Can. J. Math., Vol. XXXVI, No. 6, 1984, pp. 1067-1080

ON TORSION-FREE GROUPS OF FINITE RANK

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1. Introduction. This paper deals with two conditions which, when stated, appear similar, but when applied to finitely generated solvable groups have very different effect. We first establish the notation before stating these conditions and their implications. If H is a subgroup of a group G, let $\sqrt{H}$ denote the set

$$\{g \in G: g^n \in H \text{ for some } n \geqq 1\}.$$

We say G has the *isolator property* if $\sqrt{H}$ is a subgroup for all $H \leqq G$. Groups possessing the isolator property were discussed in [**2**]. If we define the relation $\sim$ on the set of subgroups of a given group G by the rule $H \sim K$ if and only if $\sqrt{H} = \sqrt{K}$, then $\sim$ is an equivalence relation and every equivalence class has a maximal element which may not be unique. If $H = \sqrt{H}$, we call H an isolated subgroup of G. For a nilpotent group G it is well known that if $H \leqq G$ and $HG' = G$ then $H = G$. If G is solvable then we need to state in addition that H be a subnormal subgroup of G. It is also well known that for a nilpotent group G, $HG' \sim G$ implies $H \sim G$. This result no longer holds even for a normal subgroup H when G is polycyclic for we need only look at an infinite polycyclic group G in which G' is of finite index, and take H to be the identity subgroup. We shall consider the following two conditions $(*)$ and $(+)$.

G is said to have the property $(*)$ if for all subgroups $H \leqq KSnG$, $HK' \sim K$ implies $H \sim K$. ($KSnG$ means K is a subnormal subgroup of G.) G is said to have the property $(+)$ if for all subgroups $HSnK \leqq G$, $HK' \sim K$ implies $H \sim K$.

THEOREM A. *If G is a finitely generated solvable group with the property $(*)$, then G is finite-by-nilpotent. Conversely, every finite-by-nilpotent group has the property $(*)$.*

Throughout this paper when we talk of a group having finite rank, we mean finite Prüfer rank. Thus G has finite rank if there is an integer n such that every finitely generated subgroup of G can be generated by n elements.

Received March 23, 1983 and in revised form January 3, 1984. This paper is dedicated to the memory of Philip Hall. This research was partially supported by an NSERC grant.

THEOREM B. *Let G be a finitely generated torsion-free solvable group. If G has the property* (+), *then G has finite rank. Conversely, if G has finite rank, then it has a subgroup of finite index with property* (+).

Remarks. If G is a finite-by-nilpotent group then the well known results of P. Hall on isolators show that G has the property (∗). Thus in the proof of Theorem A we simply establish the first part of the statement. The situation is similar for Theorem B. (See Theorem B and the comments following the statement of Theorem E in [**2**].) The group $C_p \wr C_\infty$, (the wreath product of a cyclic group of order p and an infinite cyclic group) has the property (+). This shows that G be torsion-free is required in Theorem B.

In the process of proving the two theorems above, we will also obtain the following.

THEOREM C. *If a finitely generated torsion-free solvable group has the isolator property then it has finite rank.*

Finally in Section 4 we define the π-isolator property when π is a finite non-empty set of primes and show that for a finitely generated solvable group G, this property is equivalent to the property (∗).

The terminology and notation used in this paper is mostly from [**4**] and hence most of the terms are not explicitly defined.

2. Proof of Theorem A. We begin with the following six preliminary lemmas and deduce Theorem A from these.

LEMMA 1. *If G has property* (∗), *then the periodic elements of G form a subgroup.*

Proof. Let T be the subgroup generated by the periodic elements; then $T \lhd G$ and T/T' is periodic so that $1.T' \sim T$. Hence $1 \sim T$ and T is therefore periodic.

LEMMA 2. *If G is a finitely generated abelian-by-finite group with property* (∗), *then G is finite-by-abelian.*

Proof. By Lemma 1, we can assume G to be torsion-free. Let A be an abelian normal subgroup of finite index. Assume first that the Hirsch length $l = l(G/G') < l(A)$. Since $|G:A|$ is finite, we can find $x_1, \ldots, x_l$ in A such that $U = \langle x_1, \ldots, x_l \rangle$ has the property $UG' \sim G$. Then by (∗), $U \sim G$ and since $U \leqq A$; $U \sim A$, a contradiction. Thus $l(G/G') = l(A)$ and G' is finite.

LEMMA 3. *Suppose G is finitely generated nilpotent-by-finite with property* (∗). *Then G is finite-by-nilpotent.*

Proof. We may again restrict our attention to the case where G is torsion-free, and show that G is nilpotent. We proceed by induction on the Hirsch length $l = l(G)$. If $l = 1$ then the result follows from Lemma 2. If $l > 1$, then $l(G/G')$ can not be one, for else $\langle x \rangle G' \sim G$ for some x in G, hence $\langle x \rangle \sim G$ and $l(G) = 1$. Let

$$J = \sqrt{G'} \quad \text{and} \quad G/J = \langle x_1 J, \ldots, x_n J \rangle.$$

Then $J_i = \langle x_i, J \rangle$ is of length less than l, and by induction hypothesis J_i is nilpotent. Thus G is a product of finitely many normal nilpotent subgroups and hence nilpotent.

LEMMA 4. *If G is a finitely generated group with property* $(*)$, *A a normal periodic abelian subgroup of G and G/A a torsion-free nilpotent group, then A is finite* (*and therefore G is finite-by-nilpotent*).

Proof. For any $x \in G \setminus A$ let $G_x = \langle x, A \rangle$ so that G_x is subnormal in G. Since A is periodic, $\langle x \rangle G'_x \sim G_x$ so that $\langle x \rangle \sim G_x$. Therefore if $a \in A$, then there exist non-zero integers n and k such that

$$a^{-1}x^n a = (a^{-1}xa)^n = x^k.$$

If we work mod A and use that G/A is torsion-free, we see that $n = k$. Thus for any a in A there exists $n = n(a)$ such that

$$a^{-1}x^n a = x^n \quad \text{or} \quad a^{x^n} = a.$$

Since A is finitely generated as a G-subgroup,

$$A = \langle a_1^G, \ldots, a_m^G \rangle.$$

Thus it suffices to show that the subgroup $\langle a_i^G \rangle$ is finitely generated. G/A operates on $S_i = \{a_i^g; g \in G\}$ as a permutation group $\bar{G}$. Let $\bar{x}$ be the image of x under $G/A \twoheadrightarrow \bar{G}$ and assume $\bar{x} \in Z(\bar{G})$; then by the first part of the proof there exists integer $n > 0$ such that

$$a_i^{x^n} = a_i^{\bar{x}^n} = a_i.$$

Since $\bar{x} \in Z(\bar{G})$, $\bar{x}^n$ operates trivially on S_i so that $\bar{x}^n = 1$. The elements of $Z(\bar{G})$ are therefore of finite order and hence $\bar{G}$ is finite, and therefore so is S_i.

LEMMA 5. *Let $G = A \rtimes \langle x \rangle$ be a split extension of a torsion-free abelian group A by an infinite cyclic group $\langle x \rangle$. If G has the property* $(*)$ *and G/G' is finitely generated, then G is nilpotent.*

Proof. Since G/G' is finitely generated, there exist $b_1, \ldots, b_r$ in A such that for all $n > 1$,

$$\langle x^n, b_1, \ldots, b_r \rangle G' \sim \langle x, b_1, \ldots, b_r \rangle G' \sim G.$$

Therefore, by $(*)$

$$U_n = \langle x^n, b_1, \ldots, b_r \rangle \sim \langle x, b_1, \ldots, b_r \rangle = U \sim G.$$

We show by using induction on r, that $U_n \sim U$ implies G is nilpotent of finite rank. In the case $r = 1$,

$$U_n = \langle x^n, b\rangle \sim \langle x, b\rangle = U \sim G.$$

Let

$$W = \langle b^{x^k}; k \in \mathbf{Z}\rangle \quad \text{and} \quad W_n = \langle b^{x^{kn}}; k \in \mathbf{Z}\rangle.$$

Then $W_n = W \cap U_n$ and since $U_n \sim G$, $W_n \sim W$. Thus

$$\langle b^x\rangle \cap W_n \neq 1$$

and it follows that W has finite rank. Hence G has finite rank and so $\Gamma_s \sim \Gamma_{s+1}$ for some s, where Γ_s is the s^{th} term of the lower central series of G. Since $\langle x, \Gamma_s\rangle$ is subnormal in G it has property $(*)$. Note that $\langle x, \Gamma_s\rangle' = \Gamma_{s+1}$. Hence

$$\langle x\rangle\langle x, \Gamma_s\rangle' = \langle x, \Gamma_{s+1}\rangle \sim \langle x, \Gamma_s\rangle$$

and therefore

$$\langle x\rangle \sim \langle x, \Gamma_s\rangle.$$

Since A is torsion free and $\Gamma_s \leqq A$, it follows that $\Gamma_s = 1$ and G is nilpotent, as required.

Assume the result has been established for $1 \leqq r < m$ and consider the case where

$$U_n = \langle x^n, b_1, \ldots, b_m\rangle \sim \langle x, b_1, \ldots, b_m\rangle = U \sim G.$$

Let

$$B = \sqrt{\langle b_m{}^{\langle x\rangle}\rangle},$$

the isolator in G of the normal subgroup of U generated by b_m. Then G/B is nilpotent by the induction hypothesis. In particular $B \geqq \Gamma_i$ for some $i > 0$ and $\langle B, x\rangle$ is subnormal in G. Let $V = \langle B, x\rangle$. Then

$$V' = [B, \langle x\rangle] = [B, G] \geqq [\Gamma_i, G] = \Gamma_{i+1}.$$

Since G/G' is finitely generated, so is G/Γ_{i+1} and hence V/V'. Since V is subnormal in G, it has property $(*)$. Also V is the split extension of B by $\langle x\rangle$. Thus the hypotheses of the lemma hold for V. Now $\langle b_m, x\rangle V' \sim V$ since every element g of V is of the form $g = ax^k$ for some a in B. Thus

$$a^t \in \langle b_m^{\langle x\rangle}\rangle \leqq \langle b_m, x\rangle \quad \text{for some } t > 0,$$

and hence $g^t \in \langle b_m, x\rangle V'$. By property $(*)$, $\langle x, b_m\rangle \sim V$. By the first step of the induction hypothesis, V is nilpotent. Thus for every integer $k > 0$,

$$\Gamma_{i+k} = [\Gamma_i, \underbrace{G, \ldots, G}_{k}] \leqq [B, \underbrace{G, \ldots, G}_{k}] = [B, \underbrace{V, \ldots, V}_{k}] \leqq \Gamma_{k+1}(V),$$

the $k + 1$ step of the lower central series of V. Since V is nilpotent, $\Gamma_{k+1}(V) = 1$ for some k, hence G is nilpotent.

LEMMA 6. *Let G be a finitely generated group with property* $(*)$. *If G has the series $1 \lhd A \lhd G$ where A is torsion-free abelian and G/A is torsion-free nilpotent, then G is nilpotent.*

Proof. Use induction on the Hirsch length $l = l(G/A)$. For $l = 1$, the result follows from Lemma 5. If $l > 1$, set $Z/A = Z(G/A)$, and choose $x \in Z \setminus A$ such that $G/\langle x, A\rangle$ is torsion-free. Let $D = \langle x, A\rangle'$. By induction $G/\sqrt{D}$ is nilpotent. Now apply Lemma 4 and deduce that G/D is finite-by-nilpotent. Let $\tilde{G}$ be defined by

$$\tilde{G}/D = C_G(\sqrt{D}/D).$$

Then $\langle x, A\rangle \leqq \tilde{G}$, $|G:\tilde{G}|$ is finite and $\tilde{G}/D$ is nilpotent. $\langle x, A\rangle/D$ is finitely generated, being a subgroup of a finitely generated nilpotent group. By Lemma 5 $\langle x, A\rangle$ is nilpotent. Thus by ([**1**], Theorem 7), $\tilde{G}$ is nilpotent. Finally, by Lemma 3, G is nilpotent.

Proof of Theorem A. We use induction on the solvability length of G. If G is abelian, then there is nothing to prove. Using induction we can assume G to have a series $1 \lhd A \lhd G$, where A is abelian and G/A is finite-by-nilpotent. Since this implies G/A is nilpotent-by-finite, there exists a subgroup $\tilde{G}$ of finite index in G such that $\tilde{G}/A$ is torsion-free nilpotent. Let T be the torsion subgroup of A. By Lemma 6, $\tilde{G}/T$ is nilpotent, then by Lemma 4, T is finite and by Lemma 3, G/T is finite-by-nilpotent. Therefore G is finite-by-nilpotent.

3. Proof of theorems B and C. As in the second section, the results are deduced from a sequence of preliminary results which are stated in the form of lemmas.

LEMMA 7. *Let $G = \langle a, x\rangle$ have property* $(+)$. *If $A = \langle a^G\rangle$ is torsion-free abelian, then A has finite rank.*

Proof. If A is not of finite rank, then the set $\{a^{x^i}; i \in \mathbf{Z}\}$ is independent in A. Let a_i denote a^{x^i}, $i \in \mathbf{Z}$; choose any integer $n > 1$, and let

$$H = \langle x^n, (a_r a_{r+1} \ldots a_{r+n-1}), r \in \mathbf{Z}\rangle.$$

Then (i) $H \lhd G$, (ii) $HG' \sim G$, (iii) $H \not\sim G$. To see (i), note that x normalizes H and for any integer r,

$$[a_r, x^n] = [a_r a_{r+1} \ldots a_{r+n-1}, x] \in H.$$

To see (ii), note that

$$\prod_{i=0}^{n-1} [a_i, x]^{i+1} = (a_0^{-1}a_1)(a_1^{-1}a_2)^2 \dots (a_{n-1}^{-1}a_n)^n$$

$$= (a_0a_1 \dots a_{n-1})^{-1} a_n^n \in G'.$$

Hence $a_n^n \in HG'$ so that $A^n \leqq HG'$. Since $G'\langle x^n, A^n\rangle \sim G$, it follows that $HG' \sim G$. To see (iii), we simply observe that

$$\langle a_0\rangle \cap H = 1.$$

LEMMA 8. *A torsion-free solvable group satisfying the property* $(+)$ *is an* R*-group (an* R*-group is one where* $x^n = y^n$, $n > 0$ *implies* $x = y$).

Proof. Suppose for some x, y in G, and a positive integer n, $x^n = y^n$. Let $J = \langle x, y\rangle$; then

$$\langle x^n, J'\rangle \sim \langle x, y\rangle$$

so that by $(+)$, $\langle x^n\rangle \sim J$. Thus $J/Z(J)$ is a finitely generated periodic solvable group and hence finite. By Schur's theorem (see [**4**], Theorem 4.12), J' is finite. Since J is torsion-free, $J' = 1$ and $x = y$.

COROLLARY 9. *Let* G *be a torsion-free solvable group satisfying* $(+)$. *Then*

(i) *every term of the upper central series of* G *is isolated*;

(ii) *for any integer* $r > 0$, *every maximal nilpotent-class-*r *subgroup of* G *is isolated*;

(iii) *for any integer* $r > 0$ *every maximal normal nilpotent-class-*r *subgroup of* G *is isolated.*

Proof. Use Lemma 8 and that property $(+)$ is inherited by torsion-free quotients.

LEMMA 10. *Let* $G = \langle x_1, \dots, x_m\rangle$ *be an abelian* p*-group* (p *a prime integer*) *that is the direct sum of* n *cyclic subgroups. Then there is a subset* $\{x_{i_1}, \dots, x_{i_n}\}$ *of the set* $\{x_1, \dots, x_m\}$ *such that*

$$G = \langle x_{i_1}, \dots, x_{i_n}\rangle.$$

This is an easy exercise and we omit the proof.

LEMMA 11. *Let* $G = \langle x_1, \dots, x_m\rangle$ *be an abelian group that is the direct sum of* n *cyclic subgroups. If the order* $|G|$ *of* G *is* $p_1^{\alpha_1} \dots p_k^{\alpha_k}$ *where the* p_i*'s are distinct primes, then there is a subset* $\{x_{i_1}, \dots, x_{i_j}; j \leqq kn\}$ *of the set* $\{x_1, \dots, x_m\}$ *such that*

$$G = \langle x_{i_1}, \dots, x_{i_j}\rangle.$$

https://doi.org/10.4153/CJM-1984-061-1 Published online by Cambridge University Press

Proof. Let Q_i be the subgroup of G of order $|G|/p_i^{\alpha_i}$. By Lemma 10,

$$G = \langle Q_i, x_{i_1}, \dots, x_{i_n} \rangle$$

for suitable x_{i_j} in $\{x_1, \dots, x_m\}$. Thus $\langle x_{i_1}, \dots, x_{i_n} \rangle$ contains the sylow p_i-subgroup of G. Repeat the process for all values of $i = 1, \dots, k$ to obtain

$$G = \langle x_{ij};\ i = 1, \dots, k;\ j = 1, \dots, n \rangle$$

as required.

LEMMA 12. *Suppose that $N = \langle x_i;\ i \in \mathbf{Z} \rangle$ is an abelian group of rank n, and α is a positive integer such that for all $i \in \mathbf{Z}$,*

$$x_i^{\alpha} \in \langle x_{i+1}, \dots, x_{i+n} \rangle.$$

Then there exists an integer $f = f(\alpha, n)$ such that for any $y \in N$,

$$y \in \langle x_{i_1}, \dots, x_{i_f} \rangle$$

for suitable choice of integers $i_1, \dots, i_f$.

Proof. Let

$$\alpha = p_1^{\alpha_1} \dots p_k^{\alpha_k}.$$

We will show that $(k + 1)n$ will suffice as a bound for f. If $y \in N$ then

$$y = x_{r_1}^{\epsilon_1} \dots x_{r_m}^{\epsilon_m}$$

where $\epsilon_i = \pm 1$ and $r_1 \leqq r_2 \leqq \dots \leqq r_m$. Let

$$X = \langle x_{r_m}, x_{r_m - 1}, \dots, x_{r_m - n + 1} \rangle.$$

Then for each integer r such that $r \leqq r_m$, $(x_r)^{\alpha^{\lambda}} \in X$, for some integer λ. Consider the group

$$G = \langle x_{r_1}, \dots, x_{r_m} \rangle / X.$$

It is an abelian group of rank n and order dividing a power of α. Thus by Lemma 11, there is a subset Y of size at most kn of the set $\{x_{r_1}, \dots, x_{r_m}\}$ generating G. Thus

$$\langle x_{r_1}, \dots, x_{r_m} \rangle = \langle X, Y \rangle,$$

as required.

LEMMA 13. *Suppose that $N = \langle x_{\mathbf{u}};\ \mathbf{u} \in \mathbf{Z}^r \rangle$ is an abelian group of rank n, and $\alpha_1, \dots, \alpha_r$ are positive integers such that for all $\mathbf{c} = (c_1, \dots, c_r) \in \mathbf{Z}^r$,*

$$(x_{\mathbf{c}})^{\alpha_i} \in \langle x_{(c_1, \dots, c_{i-1}, c_i + k, c_{i+1}, \dots, c_r)};\ k = 1, \dots, n \rangle.$$

Then there is an integer $f = f(\alpha_1, \dots, \alpha_r, n)$ such that for any $y \in N$, there exist vectors $\mathbf{v}_1, \dots, \mathbf{v}_f$ in $\mathbf{Z}^r$ so that $y \in \langle x_{\mathbf{v}_1}, \dots, x_{\mathbf{v}_f} \rangle$.

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Proof. We shall show that $f_1 \dots f_r$ will suffice as bound for f, where $f_i = f(\alpha_i, n)$ as in Lemma 12. If $y \in N$, then

$$y = x_{\mathbf{u}_1}^{\epsilon_1} \dots x_{\mathbf{u}_m}^{\epsilon_m};$$

where $\epsilon_i = \pm 1$, and $\mathbf{u}_i \in \mathbf{Z}^r$. We will show that there exist vectors $\mathbf{v}_1, \dots, \mathbf{v}_f$ in $\mathbf{Z}^r$ such that

$$X_0 = \langle x_{\mathbf{u}_1}, \dots, x_{\mathbf{u}_m} \rangle \leqq \langle x_{\mathbf{v}_1}, \dots, x_{\mathbf{v}_f} \rangle.$$

Change the labelling of $\mathbf{u}_1, \dots, \mathbf{u}_m$ if necessary and assume that

$$u_{11} \leqq u_{21} \leqq \dots \leqq u_{m1}$$

where u_{ij} is the j^{th} component of $\mathbf{u}_i$. By Lemma 12, there exist suitable integers $v_{11}, \dots, v_{f_1 1}$, such that given any $\mathbf{v}'$in $\mathbf{Z}^{r-1}$,

$$\langle x_{(u_{i1}, \mathbf{v}')}; i = 1, \dots, m \rangle \leqq \langle x_{(v_{j1}, \mathbf{v}')}; j = 1, \dots, f_1 \rangle.$$

Thus each

$$x_{\mathbf{u}_i} \in \langle x_{(v_{j1}, u'_i)}; j = 1, \dots, f_1 \rangle = X_1$$

where $\mathbf{u}'_i \in \mathbf{Z}^{r-1}$ is obtained from $\mathbf{u}_i$ by dropping the first coordinate. Now repeat the above process for each of the other coordinates starting with the appropriate subgroup X_i, finally obtaining the group X_r given by

$$X_r = \langle x_{(v_{j1}, v_{k2}, \dots, v_{lr})}; j = 1, \dots, f_1; k = 1, \dots, f_2; \dots; l = 1, \dots, f_r \rangle.$$

Proof of Theorem B. Use induction on the solvability length of G. If G is abelian then there is nothing to prove. So assume the result holds for finitely generated torsion-free groups of solvability length d and suppose G has length $d + 1$. Let $A = \sqrt{G^{(d)}}$ where $G^{(d)}$ is the last non-trivial term of the derived series of G. By Corollary 9, A is abelian and normal in G, by induction G/A is of finite rank, and hence torsion-free nilpotent-by-abelian-by-finite. Let N be such that N/A is nilpotent and G/N abelian-by-finite. Then by Corollary 9, G/N is in fact abelian, and again by Corollary 9, by taking $N = \sqrt{G'}$, we may further assume that G/N is finitely generated free abelian. Since N/A is torsion-free nilpotent with finite rank, there is a finitely generated subgroup X of N such that $XA \sim N$ and $\langle X^G \rangle = N$ and there is an upper central series

$$A = AN_0 < AN_1 < \dots < AN_d \sim N$$

where $N_i = \langle X_i^G \rangle$ for suitable finite sets $1 = X_0 \subseteq X_1 \subseteq \dots \subseteq X_d$ in G. Now the group AXN_d/AN_d, being finitely generated periodic and nilpotent, has finite exponent, say e. Then the group $\langle X^G \rangle / AN_d$ is generated by elements of exponent e; moreover it is nilpotent, thus the group has bounded exponent. It has finite rank, thus its order is finite.

Thus there is a central series

$$A = AN_0 \lhd AN_1 \lhd \ldots \lhd AN_d \lhd AN_{d+1} \lhd \ldots \lhd AN_c = N$$

where $N_i = \langle X_i^G \rangle$ for suitable finite sets $1 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_c$ in G. We shall now establish that for any a in A, $\langle a^G \rangle$ has finite rank. As a first step we show that $\langle a^{N_i} \rangle$ has finite rank for all $i = 0, 1, \ldots, c$. Suppose this has been established for $i < \nu \leqq c$; then since N_ν is the normal closure of a finitely generated group, it suffices to show that if

$$N_{\nu-1} \leqq M \lhd G,$$

$x \in N_\nu$ and $\langle a^M \rangle$ has finite rank, then $\langle a^{\langle M, x^G \rangle} \rangle$ has finite rank. Now, there exist $t_1, \ldots, t_r$ in G such that

$$T = \{t_1^{\alpha_1} \ldots t_r^{\alpha_r}; (\alpha_1, \ldots, \alpha_r) \in \mathbf{Z}^r\}$$

is a set of coset representatives of N in G. If $y \in N$, then

$$Mx^y = Mx.$$

Thus if we use the notation

$$B(M) = \sqrt{\langle b^M \rangle} \quad \text{and} \quad x_\alpha = x^{t_1^{\alpha_1} \ldots t_r^{\alpha_r}},$$

it suffices to show that for any $b \in A$, $\langle B(M)^{\langle x_\alpha \rangle}, \alpha \in \mathbf{Z}^r \rangle$ has finite rank. $B(M)$ has finite rank by induction hypotheses. There exist therefore $b_1, \ldots, b_n$ in $B(M)$ such that $\langle b_1, \ldots, b_n \rangle \sim B(M)$ and hence

$$\langle b_1^{\langle x_\alpha \rangle}, \ldots, b_n^{\langle x_\alpha \rangle}, \alpha \in \mathbf{Z}^r \rangle \sim B(M)^{\langle x_\alpha, \alpha \in Z^r \rangle}.$$

Thus it suffices to show that for any $b \in A$, $\langle b^{\langle x_\alpha \rangle}, \alpha \in \mathbf{Z}^r \rangle$ has finite rank. By Lemma 7, $\langle b^{\langle x_\alpha \rangle} \rangle$ has finite rank for each $\alpha \in \mathbf{Z}^r$. Consider the group $M \langle x^T \rangle / M$, generated by $\{x_\alpha \cdot M; \alpha \in \mathbf{Z}^r\}$. It is abelian of finite rank and the hypotheses of Lemma 13 apply. Thus for any $g \in M\langle x^T \rangle / M$,

$$g = x_{\mathbf{v}_1}^{\lambda_1} \ldots x_{\mathbf{v}_f}^{\lambda_f} \cdot M$$

for suitable $\mathbf{v}_i$ in $\mathbf{Z}^r$.

By repeated application of Lemma 7, $\langle A_1^{\langle t_i \rangle} \rangle$ and $\langle A_1^{\langle x_\mathbf{v} \rangle} \rangle$ have finite rank for any subgroup A_1 of A having finite rank. For any b in A, define subgroups B_{ij}, $i = 0, 1, \ldots, f$; $j = 0, 1, \ldots, r$ as follows.

$$B_{00} = \langle b \rangle;$$

$$B_{ij} = \langle B_{i(j-1)}^{\langle t_j \rangle} \rangle \quad \text{for } j = 1, \ldots, r;$$

$$B_{i(r+k)} = \langle B_{i(r+k-1)}^{\langle t_{r-k} \rangle} \rangle \quad \text{for } k = 1, \ldots, r-1, \quad \text{and}$$

$$B_{i0} = \langle B_{(i-1)(2r-1)}^{\langle x \rangle} \rangle \quad \text{for } i = 1, \ldots, f.$$

Then B_{fr} has finite rank. Now for any $\mathbf{v}_1, \ldots, \mathbf{v}_f$ in $\mathbf{Z}^r$,

$$b^{x^{\lambda_1}_{y_1} \cdots x^{\lambda_f}_{y_f}} \in B_{fr}$$

since

$$B_{ir}^{t_r^{-\alpha_r} \cdots t_1^{-\alpha_1} x^{\lambda} t_1^{\alpha_r} \cdots t_1^{\alpha_r}} \leq B_{(i+1)r}$$

for all integers $\alpha_1, \ldots, \alpha_r, \lambda$; and all $0 \leqq i < f$. Thus $\langle b^{\langle x^T \rangle} \rangle$ and hence $\langle b^{M\langle x^T \rangle} \rangle$ has finite rank. Thus $\langle a^N \rangle$ has finite rank for any a in A. Since G/N is finitely generated abelian, $\langle a^G \rangle = \langle a^{NT} \rangle$ has finite rank by repeated application of Lemma 7.

If $1 \neq B$ is a normal isolated subgroup of G contained in A and such that the rank of B is minimal, then we can consider the action of $G/C_G(B)$ as rationally irreducible on B. Thus by the well known result of Malcev (see [**4**], Theorem 3.21), $G/C_G(B)$ is abelian by finite. Now $C_G(B)$ is isolated by Lemma 8 so that $G/C_G(B)$ is a finitely generated torsion-free abelian-by-finite group with property $(+)$. It follows, again by Lemma 8, that $G/C_G(B)$ is abelian. Thus $N \leqq C_G(B)$ for all isolated subgroups B of minimal rank subject to $B \lhd G$ and $B \leqq A$. Now repeat this argument on the quotient G/B and conclude that N is a ZA (or hypercentral) – group. Now $G = \langle N, t_1, \ldots, t_r \rangle$, $AX \sim N$ and $N = \langle X^G \rangle$ where X is finitely generated. For $i = 1, \ldots, r$ let

$$X_i = \langle X^{t_i^{-1}}, X, X^{t_i} \rangle$$

so that X_i is finitely generated nilpotent. Hence

$$C_i = \sqrt{(X_i \cap A)^G}$$

has finite rank. Replace G by $G/\sqrt{C_1 \ldots C_r}$ if necessary and assume $X_i \cap A = 1$. For any $x \in X$,

$$(x^{t_i})^{\lambda} \in AX \text{ for some positive integer } \lambda$$

since $AX \sim N$. Thus

$$(x^{t_i})^{\lambda} = ax_1 \text{ for some } a \in A,\ x_1 \in X.$$

Hence $a \in A \cap X_i = 1$, so that

$$(x^{t_i})^{\lambda} \in X \text{ and } X^{t_i} \sim X^{t_i} \cap X.$$

Similarly

$$X^{t_i^{-1}} \sim X^{t_i^{-1}} \cap X$$

and therefore

$$X \sim X \cap X^{t_i} = X^{t_i} \cap X \sim X^{t_i}$$

so that $Y = \sqrt{X}$ satisfies $Y^{t_i} = Y$ for all $i = 1, \ldots, r$. Thus $Y \lhd G$. But

$\langle X^G \rangle = N$; therefore $Y = N$ has finite rank, as required. This completes the proof.

COROLLARY 14. *Let G be a finitely generated solvable group. If G has the property $(+)$, then G is periodic-by-finite rank.*

Proof. The proof of Lemma 1 shows that the periodic elements of a group with property $(+)$ form a subgroup T. G/T is torsion free, and of finite rank by Theorem B.

Proof of Theorem C. In the case G is nilpotent-by-abelian, the result was proved in [**2**]; here the result will be established in four steps. Let G be a torsion-free solvable group with the isolator property.

Step 1. G is an R-group. Let $x, y \in G$. If $x^n = y^n$ for some $n > 0$ and $[x, y] = 1$, then $x = y$ since G is torsion-free; so it suffices to show that $[x, y] = 1$. Consider the subgroup $H = \langle x, x^y \rangle$. Then

$$Z = Z(H) \geqq \langle x^n \rangle$$

and the isolator of Z in H is H. Thus H/Z is a finitely generated periodic solvable group and hence finite. Thus by Schur's theorem (see [**4**], Theorem 4.12), H' is finite and therefore 1, and $x = x^y$.

Step 2. If $H = \sqrt{H} \leqq G$ then the normalizer $N(H) = \sqrt{N(H)}$. Let H be an isolated subgroup of G and let $k \in \sqrt{N(H)}$. Then

$$H \lhd \langle H, k^r \rangle = J \quad \text{for some } r > 0.$$

Since $\langle H, k \rangle \sim J^{k^i}$ for all $i \in \mathbf{Z}$, $\langle H, k \rangle \sim M$ where

$$M = \bigcap_{i=0}^{r-1} J^{k^i}$$

which is normal in $\langle M, k \rangle$. If

$$(M \cap H)^k \sim (M \cap H),$$

then $(M \cap H)^k \sim H$ since $M \cap H \sim H$, and hence $H^k \sim H$; and because H is isolated, $H^k = H$. So, replacing H by $M \cap H$, assume that $H \lhd M \lhd \langle M, k \rangle$, $M \sim \langle M, k \rangle$ and M/H is infinite cyclic. Now $H^{k^i} \lhd M$ for all $i \in \mathbf{Z}$ and the quotient is torsion-free abelian so that if

$$N = \bigcap_{i=0}^{r-1} H^{k^i},$$

then M/N is torsion-free abelian. Since $N \lhd \langle M, k \rangle$ and $M \sim \langle M, k \rangle$, by step 1, $\langle M, k \rangle / N$ is abelian, hence $N \leqq H \lhd \langle M, k \rangle$, as required.

Step 3. If H_1, H_2 are respectively the isolators of subgroups K_1, K_2 in G, then

https://doi.org/10.4153/CJM-1984-061-1 Published online by Cambridge University Press

$$[H_1, H_2] \sim [K_1, K_2].$$

Let $K = \sqrt{[K_1, K_2]}$. By step 2, $K \lhd \langle H_1, H_2 \rangle$ and by step 1, $\langle H_1, H_2 \rangle / K$ is an R-group, where centralizers are isolated. Thus $[H_1, H_2] \leqq K$ as required.

Step 4. Completion of the proof. Suppose $MSnG$ and $MG' \sim G$. We may assume that M is isolated and there is a series

$$M = M_0 \lhd M_1 \lhd \ldots \lhd M_{r+1} = G$$

where each M_i is isolated and r is the least integer with a series of this form. If $M \neq G$, then $M_r \neq G$, so replace M by M_r, if necessary, and assume that $M \lhd G$. Put $G = H_1 = H_2$ and $MG' = K_1 = K_2$ in step 3 to get

$$G' \sim [MG', MG'] \leqq MG''.$$

In particular $G \sim (M \cap G')G''$ so that by inductive argument we may assume

$$M \cap G' \sim G'.$$

Since M is isolated, $M \cap G' = G'$ so that $M = MG' \sim G$ and $M = G$ as required. Since the isolator property is inherited by subgroups, G has property $(+)$. Therefore it has finite rank by Theorem B.

4. Final remarks. We have seen that within the class of finitely generated torsion-free solvable groups, the isolator property implies property $(+)$. The converse is not true as illustrated in the example at the end of this section; however such groups almost have the isolator property since a torsion-free solvable group with finite rank has a subgroup of finite index possessing the isolator property ([**2**], Theorem B).

Turning to the property $(*)$, it is just one of several conditions that force a finitely generated solvable group to be finite-by-nilpotent. Subnormal intersection property (see [**4**], Theorem 10.54) is a well known example of such a property. Another one is the π-isolator property where π is any non-empty finite set of primes. For any such set π, a group G is said to have the π-isolator property if for every subgroup $H \leqq G$, the set

$$\sqrt[\pi]{H} = \{g \in G, g^n \in H; \text{ for some } \pi\text{-number } n\}$$

is a subgroup of G. By a π-number n we mean a natural number n with all its prime factors in π.

THEOREM D. *If a finitely generated solvable group G has the π-isolator property for some finite non-empty set π of primes, then G is finite-by-nilpotent.*

Proof. Let G be a finitely generated solvable group with the π-isolator property where π is a non-empty finite set of primes. Then by Theorem A of [**3**], G is nilpotent-by-finite.

Step 1. If G is finite then $G = G_\pi \rtimes G_{\pi'}$ where $G_\pi = \sqrt[\pi]{H}$ is the unique Hall π-subgroup and $G_{\pi'}$ is a Hall π'-subgroup of G which exists because G is π-separable. Thus it remains to show that every π-element in G commutes with every π'-element. Take any π'-element x and any π-element g in G. Then $[g, x^{-1}] = h$ is a π-element. Thus

$$h \in \sqrt[\pi]{\langle x \rangle} \quad \text{and} \quad hx = x^g \in \sqrt[\pi]{\langle x \rangle}.$$

But x^g is a π'-element; so $x^g \in \langle x \rangle$; therefore $h \in \langle x \rangle$. Hence $h = 1$ and $x^g = x$ as required.

Step 2. We may assume G to be torsion-free. Let $P = \sqrt[\pi]{1}$, then P is a finite normal subgroup of G so replace G by G/P if necessary and assume $P = 1$. Let N be the Fitting subgroup of G. Then the torsion subgroup $T(N)$ is a finite normal subgroup of G, so again replace G by $G/T(N)$ if necessary and assume N is torsion-free. Let Q be the subgroup of G generated by all periodic (π'-elements) of G. If Q is periodic, then it is finite so replace G by G/Q and assume G is torsion-free. So it suffices to show that Q is periodic. If not, then $M = Q \cap N \neq 1$. But then by step 1, $Q/M^p (p \in \pi)$ does not have the π-isolator property since it is finite, generated by π'-elements and is not a π'-group.

Step 3. Completion of the proof. We may assume now that G is torsion-free. Suppose $x, y \in G$ and $x^n = y^n$ for some $n > 0$. Let $J = \langle x, x^y \rangle$. Then

$$Z = Z(J) \geqq \langle x^n \rangle$$

and J/Z is finite by step 2. Thus by Schur's Lemma, J' is finite, hence 1 and G is an R-group. Since a torsion-free nilpotent-by-finite R-group is nilpotent, we have the required result.

COROLLARY 15. *Let G be a finitely generated solvable group. Then G has the isolator property for some finite set π of primes if and only if G has the property* $(*)$.

Example. Let t be a root of $p(x) = x^4 - x^2 - 1$, an irreducible polynomial over $\mathbf{Q}$. Then

$$[\mathbf{Z}[t]:\mathbf{Z}] = 4 \quad \text{and} \quad [\mathbf{Z}[t^2]:\mathbf{Z}] = 2.$$

Let

$$A = \mathbf{Z} + \mathbf{Z}t + \mathbf{Z}t^2 + \mathbf{Z}t^3,$$

the additive group $\mathbf{Z}[t]$. Let $G = \langle A, \tau \rangle$ where the action of τ on A under conjugation is that of multiplication by t. Then G does not have the isolator property since the Hirsch length of the subgroup $\langle a, \tau^2 \rangle$ is three for any $a \neq 0$ in A, where as that of $\langle a, \tau \rangle$ is five. G has the property $(+)$, for let $HSnK \leqq G$, and $HK' \sim K$. If K is abelian, then $K' = 1$ and $H \sim K$. So assume $K \nleqq A$, and that K is not abelian. Then $K' \sim K \cap A$ since

$$A_1(t^n - 1) \sim A_1$$

for every τ^n invariant subgroup A_1 of A (for any $n > 0$). Hence if $HK' \sim K$, then

$$[K \cap A, H] = [K \cap A, HK'] \sim K \cap A \quad \text{and}$$

$$[K \cap A, H, \ldots, H] \sim K \cap A$$

so if $HSnK$, then $H \cap A \sim K \cap A$ and $H \sim K$.

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