha(\boldsymbol{x}) + c_1(t-\tau)), \\ y_2 = (r(\boldsymbol{x}) + c_0\tau)\sin(\alpha(\boldsymbol{x}) + c_1(t-\tau)), \end{cases} \qquad \tau \in [0, t]. \tag{3.5}$$

Let $\xi = \xi(\boldsymbol{x}, \boldsymbol{y}) = \frac{|\boldsymbol{y}| - |\boldsymbol{x}|}{c_0}$, $\boldsymbol{y} \in \ell(t, \boldsymbol{x})$, be the total time of radial motion, $0 \le \xi \le t$, such that the remaining time, $t - \xi$, is the total time of circular motion.

From Theorem 2.1, the density functions $p_i(\boldsymbol{y}, t; n \mid \boldsymbol{x})$ of $\mathbb{X}^{\boldsymbol{x}}(t)$ are given by

$$p_i(\boldsymbol{y}, t; n \mid \boldsymbol{x})\, d\boldsymbol{y} = q_i(\xi, t-\xi; n)\theta(\boldsymbol{x}, \boldsymbol{y})\delta_\ell(d\boldsymbol{y}), \qquad i \in \{0, 1\}, \ n \ge 1, \tag{3.6}$$

https://doi.org/10.1017/jpr.2019.58 Published online by Cambridge University Press

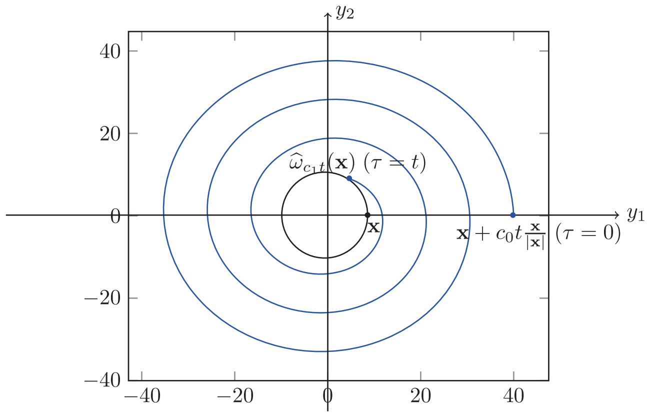

FIGURE 2: The support of the distribution of $\mathbb{X}(t)$: the Archimedean spiral $\ell(\boldsymbol{x}, t)$ defined by (3.5) with $\boldsymbol{x} = (10, 0)$, $c_0 = 2$, $c_1 = 3$, and time $t = 10$.

where

$$\theta(\boldsymbol{x}, \boldsymbol{y}) = \frac{|\boldsymbol{y}|}{\sqrt{c_0^2 + c_1^2}} \exp\left(-\lambda_0\xi - \lambda_1(t - \xi)\right),$$

and $q_i(\xi, \eta; n)$ are defined by (1.3). If there are no switchings, we have

$$p_0(\boldsymbol{y}, t; 0 \mid \boldsymbol{x}) = \mathrm{e}^{-\lambda_0 t}\delta(\boldsymbol{y} - \widehat{r}_{c_0t}(\boldsymbol{x})),$$
$$p_1(\boldsymbol{y}, t; 0 \mid \boldsymbol{x}) = \mathrm{e}^{-\lambda_1 t}\delta(\boldsymbol{y} - \widehat{\omega}_{c_1t}(\boldsymbol{x})).$$

Here,

$$\widehat{r}_{c_0t}(\boldsymbol{x}) = \boldsymbol{x}\left(1 + c_0t\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)$$

is the radial displacement and $\widehat{\omega}_\alpha(\boldsymbol{x})$ denotes the rotation of $\boldsymbol{x}$.

The density functions $p_i(\boldsymbol{y}, t \mid \boldsymbol{x})$, $i \in \{0, 1\}$, $\boldsymbol{y} \in \ell(\boldsymbol{x}, t)$, can be obtained by summing up in (3.6) similarly to (2.14) and (2.15); see Figure 3.

4. Self-similarity

The process $\mathbb{X}^x = \mathbb{X}^x(t) \in \mathbb{R}^1_+$ is called positive self-similar if there exists a constant $\gamma > 0$ such that, for any $x > 0$ and $R > 0$,

$$R \cdot \mathbb{X}^x(R^{-\gamma}t) \text{ is equal in law to } \mathbb{X}^{Rx}(t), \qquad t \geq 0; \tag{4.1}$$

see the definition in [17, Chapter 13].

The following theorem characterizes piecewise deterministic positive (1D) self-similar processes.

Theorem 4.1. *Let $\mathbb{X}^x = \mathbb{X}^x(t) \in \mathbb{R}^1_+$, $x > 0$, be the positive piecewise deterministic time-homogeneous process with two alternating patterns ϕ_0, ϕ_1 based on a common rectifying diffeomorphism Φ, (2.13), such that $\phi_0 = \Phi^{-1}(\Phi(x) + c_0t)$ and $\phi_1 = \Phi^{-1}(\Phi(x) + c_1t)$ with $c_0, c_1 > 0$.*

The process $\mathbb{X}^x$ is positive self-similar with index $\gamma > 0$ if and only if the underlying patterns are given by $\Phi(x) = x^\gamma$, $x \in \mathbb{R}^1_+$.

https://doi.org/10.1017/jpr.2019.58 Published online by Cambridge University Press

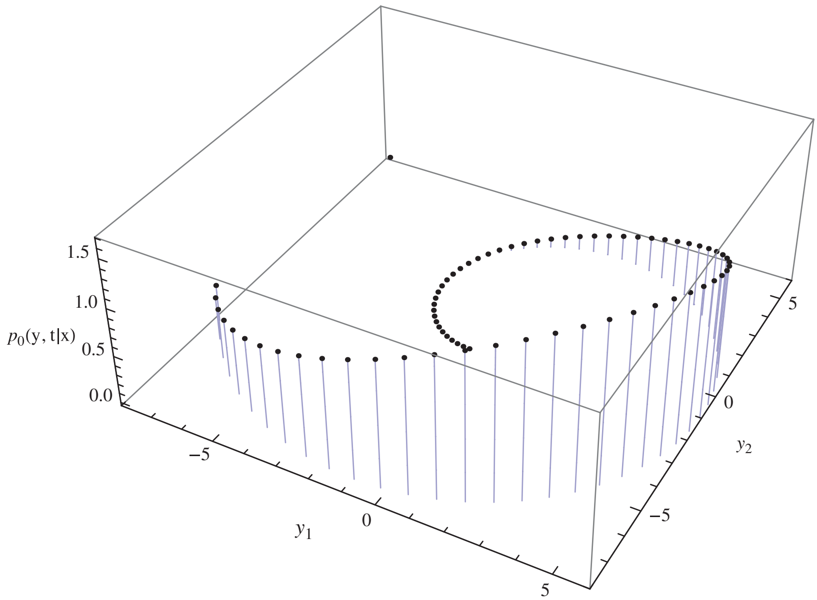

FIGURE 3: The regular part of the density function $p_0(\,\cdot\,, 10\,|\,\boldsymbol{x})$ with $c_0 = c_1 = 1$, $\lambda_0 = \lambda_1 = 2$, and the initial point $\boldsymbol{x} = (1, 1)$.

Proof. Let $\mathbb{X}^x$ be the piecewise deterministic time-homogeneous process based on the two patterns

$$\phi_i(t; x) = (x^\gamma + c_i t)^{1/\gamma}, \qquad t \geq 0, \ x > 0, \ i \in \{0, 1\}, \tag{4.2}$$

with $c_0, c_1 > 0$.

Note that the flows $\phi_i(t; x)$, $i \in \{0, 1\}$, defined by (4.2) satisfy the scaling relation

$$\phi_i(R^{-\gamma} t; R^{-1} x) = R^{-1} \phi_i(t; x), \qquad x > 0, \ t \geq 0, \ i \in \{0, 1\}. \tag{4.3}$$

Moreover, under the time scaling $t \to R^{-\gamma} t$ the switching intensities are transformed as

$$\lambda_0 \to R^\gamma \lambda_0, \qquad \lambda_1 \to R^\gamma \lambda_1. \tag{4.4}$$

Therefore, the piecewise deterministic process $\mathbb{X}^x(t)$, $t \geq 0$, which follows the patterns (4.2), switching from one to another with alternating intensities λ_0, λ_1, is the positive self-similar continuous process with index γ, (4.1).

Note that this can also be verified by using explicit formulae for the distribution. Let $\Phi(x) = x^\gamma$, $x > 0$. Under the space–time scaling $x \to R^{-1} x$, $t \to R^{-\gamma} t$ the variable $\xi = \varphi(\Phi(y) - \Phi(x), t)$, (2.9), used in Theorem 2.1, is transformed as $\xi \to R^{-\gamma} \xi$. Hence, by Theorem 2.1 and Equations (4.4) and (4.3), the transition densities $p_i^{\mathbb{X}}(\,\cdot\,, t; n\,|\,x)$ satisfy the relation

$$R^{-1} p_i^{\mathbb{X}}(R^{-1} y, R^{-\gamma} t, n\,|\,R^{-1} x)|_{\lambda_0 \to R^\gamma \lambda_0,\, \lambda_1 \to R^\gamma \lambda_1} \equiv p_i^{\mathbb{X}}(y, t, n\,|\,x), \qquad n \geq 0, \ t > 0.$$

The same is fulfilled for $p_i^{\mathbb{X}}(y, t\,|\,x)$:

$$R^{-1} p_i^{\mathbb{X}}(R^{-1} y, R^{-\gamma} t\,|\,R^{-1} x)|_{\lambda_0 \to R^\gamma \lambda_0,\, \lambda_1 \to R^\gamma \lambda_1} \equiv p_i^{\mathbb{X}}(y, t\,|\,x),$$

$i \in \{0, 1\}$, which corresponds to (4.1).

https://doi.org/10.1017/jpr.2019.58 Published online by Cambridge University Press

To prove the inverse assertion note that by definition (4.1) (with $x \downarrow 0$) one can see that the underlying patterns satisfy

$$\phi_i(t; 0) = (c_i t)^{1/\gamma},$$

where $c_i = \phi_i(1; 0)^\gamma > 0$.

Due to the semi-group property $\phi_i(t-s; \phi_i(0; s)) = \phi_i(t; 0)$, we have

$$\phi_i(t-s; (c_i s)^{1/\gamma}) = (c_i t)^{1/\gamma}, \qquad 0 < s < t.$$

Hence,

$$\phi_i(t - x^\gamma / c_i; x) = (c_i t)^{1/\gamma}.$$

Therefore (under the shift $t \to t + x^\gamma / c_i$) we have

$$\phi_i(t; x) = (x^\gamma + c_i t)^{1/\gamma}. \qquad \square$$

Remark 4.1. If the 'velocities' c_0, c_1 are positive, the process $\mathbb{X}^x$ is a subordinator (defined for all $t \geq 0$).

In the case of a negative velocity the process $\mathbb{X}^x$ is defined until hitting zero at time $\zeta^x = \inf\{t > 0 \mid X^x(t) = 0\} = \inf\{t > 0 \mid T(t) = -x^\gamma\}$. The distribution of ζ^x is known explicitly; see, e.g., [2].

Remark 4.2. Consider the time-homogeneous process $\mathbb{X}^x$ determined by the alternating patterns ϕ_0, ϕ_1 with common diffeomorphism $\Phi(x) = \mathrm{e}^x$, $x \in \mathbb{R}^1$:

$$\phi_i(t; x) = \log(\mathrm{e}^x + c_i t), \qquad t \geq 0, \ \mathrm{e}^x + c_i t > 0, \ i \in \{0, 1\}.$$

If $c_0, c_1 \geq 0$, the process $\mathbb{X}^x(t)$ is defined for all $t \geq 0$. In the case of negative c_i the process is killed and sent to the cemetery state $-\infty$ at time $t_* = \inf\{t > 0 \mid \mathbb{T}(t) = -\mathrm{e}^x\}$, where $\mathbb{T}(t)$ is the respective telegraph process.

The process $\mathbb{X}^x(t)$ possesses the property of *additive* self-similarity: under time scaling the process takes a spatial shift,

$$\mathbb{X}^{x-R}(\mathrm{e}^{-R} t) \text{ is equal in law to } \mathbb{X}^x(t) - R.$$

Indeed, under transformations $t \to \mathrm{e}^{-R} t$ and $x \to x - R$ the switching intensities are transformed as $\lambda_0 \to \mathrm{e}^R \lambda_0$, $\lambda_1 \to \mathrm{e}^R \lambda_1$, and $\xi \to \mathrm{e}^{-R} \xi$. By Theorem 2.1, the distributions of $\mathbb{X}^{x-R}(\mathrm{e}^{-R} t)$ and $\mathbb{X}^x(t) - R$ coincide.

Appendix A. The auxiliary result

Lemma A.1. *Let $z \in I_t$ be fixed, and $\xi = \varphi(z, t)$, $0 \leq \xi \leq t$, be the (unique) solution of the equation $z - tc_1 = \xi(c_0 - c_1)$, (2.9). Then $z - c_0\tau \in I_{t-\tau}$ and $z - c_1\tau \in I_{t-\tau}$ for sufficiently small τ, $\tau > 0$.*

Further, for all $z \in I_t$ the solution $\xi = \varphi(z, t)$ of (2.9) satisfies the following identities:

$$\begin{aligned} \varphi(z - c_0\tau, t - \tau) &\equiv \xi - \tau \quad && \text{if } \tau \in [0, \xi], \\ \varphi(z - c_1\tau, t - \tau) &\equiv \xi && \text{if } \tau \in [0, t - \xi]. \end{aligned} \tag{A.1}$$

Proof. By substitution of $z - c_0\tau$ and $z - c_1\tau$ with z and $t - \tau$ with t into (2.9) one can obtain

$$z - c_0\tau = \widetilde{\xi} c_0 + (t - \tau - \widetilde{\xi}) c_1, \qquad \widetilde{\xi} = \varphi(z - c_0\tau, t - \tau), \tag{A.2}$$

https://doi.org/10.1017/jpr.2019.58 Published online by Cambridge University Press

and

$$z - c_1\tau = \tilde{\xi}c_0 + (t - \tau - \tilde{\xi})c_1, \qquad \tilde{\xi} = \varphi(z - c_1\tau, t - \tau). \tag{A.3}$$

Equation (A.2) is satisfied by $\widetilde{\xi} = \xi - \tau$ if $\tau \le \xi$, and (A.3) is satisfied by $\widetilde{\xi} = \xi$ if $\tau \le t - \xi$.

Further, note that, by definition, $z - c_0\tau \notin I_{t-\tau}$ if $\tau > \xi$ and $z - c_1\tau \notin I_{t-\tau}$ if $\tau > t - \xi$. Hence, the lemma is proved. □

Appendix B. Proof of Proposition 2.1

System (2.7), $n = 1$, and (2.8) give the density functions $p_0^{\mathbb{T}}(z, t; 1)$ and $p_1^{\mathbb{T}}(z, t; 1)$. Indeed,

$$\begin{aligned} p_0^{\mathbb{T}}(z, t; 1) &= \int_0^t \lambda_0 \mathrm{e}^{-\lambda_0\tau}\mathrm{e}^{-\lambda_1(t-\tau)}\delta(z - \tau c_0 - (t-\tau)c_1)\,\mathrm{d}\tau \\ &= \frac{\lambda_0}{\|c_0 - c_1\|}\exp\left(-\lambda_0\xi - \lambda_1(t-\xi)\right)\mathbf{1}_{\{0<\xi<t\}} \\ &= \lambda_0\theta(\xi, t-\xi), \end{aligned}$$

where $\xi = \varphi(z, t)$, $\xi \in (0, t)$, is the solution of (2.9). Similarly, $p_1^{\mathbb{T}}(z, t; 1) = \lambda_1\theta(\xi, t-\xi)$. This corresponds to (1.2), $n = 1$, with $q_i(\xi, \eta; 1)$ defined by (1.3) $(k = 0)$ and θ defined by (2.10).

By recalling Lemma A.1 in Appendix A and (2.10),

$$\begin{aligned} \mathrm{e}^{-\lambda_0\tau}\theta(\varphi(z - c_0\tau, t-\tau), t - \tau - \varphi(z - c_0\tau, t-\tau)) &= \mathrm{e}^{-\lambda_0\tau}\theta(\widetilde{\xi}, t - \tau - \widetilde{\xi})\big|_{\widetilde{\xi}=\xi-\tau} \\ &\equiv \theta(\xi, t-\xi)\mathbf{1}_{\{\tau<\xi\}}, \\ \mathrm{e}^{-\lambda_1\tau}\theta(\varphi(z - c_1\tau, t-\tau), t - \tau - \varphi(z - c_1\tau, t-\tau)) &= \mathrm{e}^{-\lambda_1\tau}\theta(\widetilde{\xi}, t - \tau - \widetilde{\xi})\big|_{\widetilde{\xi}=\xi} \\ &\equiv \theta(\xi, t-\xi)\mathbf{1}_{\{\tau<t-\xi\}}. \end{aligned} \tag{B.1}$$

Moreover, by applying (A.1) and (B.1) one can obtain the following identities, which are sufficient to finish the proof:

$$\begin{aligned} &\int_0^t \mathrm{e}^{-\lambda_0\tau}\varphi(z - c_0\tau, t-\tau)^m(t - \tau - \varphi(z - c_0\tau, t-\tau))^k \\ &\qquad\qquad \times \theta(\varphi(z - c_0\tau, t-\tau), t - \tau - \varphi(z - c_0\tau, t-\tau))\,\mathrm{d}\tau \\ &\qquad = \theta(\xi, t-\xi)\int_0^\xi (\varphi(z, t) - \tau)^m(t - \varphi(z, t))^k\,\mathrm{d}\tau = \theta(\xi, t-\xi)\frac{\xi^{m+1}}{m+1}(t-\xi)^k, \end{aligned}$$

$$\begin{aligned} &\int_0^t \mathrm{e}^{-\lambda_1\tau}\varphi(z - c_1\tau, t-\tau)^m(t - \tau - \varphi(z - c_1\tau, t-\tau))^k \\ &\qquad\qquad \times \theta(\varphi(z - c_1\tau, t-\tau), t - \tau - \varphi(z - c_1\tau, t-\tau))\,\mathrm{d}\tau \\ &\qquad = \theta(\xi, t-\xi)\int_0^{t-\xi} \varphi(z, t)^m(t - \tau - \varphi(z, t))^k\,\mathrm{d}\tau = \theta(\xi, t-\xi)\xi^m\frac{(t-\xi)^{k+1}}{k+1}, \end{aligned}$$

$\xi = \varphi(z, t)$; cf. [16, Chapter 4].

Acknowledgements

We express our thanks to the referees for useful comments that helped to improve the paper. A.D.C. and B.M. are members of the research group GNCS of INdAM.

References

[1] ARNOLD, V. I. (1992). *Ordinary Differential Equations*. Springer, Berlin.
[2] BOGACHEV, L. AND RATANOV, N. (2011). Occupation time distributions for the telegraph process. *Stoch. Process. Appl.* **121,** 1816–1844.
[3] BOXMA, O., KASPI, H., KELLA, O. AND PERRY, D. (2005). On/off storage systems with state-dependent input, output and switching rates. *Probab. Engrg. Inform. Sci.* **19,** 1–14.
[4] DAVIS, M. H. A. (1984). Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. *J. R. Statist. Soc. B*, **46,** 353–388.
[5] DAVIS, M. H. A. (1993). *Markov Models and Optimization.* Chapman & Hall, London.
[6] DE GREGORIO, A. (2010). Stochastic velocity motions and processes with random time. *Adv. Appl. Prob.* **42,** 1028–1056.
[7] DI CRESCENZO, A. AND MARTINUCCI B. (2010). A damped telegraph random process with logistic stationary distribution. *J. Appl. Prob.* **47,** 84–96.
[8] EMBRECHTS, P. AND SCHMIDLI, H (1994). Ruin estimation for a general insurance risk model. *Adv. Appl. Prob.* **26,** 404–422.
[9] GARRA, R., ORSINGHER, E. AND RATANOV, N. (2017). Planar piecewise linear random motions with jumps. *Math. Meth. Appl. Sci.* **40,** 7673–7685.
[10] GNEDENKO, B. V. AND KOVALENKO, I. N. (1989). *Introduction to Queueing Theory*, 2nd edn. Birkhaüser, Boston.
[11] JACOBSEN, M. (2006). *Point Process Theory and Applications. Marked Point and Piecewise Deterministic Processes.* Birkhäuser, Boston.
[12] KAC, M. (1974). A stochastic model related to the telegrapher's equation. *Rocky Mountain J. Math.* **4,** 497–509.
[13] KOLESNIK, A. D. (1998). The equations of Markovian random evolution on the line. *J. Appl. Prob.* **35,** 27–35.
[14] KOLESNIK, A. D. (2007). A note on planar random motion at finite speed. *J. Appl. Prob.* **44,** 838–842.
[15] KOLESNIK, A. D. AND ORSINGHER, E. (2005) A planar random motion with an infinite number of directions controlled by the damped wave equation. *J. Appl. Prob.* **42,** b1168–1182.
[16] KOLESNIK, A. D. AND RATANOV, N. (2013). *Telegraph Processes and Option Pricing.* Springer, Heidelberg.
[17] KYPRIANOU, A. E. (2014). *Fluctuations of Lévy Processes with Applications. Introductory lectures*, 2nd edn. Springer, Heidelberg.
[18] LASOTA, A., MACKEY, M. C. AND TYRCHA, J. (1992). The statistical dynamics of recurrent biological events. *J. Math. Biol.* **30,** 775–800.
[19] LÓPEZ, O. AND RATANOV, N. (2012). Option pricing driven by a telegraph process with random jumps. *J. Appl. Prob.* **49,** 838–849.
[20] MARTINUCCI, B. AND MEOLI, A. (2019). On certain functionals of squared telegraph processes. *Stoch. Dynam.* Available at https://doi.org/10.1142/S0219493720500057.
[21] ORSINGHER, E. (2000). Exact joint distribution in a model of planar random motion. *Stoch. Stoch. Rep.* **69,** 1–10.
[22] ORSINGHER, E. AND RATANOV, N. (2002) Planar random motions with drift. *J. Appl. Math. Stoch. Anal.* **15,** 189–205.
[23] RATANOV, N. (2007). Jump telegraph processes and financial markets with memory. *J. Appl. Math. Stoch. Anal.* **2007,** 72326.
[24] STADJE, W. AND ZACKS, S. (2004). Telegraph processes with random velocities. *J. Appl. Prob.* **41,** 665–678.