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Fractals and domain theory

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We show that a measurement μ on a continuous dcpo D extends to a measurement $\bar{\mu}$ on the convex powerdomain $\mathbf{C}D$ iff it is a Lebesgue measurement. In particular, $\ker\mu$ must be metrisable in its relative Scott topology. Moreover, the space $\ker\bar{\mu}$ in its relative Scott topology is homeomorphic to the Vietoris hyperspace of $\ker\mu$, that is, the space of non-empty compact subsets of $\ker\mu$ in its Vietoris topology – the topology induced by any Hausdorff metric. This enables one to show that Hutchinson's theorem holds for any finite set of contractions on a domain with a Lebesgue measurement. Finally, after resolving the existence question for Lebesgue measurements on countably based domains, we uncover the following relationship between classical analysis and domain theory: for an ω-continuous dcpo D with $\max(D)$ regular, the Vietoris hyperspace of $\max(D)$ embeds in $\max(\mathbf{C}D)$ as the kernel of a measurement on $\mathbf{C}D$.

1. Introduction

In analysis, each hyperbolic iterated function system on a complete metric space (X, d) gives rise to a contraction on the complete metric space of compact sets in their Hausdorff metric $(\mathscr{P}_{com}(X), d_H)$. The resulting unique attractor of this higher order contraction can, for instance, be used to model fractals. It has been shown in two separate papers that domain theory could be used to derive this result.

First, Edalat (1995) used the upper space construction to prove it for compact metric spaces, and then Edalat and Heckmann (1998) used the formal ball model to give the proof for complete metric spaces in general (minus the convergence in the Hausdorff metric). But while analysis has a formal theory to describe the progression from (X, d) to $(\mathscr{P}_{com}(X), d_H)$, domain theory does not. In fact, because Edalat (1995) and Edalat and Heckmann (1998) deal only with two specific examples of domains, it is reasonable to ask whether such a theory even exists. In this paper, we will prove that one does exist.

The essential analogies to keep in mind as we progress are as follows: a complete metric space (X, d) will be replaced by a domain with a measurement (D, μ) such that $\ker\mu \simeq X$, the hyperspace $(\mathscr{P}_{com}(X), d_H)$ of compact sets will be replaced by the convex powerdomain $(\mathbf{C}D, \bar{\mu})$ such that $\ker\bar{\mu} \simeq \mathscr{P}_{com}(X)$, and the Banach fixed point theorem will be eliminated and replaced by one of the standard measurement based results. We might say that Edalat (1995) and Edalat and Heckmann (1998) offer applications of domain theory to an area of analysis and that the results presented here establish a connection between two different parts of mathematics.

There are at least two immediate benefits of this more abstract stance. The first, for domain theorists, is the homeomorphism $\ker \bar{\mu} \simeq \mathscr{P}_{com}(X)$ between the Scott topology on $\ker \bar{\mu}$ and the Vietoris topology on $\mathscr{P}_{com}(X)$. This allows us to prove the convergence in the Hausdorff metric left open from Edalat and Heckmann (1998), and to establish that the convex powerdomain provides a domain theoretic way of constructing $\mathscr{P}_{com}(X)$. The second benefit, for analysts, is a persuasive argument that in order to prove Hutchinson's theorem complete metrisability of the underlying space is *necessary*.

After reviewing some basic ideas about domains, measurement and the convex powerdomain, we determine exactly when it is that a measurement μ on a domain D extends to a measurement $\bar{\mu}$ on the convex powerdomain $\mathbf{C}D$. It turns out that only some measurements extend, these are called *Lebesgue measurements*. They become a remarkable class of measurements when one realises that, in addition to their extensible nature, they also capture metrisability: a space X is (completely) metrisable iff $X \simeq \ker \mu$ for some Lebesgue measurement μ on a continuous poset (dcpo). We also prove the domain theoretic version of Hutchinson's result for Lebesgue measurements, develop simple ways to recognise Lebesgue measurements in ordinary situations, and resolve the question of their existence.

2. Domain theory

Let $(P, \sqsubseteq)$ be a partially ordered set or *poset* (Abramsky and Jung 1994). A non-empty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* $\bigsqcup S$ of $S \subseteq P$ is the least of its upper bounds when it exists.

For elements x, y of a poset P, we write $x \ll y$ iff for every directed set S with a supremum, if $y \sqsubseteq \bigsqcup S$, we have $x \sqsubseteq s$, for some $s \in S$. Intuitively, $x \ll y$ means that any computational path to y must pass through x.

Definition 2.1. Let $(P, \sqsubseteq)$ be a poset. We set

- $\twoheaddownarrow x := \{y \in P : y \ll x\}$ and $\twoheaduparrow x := \{y \in P : x \ll y\}$
- $\downarrow x := \{y \in P : y \sqsubseteq x\}$ and $\uparrow x := \{y \in P : x \sqsubseteq y\}$

and say that $B \subseteq P$ is a *basis* for P if $\twoheaddownarrow x \cap B$ is directed with supremum x for each $x \in P$. A poset is *continuous* if it has a basis and ω*-continuous* if it has a countable basis.

For $X \subseteq P$, we define $*X := \bigcup_{x \in X} *x$ whenever $* \in \{\downarrow, \uparrow, \twoheaddownarrow, \twoheaduparrow\}$.

Definition 2.2. A subset U of a poset P is *Scott open* if $U = \uparrow U$ and

$$\bigsqcup S \in U \Rightarrow S \cap U \neq \varnothing,$$

for any directed $S \subseteq P$ with a supremum. The collection of all Scott open sets is called the *Scott topology*.

On a continuous poset P, the collection $\{\twoheaduparrow x : x \in P\}$ forms a basis for the Scott topology. A function f between posets is *Scott continuous* if it reflects Scott open sets. This is equivalent to saying that f is *monotone*,

$$(\forall x, y)\, x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y),$$

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and that it *preserves directed suprema*:

$$f(\bigsqcup S) = \bigsqcup f(S),$$

for all directed S with a supremum.

Definition 2.3. A *dcpo* is a poset in which every directed set has a supremum. A *domain* is a continuous dcpo.

In this paper, *all* topological statements about domains are made with respect to the *Scott topology*.

3. Measurement

Let $[0,\infty)^*$ denote the set of non-negative reals in the order dual to the usual one: $x \sqsubseteq y \Leftrightarrow x \geqslant y$.

Definition 3.1. A continuous map $\mu : D \to [0,\infty)^*$ is a *measurement* if for all $x \in D$ with $\mu x = 0$ and all open sets $U \subseteq D$,

$$x \in U \Rightarrow (\exists\, \varepsilon > 0)\, x \in \mu_\varepsilon(x) \subseteq U,$$

where $\mu_\varepsilon(x) := \{y \in D : y \sqsubseteq x \ \&\ \mu y < \varepsilon\}$ are the *ε-approximations* of x.

The *kernel* of a measurement μ is $\ker \mu := \{x \in D : \mu x = 0\}$. The set of *maximal elements* in a poset D is $\max(D) := \{x \in D : \uparrow x = \{x\}\}$.

Proposition 3.2. Let (D, μ) be a domain with a measurement. Then

(i) An element with measure zero is maximal, that is, $\ker \mu \subseteq \max(D)$.
(ii) For all $x \in \ker \mu$, if (x_n) is a sequence with $x_n \ll x$, then

$$\lim_{n\to\infty} \mu x_n = \mu x \Rightarrow \bigsqcup_{n \geqslant 1} x_n = x,$$

and this supremum is directed.

Proof. (i) Let $x \in \ker \mu$. If $x \sqsubseteq y$, then $y \in \ker \mu$. But since μ is a measurement, $\twoheaddownarrow y \subseteq \twoheaddownarrow x$, which gives $y \sqsubseteq x$ by the continuity of D, and thus $x = y$. This proves $x \in \max(D)$.

(ii) If $a = x_n$ and $b = x_m$, then $\twoheaduparrow a \cup \twoheaduparrow b$ is a Scott open set containing x. Because μ is a measurement and $\mu x_n \to 0$, eventually some $c := x_k$ lands in $\twoheaduparrow a \cup \twoheaduparrow b$, which means (x_n) is directed. As a directed set, it has a supremum, which by (i) is maximal, and so equal to x. □

Though we have not done so here, the definition works equally well on a continuous poset – a fact we will use briefly at the end of this paper. All the results of this section hold more generally as well. The reader unfamiliar with the following examples will find them discussed in more detail in Martin (2000b).

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Example 3.3. Domains and their canonical measurements.

(i) $(\Sigma^\infty, 1/2^{|\cdot|})$ the domain of streams in the prefix order with

$$\mu s = \frac{1}{2^{|s|}},$$

where $|\cdot| : \Sigma^\infty \to [0,\infty]$ is the length of a string.

(ii) $(\mathscr{P}\omega, |\cdot|)$ the powerset of the naturals ordered by inclusion with

$$|x| = 1 - \sum_{n\in x} \frac{1}{2^{n+1}}.$$

(iii) $([\mathbb{N} \rightharpoonup \mathbb{N}], |\mathrm{dom}|)$ the partial functions on the naturals ordered by extension with

$$\mu f = |\mathrm{dom}(f)|$$

using the measurement on $\mathscr{P}\omega$ from (ii).

(iv) $(\mathbf{I}\mathbb{R}, \mu)$ the interval domain with the length measurement $\mu[a,b] = b - a$.

(v) $(\mathbf{U}X, \mathrm{diam})$ the upper space of a locally compact metric space (X, d) with

$$\mathrm{diam}\, K = \sup\{d(x,y) : x, y \in K\}.$$

(vi) $(\mathbf{B}X, \pi)$ the formal ball model (Edalat and Heckmann 1998) of a complete metric space (X, d) with $\pi(x, r) = r$.

In each case above, we have a pair (D, μ) with $\ker\mu = \max(D)$.

4. Contractions on domains

Definition 4.1. Let D be a continuous dcpo with a measurement μ. A monotone map $f : D \to D$ is a *contraction* if there is a constant $0 \leqslant c < 1$ with

$$\mu f(x) \leqslant c \cdot \mu x$$

for all $x \in D$. The constant c is called a Lipschitz constant.

A proof of the next result can be found in Martin (2000b).

Theorem 4.2. Let D be a domain with a measurement μ such that

$$(\forall\, x, y \in D)(\exists\, z \in D)\; z \sqsubseteq x, y.$$

If $f : D \to D$ is a contraction and there is an $x \in D$ with $x \sqsubseteq f(x)$, then

$$x^\star = \bigsqcup_{n\geqslant 0} f^n(x) \in \ker\mu$$

is the unique fixed point of f on D. Further, $x^\star$ is an attractor in two different senses:

(i) for all $x \in \ker\mu$, we have $f^n(x) \to x^\star$ in the Scott topology on $\ker\mu$; and

(ii) for all $x \sqsubseteq x^\star$, we have $\bigsqcup_{n\geqslant 0} f^n(x) = x^\star$, and this supremum is a limit in the Scott topology on D.

We can use the upper space $(\mathbf{U}X, \mathrm{diam})$ to prove the Banach contraction theorem for compact metric spaces by applying the result above, or the formal ball model $(\mathbf{B}X, \pi)$ to prove it for any complete metric space X.

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Example 4.3. Let $f : X \to X$ be a contraction on a complete metric space X with Lipschitz constant $c < 1$. The mapping $f : X \to X$ extends to a monotone map $\bar{f} : \mathbf{B}X \to \mathbf{B}X$ on the formal ball model $\mathbf{B}X$ (Edalat and Heckmann 1998) given by

$$\bar{f}(x, r) = (fx, c \cdot r),$$

which satisfies

$$\pi\bar{f}(x, r) = c \cdot \pi(x, r),$$

where $\pi : \mathbf{B}X \to [0, \infty)^*$, $\pi(x, r) = r$, is the standard measurement on $\mathbf{B}X$. For all $(x, r), (y, s) \in \mathbf{B}X$, there is $z = (x, r + s + d(x, y)) \in \mathbf{B}X$ with $z \sqsubseteq (x, r), (y, s)$. Given $x \in X$, we can choose r so that $(x, r) \sqsubseteq \bar{f}(x, r)$. By Theorem 4.2, $\bar{f}$ has a unique attractor, which implies that f does too.

There are also measurement based fixed point theorems that guarantee the existence of unique *non-maximal* fixed points for monotone maps (Martin 2001), as well as those which apply to *non-monotonic* mappings (Martin 2000b).

5. The convex powerdomain

A useful technique for constructing domains is to take the *ideal completion* of an *abstract basis.*

Definition 5.1. An *abstract basis* is given by a set B together with a transitive relation $<$ on B which is *interpolative,* that is,

$$M < x \Rightarrow (\exists\, y \in B)\, M < y < x$$

for all $x \in B$ and all finite subsets M of B.

Abstract bases are covered in Abramsky and Jung (1994), which is where one finds the following.

Definition 5.2. An *ideal* in $(B, <)$ is a non-empty subset I of B such that:

(i) I is a lower set: $(\forall\, x \in B)(\forall\, y \in I)\, x < y \Rightarrow x \in I$.
(ii) I is directed: $(\forall\, x, y \in I)(\exists\, z \in I)\, x, y < z$.

The collection of ideals of an abstract basis $(B, <)$ ordered under inclusion is a partially ordered set called the *ideal completion* of B. We denote this poset by $\bar{B}$.

The set $\{y \in B : y < x\}$ for $x \in B$ is an ideal that leads to a natural mapping from B into $\overline{B}$ given by $i(x) = \{y \in B : y < x\}$.

Proposition 5.3. If $(B, <)$ is an abstract basis, then:

(i) Its ideal completion $\bar{B}$ is a dcpo.
(ii) For $I, J \in \bar{B}$,

$$I \ll J \Leftrightarrow (\exists\, x, y \in B)\, x < y\ \&\ I \subseteq i(x) \subseteq i(y) \subseteq J.$$

(iii) $\bar{B}$ is a continuous dcpo with basis $i(B)$.

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If one takes any basis B of a domain D and restricts the approximation relation $\ll$ on D to B, they are left with an abstract basis $(B, \ll)$ whose ideal completion is D. Thus, all domains arise as the ideal completion of an abstract basis. We now use this technique to construct a domain called the *convex powerdomain*. This is discussed in more detail in Abramsky and Jung (1994).

Definition 5.4. Let D be a continuous dcpo. For subsets $A, B \subseteq D$, we define relations:

— $A \ll_{\mathrm{L}} B \Leftrightarrow (\forall a \in A)(\exists b \in B)\, a \ll b$
— $A \ll_{\mathrm{U}} B \Leftrightarrow (\forall b \in B)(\exists a \in A)\, a \ll b$
— $A \ll_{\mathrm{EM}} B \Leftrightarrow A \ll_{\mathrm{L}} B \;\&\; A \ll_{\mathrm{U}} B$.

In the same way, we derive $\sqsubseteq_{\mathrm{L}}, \sqsubseteq_{\mathrm{U}}$ and $\sqsubseteq_{\mathrm{EM}}$ from the order $\sqsubseteq$ on D.

Definition 5.5. The non-empty finite subsets of a space X are denoted $\mathscr{P}_{fin}(X)$, while its non-empty compact subsets are written as $\mathscr{P}_{com}(X)$.

The set $\mathscr{P}_{fin}(D)$ together with $\ll_{\mathrm{EM}}$ is an abstract basis.

Definition 5.6. The *convex powerdomain* $\mathbf{C}D$ of a continuous dcpo D is the ideal completion of the abstract basis $(\mathscr{P}_{fin}(D), \ll_{\mathrm{EM}})$.

Definition 5.7. For a Scott compact $K \in \mathscr{P}_{com}(D)$, we set

$$K^* = \{F \in \mathscr{P}_{fin}(D) : F \ll_{\mathrm{EM}} K\}.$$

Notice that this operation is also defined for elements of $\mathscr{P}_{fin}(D)$.

Proposition 5.8. For a continuous dcpo D, we have:

(i) If $K \in \mathscr{P}_{com}(D)$, then $K^* = \{F \in \mathscr{P}_{fin}(D) : F \ll_{\mathrm{EM}} K\} \in \mathbf{C}D$.
(ii) For ideals $I, J \in \mathbf{C}D$,

$$I \ll J \Leftrightarrow (\exists F, G \in \mathscr{P}_{fin}(D))\, F \ll_{\mathrm{EM}} G \;\&\; I \subseteq F^* \subseteq G^* \subseteq J.$$

(iii) For $F \in \mathscr{P}_{fin}(D)$ and $I \in \mathbf{C}D$, $F \in I \Leftrightarrow F^* \ll I$.
(iv) For $F, G \in \mathscr{P}_{fin}(D)$, $F^* \sqsubseteq G^*$ in $\mathbf{C}D \Leftrightarrow F \sqsubseteq_{\mathrm{EM}} G$.

Here is how we extend continuous maps on D to ones on $\mathbf{C}D$.

Definition 5.9. If $f : D \to D$ is monotone, we extend it to a mapping

$$\bar{f} : \mathbf{C}D \to \mathbf{C}D$$

by setting

$$\bar{f}(I) = \bigcup_{F \in I} f(F)^*$$

for an ideal $I \in \mathbf{C}D$.

In addition, $\mathbf{C}D$ has a union operation we will need later on.

Definition 5.10. The function $+ : \mathbf{C}D \times \mathbf{C}D \to \mathbf{C}D$ is given by

$$I + J = \{\, H \in \mathscr{P}_{fin}(D) \,|\, \exists\, F \in I, G \in J : H \ll_{\mathrm{EM}} F \cup G \,\}.$$

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Lemma 5.11. Let D be a continuous dcpo. Then:

(i) If $f : D \to D$ is monotone, then $\bar{f} : \mathbf{C}D \to \mathbf{C}D$ is Scott continuous.
(ii) The operation $+ : \mathbf{C}D \times \mathbf{C}D \to \mathbf{C}D$ is Scott continuous, commutative, associative and idempotent. For $K, L \in \mathscr{P}_{com}(D)$, $K^* + L^* = (K \cup L)^*$.

Proof. To see (i), note that for $F \in \mathscr{P}_{fin}(D)$, $F^* \ll I \Leftrightarrow F \in I$. Thus, the definition of $\bar{f}$ may be recast as

$$\bar{f}(I) = \bigsqcup \{\bar{f}(F^*) : F^* \ll I, F \in \mathscr{P}_{fin}(D)\}.$$

But this is the general technique by which a monotone map defined on a basis is extended to a Scott continuous map on the entire domain.

Part (ii) is in Edalat and Heckmann (1998). $\square$

Here is the question around which the present work revolves: if we have a measurement μ on a domain D, how can we obtain a measurement on $\mathbf{C}D$?

6. A measurement on the convex powerdomain

By induction, a continuous map $\mu : D \to [0, \infty)^*$ is a measurement iff for all *finite* $F \subseteq \ker \mu$ and all open sets $U \subseteq D$,

$$F \subseteq U \Rightarrow (\exists\, \varepsilon > 0)(\forall x \in F)\, \mu_\varepsilon(x) \subseteq U.$$

If we require this to hold not just for finite sets F, but for *all* compact sets K, we have exactly a Lebesgue measurement.

Definition 6.1. A *Lebesgue measurement* $\mu : D \to [0, \infty)^*$ is a continuous map such that for all compact sets $K \subseteq \ker \mu$ and all open sets $U \subseteq D$,

$$K \subseteq U \Rightarrow (\exists\, \varepsilon > 0)(\forall x \in K)\, \mu_\varepsilon(x) \subseteq U.$$

Not all measurements are Lebesgue (see Martin (2000a, Example 5.3.2)). Lebesgue measurements are the measurements that extend to the convex powerdomain.

Definition 6.2. Let $\mu : D \to [0, \infty)^*$ be a monotone map on a continuous dpco. We first extend it to the abstract basis $(\mathscr{P}_{fin}(D), \ll_{EM})$ via

$$\mu_f : \mathscr{P}_{fin}(D) \to [0, \infty)^*$$

$$F \mapsto \max\{\mu x : x \in F\},$$

and then to the convex powerdomain $\mathbf{C}D$ by

$$\bar{\mu} : \mathbf{C}D \to [0, \infty)^*$$
$$I \mapsto \inf\{\mu_f(F) : F \in I\}.$$

When we speak of a measurement μ extending to $\mathbf{C}D$, we mean that the mapping $\bar{\mu}$ is a measurement.

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Lemma 6.3. If $\mu : D \to [0,\infty)^*$ is Scott continuous on a domain, then:

(i) The map $\bar{\mu} : \mathbf{C}D \to [0,\infty)^*$ is Scott continuous.
(ii) For all $F \in \mathscr{P}_{fin}(D)$, $\bar{\mu}(F^*) = \mu_f(F)$.
(iii) If $K \in \mathscr{P}_{com}(\ker\mu)$, then $\bar{\mu}(K^*) = 0$.

Proof. (i) If μ is monotone, then μ_f is monotone on an abstract basis. The map $\bar{\mu}$ is defined by $\bar{\mu}(I) = \bigsqcup_{F\in I} \mu_f(F)$. Thus, it is the greatest Scott continuous map on $\mathbf{C}D$ satisfying $\bar{\mu}(F^*) \sqsubseteq \mu_f(F)$ for all $F \in \mathscr{P}_{fin}(D)$. This technique works for any abstract basis; the details may be found in Abramsky and Jung (1994).

(ii) Let $F \in \mathscr{P}_{fin}(D)$ and choose $x \in F$ with $\mu x = \mu_f(F)$. From (i), we have $\mu_f(F) \leqslant \bar{\mu}(F^*)$. Now let $n \geqslant 1$ be arbitrary. For each $x_i \in F$, use the continuity of μ to choose $a_i \ll x_i$ with

$$\mu x_i \leqslant \mu a_i < \mu x + 1/n,$$

which is possible since $\mu x_i \leqslant \mu x$. Then for the finite set $G_n = \{a_i : x_i \in F\}$, we see that $G_n \ll_{\text{EM}} F$, which gives

$$\bar{\mu}(F^*) \leqslant \mu_f(G_n) < \mu x + 1/n = \mu_f(F) + 1/n,$$

and hence $\bar{\mu}(F^*) \leqslant \mu_f(F)$.

(iii) Let $n \geqslant 1$ be fixed. For each $k \in K$, there is $a_k \ll k$ with $\mu a_k < 1/n$. Then a finite number of the a_k cover K by compactness. This yields a finite set F with $F \ll_{\text{EM}} K$ and $\mu_f(F) < 1/n$. Hence, $\bar{\mu}(K^*) < 1/n$ for each $n \geqslant 1$. □

Naturally, we now wonder when it is that $\bar{\mu}$ is a measurement on $\mathbf{C}D$. Before we can answer this, we need an important lemma.

Lemma 6.4. Let $\mu : D \to [0,\infty)^*$ be a Lebesgue measurement on a continuous dcpo. Suppose that $F \in \mathscr{P}_{fin}(D)$ and $K \in \mathscr{P}_{com}(\ker\mu)$ with $F \ll_{\text{EM}} K$. Then there is $\lambda > 0$ such that for every $G \in \mathscr{P}_{fin}(D)$,

$$G \ll_{\text{EM}} K \;\&\; \mu_f(G) < \lambda \Rightarrow F \ll_{\text{EM}} G.$$

Proof. For each $x_i \in F$, choose $k_i \in K$ with $x_i \ll k_i$. Because μ is a measurement, $(\exists\, \varepsilon_i > 0)\, k_i \in \mu_{\varepsilon_i}(k_i) \subseteq \mathord{\uparrow\!\!\uparrow} x_i$. In addition, μ is a Lebesgue measurement and $K \subseteq \mathord{\uparrow\!\!\uparrow} F$ is compact, so

$$(\exists \delta > 0)(\forall k \in K)\, k \in \mu_\delta(k) \subseteq \mathord{\uparrow\!\!\uparrow} F.$$

Let $0 < \lambda < \min(\{\varepsilon_i : x_i \in F\} \cup \{\delta\})$. If $G \in \mathscr{P}_{fin}(D)$ with $G \ll_{\text{EM}} K$ and $\mu_f(G) < \lambda$, we claim that $F \ll_{\text{EM}} G$.

To see that $F \ll_{\text{L}} G$, let $x_i \in F$. Then we know $x_i \ll k_i \in K$. Since $G \ll_{\text{EM}} K$, there is $y \in G$ with $y \ll k_i$. Because $\mu y \leqslant \mu_f(G) < \lambda < \varepsilon_i$, we see that $y \in \mu_{\varepsilon_i}(k_i) \subseteq \mathord{\uparrow\!\!\uparrow} x_i$, which gives $x_i \ll y \in G$.

For $F \ll_{\text{U}} G$, let $y \in G$. Since $G \ll_{\text{EM}} K$, there is $k \in K$ with $y \ll k$. Then $\mu y \leqslant \mu_f(G) < \lambda < \delta$, so $y \in \mu_\delta(k) \subseteq \mathord{\uparrow\!\!\uparrow} F$. Hence, there is $x \in F$ with $x \ll y$. □

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Theorem 6.5. For a Scott continuous $\mu : D \to [0,\infty)^*$ on a continuous dcpo D, the following are equivalent:

(i) The mapping μ is a Lebesgue measurement.
(ii) The canonical extension of μ to the convex powerdomain

$$\bar{\mu} : \mathbf{C}D \to [0,\infty)^*$$

is a measurement.

In either case, $\ker \bar{\mu} = \{K^* : K \in \mathscr{P}_{com}(\ker \mu)\}$.

Proof. (i) $\Rightarrow$ (ii) Let $A \ll I$ in $\mathbf{C}D$ with $\bar{\mu}(I) = 0$. By the directedness of I, there is a sequence of finite sets (F_n) such that

$$F_n \in I,\ \mu_f(F_n) < 1/n \ \&\ F_n \ll_{\mathrm{EM}} F_{n+1}$$

for all $n \geqslant 1$. Set $K = \bigcap_{n\geqslant 1} \uparrow F_n$. This set is non-empty and compact by the Hofmann-Mislove Theorem. In addition, notice that we also have $F_n \ll_{\mathrm{EM}} K \subseteq \ker \mu$ for all $n \geqslant 1$.

First we prove that $I \subseteq K^*$. Let $F \in I$ be arbitrary. Using the directedness of I, choose $M_1 \in I$ with $F, F_1 \ll_{\mathrm{EM}} M_1$, and given any M_n, choose $M_{n+1} \in I$ with $M_n, F_{n+1} \ll_{\mathrm{EM}} M_{n+1}$. Let $M = \bigcap \uparrow M_n$ and notice again that $M \in \mathscr{P}_{com}(\ker \mu)$. Because $F \ll_{\mathrm{EM}} M$, $F_n \ll_{\mathrm{EM}} M$ for all $n \geqslant 1$, and $\mu_f(F_n) \to 0$, Lemma 6.4 implies that $F \ll_{\mathrm{EM}} F_i$ for all i sufficiently large. But we also know that $F_i \ll_{\mathrm{EM}} K$, so transitivity of $\ll_{\mathrm{EM}}$ gives $F \in K^*$. Hence $I \subseteq K^*$.

Finally, $\bar{\mu}$ is a measurement. Since $A \ll I$, there are $F, G \in \mathscr{P}_{fin}(D)$ with $F \ll_{\mathrm{EM}} G$ and $A \subseteq F^* \subseteq G^* \subseteq I$, by Proposition 5.8. Because $F \in G^* \subseteq K^*$, we have $F \ll_{\mathrm{EM}} K$. Using Lemma 6.4, choose a $\lambda > 0$ with respect to $F \ll_{\mathrm{EM}} K$. We will prove that

$$I \in \bar{\mu}_\lambda(I) \subseteq \Uparrow A.$$

If $J \subseteq I$ and $\bar{\mu}(J) < \lambda$, there is an $H \in J$ with $\mu_f(H) < \lambda$. But then we see $H \in I \subseteq K^*$ and $\mu_f(H) < \lambda$, so by the choice of λ, $F \ll_{\mathrm{EM}} H$. Hence,

$$A \subseteq F^* \subseteq H^* \subseteq J \ \&\ F \ll_{\mathrm{EM}} H,$$

which gives $J \in \Uparrow A$. Thus, $\bar{\mu}$ is a measurement.

(ii) $\Rightarrow$ (i) Let $K \subseteq \ker \mu$ be Scott compact and $U \subseteq D$ be Scott open with $K \subseteq U$. By the compactness of K, there is a finite set $F \subseteq U$ with $K \subseteq \Uparrow F$ and $F \ll_{\mathrm{EM}} K$. Thus, $F^* \ll K^*$, using Proposition 5.8(iii). By Lemma 6.3(iii), $K^* \in \ker \bar{\mu}$, and since $\bar{\mu}$ is a measurement, there is a $\lambda > 0$ with

$$K^* \in \bar{\mu}_\lambda(K^*) \subseteq \Uparrow F^*.$$

We claim that $k \in \mu_\lambda(k) \subseteq U$ for all $k \in K$.

First suppose that $k \in K$ and $x \ll k$ with $\mu x < \lambda$. By compactness of K and continuity of μ, there is a finite set G with $x \in G$, $\mu_f(G) < \lambda$ and $G \ll_{\mathrm{EM}} K$. But then

$$G^* \ll K^* \ \&\ \bar{\mu}(G^*) = \mu_f(G) < \lambda,$$

which means $F^* \ll G^*$. Hence, $F \sqsubseteq_{\mathrm{EM}} G$, by Proposition 5.8(iv). Thus, there is a $y \in F$ with $y \sqsubseteq x$ since $x \in G$. Then $x \in \uparrow F \subseteq U$.

In general, if $x \in \mu_\lambda(k)$, use the directedness of $\mathord{\downarrow\!\!\downarrow} x$ and continuity of μ to choose $a \ll x$ with $\mu a < \lambda$. By the previous argument, $a \in U$, and since $U = \uparrow U$, we get $x \in U$. Hence, $k \in \mu_\lambda(k) \subseteq U$ for all $k \in K$, which means μ is a Lebesgue measurement.

Now we calculate $\ker \bar{\mu}$. The inclusion $\{K^* : K \in \mathscr{P}_{com}(\ker \mu)\} \subseteq \ker \bar{\mu}$ is clear by Lemma 6.3(iii). For the other, suppose that $\bar{\mu}(I) = 0$. Then, as in the proof of (i) $\Rightarrow$ (ii), there is a compact $K \subseteq \ker \mu$ with $I \subseteq K^*$. But $\bar{\mu}$ is a measurement, so $I \in \max(\mathbf{C}D)$. Hence $I = K^*$, and we are finished. □

In fact, the relationship between the kernel of $\bar{\mu}$ and the compact subsets of $\ker \mu$ is much stronger than the last theorem shows.

7. A model of Vietoris hyperspace

We now exhibit the fundamental topological relationship that exists between $\ker \bar{\mu}$ and $\ker \mu$.

Definition 7.1. The *Vietoris hyperspace* of a Hausdorff space X is the set of all non-empty compact subsets $\mathscr{P}_{com}(X)$ with the. *Vietoris topology:* it has a basis given by all sets of the form

$$\sigma(U_1, \cdots, U_n) := \{K \in \mathscr{P}_{com}(X) : K \subseteq \bigcup_{i=1}^{n} U_i \text{ and } K \cap U_i \neq \emptyset, 1 \leqslant i \leqslant n\},$$

where U_i is a non-empty open subset of X, for each $1 \leqslant i \leqslant n$.

Notice that if $\mathscr{B}$ is a basis for the topology on X, then the collection $\{\sigma(B_1, \cdots, B_n) : B_i \in \mathscr{B}, 1 \leqslant i \leqslant n\}$ is a basis for the Vietoris topology on $\mathscr{P}_{com}(X)$. We make use of this in the proof of Theorem 7.3 below.

Lemma 7.2. The kernel of a Lebesgue measurement is Hausdorff.

Proof. If *distinct points* $a, b \in \ker \mu$ cannot be separated by open sets, we can use the continuity of D and the measurement μ to find a sequence $x_n \in \ker \mu$ with $x_n \to a$ and $x_n \to b$. Now let $z \ll a$.

Because $\mathord{\uparrow\!\!\uparrow} z$ is Scott open around a, there is an integer K_1 with $x_i \in \mathord{\uparrow\!\!\uparrow} z$ for $i \geqslant K_1$. Since $\{x_i : i \geqslant K_1\} \cup \{a\}$ is a compact subset of $\ker \mu$ and μ is Lebesgue,

$$(\exists \lambda > 0)\, \mu_\lambda(x_i) \subseteq \mathord{\uparrow\!\!\uparrow} z$$

for all $i \geqslant K_1$. Now choose $b_\lambda \ll b$ with $\mu b_\lambda < \lambda$. Then there is also K_2 with $x_j \in \mathord{\uparrow\!\!\uparrow} b_\lambda$ for $j \geqslant K_2$. Thus, for $n \geqslant \max(K_1, K_2)$, we have

$$b_\lambda \ll x_n \;\&\; \mu_\lambda(x_n) \subseteq \mathord{\uparrow\!\!\uparrow} z,$$

which gives $z \ll b_\lambda \ll b$.

Since z was arbitrary, we have shown $\mathord{\downarrow\!\!\downarrow} a \subseteq \mathord{\downarrow\!\!\downarrow} b$, which gives $a \sqsubseteq b$. But $\ker \mu \subseteq \max(D)$, so $a = b$, contradicting $a \neq b$. □

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Theorem 7.3. If $\mu : D \to [0,\infty)^*$ is a Lebesgue measurement on a domain, then the correspondence

$$\mathscr{P}_{com}(\ker \mu) \to \ker \bar{\mu}$$

$$K \mapsto K^*$$

is a homeomorphism between the Vietoris hyperspace of $\ker \mu$ and $\ker \bar{\mu}$ in its relative Scott topology.

Proof. The surjectivity follows from Theorem 6.5. Suppose that we have $K^* = L^*$ for $L, K \in \mathscr{P}_{com}(\ker \mu)$. Let $x \in K$ and use μ to choose an increasing sequence (x_n) with $x_n \ll x$ and $\mu x_n < 1/n$. By the compactness of K, for each $n \geqslant 1$ there is a finite set F_n with $x_n \in F_n$ and $F_n \ll_{EM} K$. Then $F_n \in L^*$, so

$$(\forall n \geqslant 1)(\exists\, a_n \in L)\, x_n \ll a_n.$$

As the sequence (x_n) is increasing, it is clear that $a_n \to x$ in $\ker \mu$. But $\ker \mu$ is Hausdorff, so $L \subseteq \ker \mu$ being compact must also be closed, which puts $\lim a_n = x \in L$. Thus, $K \subseteq L$. The same argument proves $L \subseteq K$.

To see that this mapping is a homeomorphism, note that for $F = \{a_1, \cdots, a_n\} \subseteq D$, we have

$$K^* \in \Uparrow F^* \cap \ker \bar{\mu} \Leftrightarrow K \in \sigma(\Uparrow a_1 \cap \ker \mu, \cdots, \Uparrow a_n \cap \ker \mu).$$

Using the remark after Definition 7.1, this map preserves and reflects basic open sets, finishing the proof. □

Thus, if a domain D models a space X, the convex powerdomain $\mathbf{C}D$ models the Vietoris hyperspace of X.

8. Contractions on the convex powerdomain

We now show that contractions on domains extend to contractions on the convex powerdomain. First recall the following definition.

Definition 8.1. Let D be a continuous dcpo with a measurement μ. A monotone map $f : D \to D$ is a *contraction* if there is a constant $0 \leqslant c < 1$ with

$$\mu f(x) \leqslant c \cdot \mu x$$

for all $x \in D$. The constant c is called a Lipschitz constant.

From now on in this section, we assume that D is a continuous dcpo with a *Lebesgue measurement* μ. Its convex powerdomain $\mathbf{C}D$ then carries the measurement $\bar{\mu}$ studied in the last two sections. We also assume that all *contractions are Scott continuous*. Notice that $\bar{f}(F^*) = (f(F))^*$ for $F \in \mathscr{P}_{fin}(D)$ when $f : D \to D$ is Scott continuous.

Proposition 8.2. If $f, g : D \to D$ are contractions with respect to μ, then

$$h : \mathbf{C}D \to \mathbf{C}D$$

$$hx = \bar{f}x + \bar{g}x$$

is a contraction with respect to $\bar{\mu}$.

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Proof. Let f and g have Lipschitz constants c_f and c_g, respectively. We will show that h has Lipschitz constant $\max\{c_f, c_g\}$. First suppose $F \in \mathscr{P}_{fin}(D)$. By Lemma 5.11,

$$h(F^*) = \bar{f}(F^*) + \bar{g}(F^*) = (f(F))^* + (g(F))^* = (f(F) \cup g(F))^*,$$

which enables the estimate

$$\begin{aligned} \bar{\mu}(h(F^*)) &= \bar{\mu}(f(F) \cup g(F))^* \\ &= \max(\mu f(F) \cup \mu g(F)) \\ &\leqslant \max\{c_f, c_g\} \cdot \max \mu(F) \\ &= \max\{c_f, c_g\} \cdot \bar{\mu}(F^*), \end{aligned}$$

where the second and third equalities follow from Lemma 6.3(ii). Now let $I \in \mathbf{C}D$ be arbitrary. If $F \in I$, then $F^* \sqsubseteq I$, which gives

$$\bar{\mu}(h(I)) \leqslant \bar{\mu}(h(F^*)) \leqslant \max\{c_f, c_g\} \cdot \bar{\mu}(F^*) = \max\{c_f, c_g\} \cdot \mu_f(F),$$

and so by the definition of $\bar{\mu}$, we have $\bar{\mu}(h(I)) \leqslant \max\{c_f, c_g\} \cdot \bar{\mu}(I)$. □

The contraction theorem (Theorem 4.2) can now be applied to $\bar{f} + \bar{g}$. We follow this idea to its natural conclusion, which is a significant extension of Theorem 4.2.

Proposition 8.3. If $f : D \to D$ is a contraction with respect to μ, then

$$\bar{f}(K^*) = (f(K))^*$$

for all non-empty compact subsets $K \subseteq \ker \mu$.

Proof. First, if $K \in \mathscr{P}_{com}(\ker \mu)$, then $f(K) \in \mathscr{P}_{com}(\ker \mu)$, since $f|_{\ker \mu}$ is a continuous selfmap on $\ker \mu$. Thus, $(f(K))^*$ is an element of $\mathbf{C}D$.

Now we show $\bar{f}(K^*) \sqsubseteq (f(K))^*$. If $G \in \bar{f}(K^*)$, then

$$G \in \bar{f}(K^*) = \bigcup_{F \in K^*} (f(F))^*,$$

so there is $F \in \mathscr{P}_{fin}(D)$ with $F \ll_{\text{EM}} K$ and $G \ll_{\text{EM}} f(F)$. Then $G \ll_{\text{EM}} f(F) \sqsubseteq_{\text{EM}} f(K)$, which gives $G \ll_{\text{EM}} f(K)$, and hence $G \in (f(K))^*$.

Finally, $\bar{f}$ is a contraction on $\mathbf{C}D$, by Proposition 8.2 (applied with $f = g$), and $K^* \in \ker \bar{\mu}$, by Theorem 6.5, so $\bar{f}(K^*) \in \ker \bar{\mu} \subseteq \max(\mathbf{C}D)$. Thus, $\bar{f}(K^*) = (f(K))^*$. □

This brings us to the main result of this section – the domain theoretic analogue of Hutchinson's theorem (Hutchinson 1981).

Theorem 8.4. Let D be a continuous dcpo such that

$$(\forall\, x, y \in D)(\exists\, z \in D)\, z \sqsubseteq x, y.$$

If $f : D \to D$ and $g : D \to D$ are contractions for which

$$(\exists\, x \in D)\, x \sqsubseteq f(x)\ \&\ x \sqsubseteq g(x),$$

then there is a unique $K \in \mathscr{P}_{com}(\ker \mu)$ such that $f(K) \cup g(K) = K$. In addition, it is an attractor,

$$(\forall C \in \mathscr{P}_{com}(\ker \mu))\, (f \cup g)^n(C) \to K,$$

in the Vietoris topology on $\mathscr{P}_{com}(\ker \mu)$.

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Proof. First we prove that **C**D has the same property as we have assumed for D. Let $I, J \in \mathbf{C}D$ and $F \in I, G \in J$. The set $F \cup G$ is finite, so, by induction, there is $z \in D$ with $\{z\} \sqsubseteq_{\mathrm{E}M} F$ and $\{z\} \sqsubseteq_{\mathrm{E}M} G$. Thus,

$$\{z\}^* \sqsubseteq F^* \sqsubseteq I \text{ and } \{z\}^* \sqsubseteq G^* \sqsubseteq J.$$

Next, for the mapping $h = \bar{f} + \bar{g}$ we see that $\{x\}^* \sqsubseteq h\{x\}^*$, by first noting $h\{x\}^* = \{f(x), g(x)\}^*$, and then $\{x\} \sqsubseteq_{\mathrm{E}M} \{f(x), g(x)\}$, which finally gives $\{x\}^* \sqsubseteq \{f(x), g(x)\}^* = h\{x\}^*$.

Then, since h is a contraction with respect to $\bar{\mu}$ (Proposition 8.2), Theorem 4.2 ensures that it has a unique fixed point given by

$$\mathrm{fix}(h) = \bigsqcup_{n \geqslant 0} h^n\{x\}^* = K^* \in \ker \bar{\mu},$$

where $K \in \mathscr{P}_{com}(\ker \mu)$.

Now observe that for any $C \in \mathscr{P}_{com}(\ker \mu)$,

$$h(C^*) = \bar{f}(C^*) + \bar{g}(C^*) = (f(C))^* + (g(C))^* = (f(C) \cup g(C))^*,$$

where the second equality follows from Proposition 8.3. Then, since $h(K^*) = K^*$, we have $(f(K) \cup g(K))^* = K^*$, which gives $f(K) \cup g(K) = K$, using the bijection of Theorem 7.3.

For the uniqueness of K, if $C \in \mathscr{P}_{com}(\ker \mu)$ satisfies $f(C) \cup g(C) = C$, then $h(C^*) = C^*$, which by the uniqueness of K^* gives $K^* = C^*$. But then once again (Theorem 7.3 yields) $K = C$.

Finally, the fact that K is an attractor for the map

$$f \cup g : \mathscr{P}_{com}(\ker \mu) \to \mathscr{P}_{com}(\ker \mu) :: C \mapsto f(C) \cup g(C)$$

in the Vietoris topology follows from the fact that h is a contraction with respect to $\bar{\mu}$, the equality $h(C^*) = (f(C) \cup g(C))^*$ for $C \in \mathscr{P}_{com}(\ker \mu)$, and the homeomorphism $\ker \bar{\mu} \simeq \mathscr{P}_{com}(\ker \mu)$ (Theorem 7.3). □

Corollary 8.5. If $f : D \to D$ and $g : D \to D$ are contractions on a domain with least element $\perp$, then there is a unique $K \in \mathscr{P}_{com}(\ker \mu)$ such that $f(K) \cup g(K) = K$. In addition, K is an attractor for $f \cup g$ in the Vietoris topology.

Nothing but the desire for elegance prevents us from extending these results from two to n contractions. Later though, when proving Hutchinson's theorem, the extension to n maps will be worth remembering.

9. Examples of Lebesgue measurements

In order to apply the previous results, we need a simple and clear way to recognise Lebesgue measurements. Let $f : [0, \infty)^2 \to [0, \infty)$ be a function such that $f(x_n, y_n) \to 0$ whenever $x_n, y_n \to 0$.

Theorem 9.1. If $\mu : D \to [0, \infty)^*$ is a measurement such that for all pairs $x, y \in D$ with an upper bound

$$(\exists\, z \sqsubseteq x, y\,)\; \mu z \leqslant f(\mu x, \mu y),$$

then μ is a Lebesgue measurement.

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Proof. First, $\ker\mu$ is Hausdorff. Let $x, y \in \ker\mu$ be distinct. Using μ, we can find increasing sequences (a_n) and (b_n) such that $a_n \ll x$, $\bigsqcup a_n = x$ and $b_n \ll y$, $\bigsqcup b_n = y$. If $\twoheaduparrow a_n \cap \twoheaduparrow b_n = \emptyset$, the proof is done. Otherwise, there is $c_n \in \twoheaduparrow a_n \cap \twoheaduparrow b_n \cap \ker\mu$, and then, by assumption, we have

$$(\exists d_n \sqsubseteq a_n, b_n)\, \mu d_n \leqslant f(\mu a_n, \mu b_n).$$

Since $\mu a_n, \mu b_n \to 0$, we have $f(\mu a_n, \mu b_n) \to 0$, so $\mu d_n \to 0$. But $d_n \sqsubseteq a_n \ll x$ and $d_n \sqsubseteq b_n \ll y$, so Proposition 3.2(ii) gives

$$\bigsqcup d_n = x = y,$$

contradicting $x \neq y$. Thus $\ker\mu$ is Hausdorff, which ensures that *compact sets are closed* in the remainder of the proof.

Let U be an open set containing a compact set $K \subseteq \ker\mu$. To produce a contradiction, suppose that

$$(\forall n \geqslant 1)(\exists\, x_n \in K)\, \mu_{1/n}(x_n) \not\subseteq U.$$

The compactness of K lets us assume that (x_n) has a limit $x \in K$. Then there is a sequence (y_n) with $y_n \sqsubseteq x_n$, $\mu y_n < 1/n$ and $y_n \notin U$. For the contradiction, we will show that some y_n belongs to U.

Let $a_k \ll x$ with $\mu a_k < 1/k$. For each $k \geqslant 1$, let $n_k \geqslant k$ be the first integer for which $x_{n_k} \in \twoheaduparrow a_k$. Then y_{n_k} and a_k are bounded above by x_{n_k}. Thus,

$$(\forall k \geqslant 1)(\exists z_k \sqsubseteq y_{n_k}, a_k)\, \mu z_k \leqslant f(\mu y_{n_k}, \mu a_k).$$

As $k \to \infty$, $n_k \to \infty$, so $\mu y_{n_k}, \mu a_k \to 0$, which means $\mu z_k \to 0$. But $z_k \sqsubseteq a_k \ll x$. Because μ is a measurement, the sequence (z_k) is directed with supremum x, by Proposition 3.2(ii). Then some $z_k \in U$, which puts $y_{n_k} \in U$, since $U = \uparrow U$. □

The value of this result is that it identifies a condition satisfied by many of the Lebesgue measurements encountered in practice. For instance, just consider the number of examples covered by $f(s,t) = 2 \cdot \max\{s,t\}$.

Example 9.2. Lebesgue measurements.

(i) The domain of streams $(\Sigma^\infty, 1/2^{|\cdot|})$.
(ii) The powerset of the naturals $(\mathscr{P}\omega, |\cdot|)$.
(iii) The domain of partial maps $([\mathbb{N} \rightharpoonup \mathbb{N}], |\mathrm{dom}|)$.
(iv) The interval domain $(\mathbf{I}\mathbb{R}, \mu)$.
(v) The upper space $(\mathbf{U}X, \mathrm{diam})$ of a locally compact metric space (X, d).
(vi) The formal ball model $(\mathbf{B}X, \pi)$ of a complete metric space (X, d).

In fact, $f(s,t) = s + t$ applies to (i)–(v).

10. Hyperbolic iterated function systems

We are now going to apply Theorem 8.4 to obtain the classical result of Hutchinson (1981) for hyperbolic iterated function systems on complete metric spaces.

Definition 10.1. An *iterated function system* (IFS) on a space X is a non-empty finite collection of continuous selfmaps on X. We write an IFS as $(X; f_1, \ldots, f_n)$.

Definition 10.2. An IFS $(X; f_1, \ldots, f_n)$ is *hyperbolic* if X is a complete metric space and f_i is a contraction for all $1 \leqslant i \leqslant n$.

Definition 10.3. Let (X, d) be a metric space. The *Hausdorff metric* on $\mathscr{P}_{com}(X)$ is

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A), \}$$

for $A, B \in \mathscr{P}_{com}(X)$.

Hyperbolic iterated function systems are used to model fractals: given a fractal image, one searches for a hyperbolic IFS that models it. But what does it mean to model an *image*? The answer is given by Hutchinson's fundamental result (Hutchinson 1981).

Theorem 10.4 (Hutchinson). If $(X; f_1, \ldots, f_n)$ is a hyperbolic IFS on a complete metric space X, then there is a unique non-empty compact subset $K \subseteq X$ such that

$$K = \bigcup_{i=1}^{n} f_i(K).$$

Moreover, for any non-empty compact set $C \subseteq X$, we have $(\bigcup_{i=1}^{n} f_i)^k(C) \to K$ in the Hausdorff metric d_H as $k \to \infty$.

At this stage, we can see that what will be most difficult in proving such a result is the convergence in the Hausdorff metric. Luckily, this topology is *independent* of the metric d on X.

Theorem 10.5. Let (X, d) be a metric space. Then the topology induced by the Hausdorff metric d_H on $\mathscr{P}_{com}(X)$ is the Vietoris topology on $\mathscr{P}_{com}(X)$.

In Edalat (1995), the upper space $\mathbf{U}X$ is used to give a domain theoretic proof of Theorem 10.4 in the special case of a compact metric space X. Here is an alternative proof using Theorem 8.4.

Example 10.6. If we have two contractions $f, g : X \to X$ on a compact metric space X, they have Scott continuous extensions

$$\bar{f}, \bar{g} : \mathbf{U}X \to \mathbf{U}X$$

that are contractions on $\mathbf{U}X$ with respect to $\lambda = \text{diam}$. But λ is a Lebesgue measurement on a domain $\mathbf{U}X$ with bottom element $\perp = X$. Thus,

$$(\exists! K \in \mathscr{P}_{com}(\ker \lambda))\, \bar{f}(K) \cup \bar{g}(K) = K,$$

by the Corollary to Theorem 8.4. Because $\ker \lambda \simeq X$ and the mappings $\bar{f}, \bar{g}$ extend f and g, it is clear that

$$(\exists! K \in \mathscr{P}_{com}(X))\, f(K) \cup g(K) = K.$$

In addition, by Theorem 8.4, $(f \cup g)^n(C) \to K$ for any $C \in \mathscr{P}_{com}(X)$ in the Vietoris topology on $\mathscr{P}_{com}(X)$, which is convergence in the Hausdorff metric d_H, by Theorem 10.5.

In Edalat and Heckmann (1998), the formal ball model $\mathbf{B}X$ is used to give a domain theoretic proof of the existence and uniqueness of the set K in Theorem 10.4 for any complete metric space (X, d). What is missing from that discussion is the important issue that K is also an attractor with respect to the Hausdorff metric d_H.

Example 10.7. If we have two contractions $f, g : X \to X$ on a complete metric space X, they have Scott continuous extensions

$$\bar{f}, \bar{g} : \mathbf{B}X \to \mathbf{B}X$$

that are contractions on $\mathbf{B}X$ with respect to $\pi(x, r) = r$. But π is a Lebesgue measurement on a domain that has the property that for all $(x, r), (y, s) \in \mathbf{B}X$, there is an element $z = (x, r + s + d(x, y)) \in \mathbf{B}X$ with $z \sqsubseteq (x, r), (y, s)$. In addition, for any $x \in X$, choosing r so that

$$r \geqslant \frac{d(x, fx)}{1 - c_f} \text{ and } r \geqslant \frac{d(x, gx)}{1 - c_g},$$

where $c_f, c_g < 1$ are the Lipschitz constants for f and g, respectively, gives a point $(x, r) \sqsubseteq \bar{f}(x, r), \bar{g}(x, r)$. By Theorem 8.4,

$$(\exists! K \in \mathscr{P}_{com}(\ker \pi))\, \bar{f}(K) \cup \bar{g}(K) = K.$$

However, because $\ker \pi \simeq X$ and the mappings $\bar{f}, \bar{g}$ extend f and g, it is clear that

$$(\exists! K \in \mathscr{P}_{com}(X))\, f(K) \cup g(K) = K.$$

Finally, by Theorems 8.4 and 10.5, K is an attractor for $f \cup g$ on $\mathscr{P}_{com}(X)$.

If a space may be realised as the kernel of a Lebesgue measurement on a continuous *dcpo* D, then Theorem 8.4 implies that Hutchinson's result holds for any finite family of contractions that extend to D. *Necessarily,* two questions arise:

(a) Which spaces arise as the kernel of a Lebesgue measurement?
(b) When does a domain admit a Lebesgue measurement?

They are not the same question.

11. The existence of Lebesgue measurements

On the surface, it might appear that Theorem 8.4 can be applied to spaces more general than the complete metric spaces required by Hutchinson. Unfortunately, this is not the case.

Theorem 11.1 (Martin 2000a). A space is completely metrisable iff it is the kernel of a Lebesgue measurement on a continuous dcpo.

The 'completeness' comes from the fact that we are on a continuous *dcpo* (Martin 2003b); what a Lebesgue measurement captures is metrisability.

Theorem 11.2 (Martin 2000a). A space is metrisable iff it is the kernel of a Lebesgue measurement on a continuous poset.

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Thus, not only are Lebesgue measurements the measurements that extend to $\mathbf{C}D$, they also capture precisely the class of metrisable spaces. This answers our first question from the end of the last section.

The other question can now be phrased as follows: if D is a continuous dcpo with $\max(D)$ metrisable, is there a Lebesgue measurement μ with $\ker\mu = \max(D)$? For domains in general, the answer is no. For countably based domains, we now answer in the affirmative.

Definition 11.3. A continuous map $\mu : D \to E$ between domains is a *Lebesgue measurement* if for any compact $K \subseteq \ker\mu$ and open set $U \subseteq D$, we have

$$K \subseteq U \Rightarrow (\exists\, \varepsilon \in E)(\forall x \in K)\, x \in \mu_\varepsilon(x) \subseteq U$$

where $\ker\mu := \{x \in D : \mu x \in \max(E)\}$ and $\mu_\varepsilon(x) := \{y \in D : y \sqsubseteq x \;\&\; \varepsilon \ll \mu y\}$.

Notice that in the case of $E = [0,\infty)^*$ the definition above collapses to the usual definition of Lebesgue measurement.

Lemma 11.4. If $\lambda : \bullet \to E$ and $\mu : D \to \bullet$ are Lebesgue measurements with

$$\mu(\ker\mu) \subseteq \ker\lambda,$$

then $\lambda \circ \mu : D \to E$ is a Lebesgue measurement.

Proof. Let $K \subseteq \ker\lambda\mu$ compact and $U \subseteq D$ open with $K \subseteq U$. Then we must have $K \subseteq \ker\mu$, so the fact that μ is Lebesgue applies to give $\varepsilon \in \mathrm{codom}(\mu)$ with $x \in \mu_\varepsilon(x) \subseteq U$ for all $x \in K$.

Because $K \subseteq \ker\mu$, we have $L := \mu(K) \subseteq \ker\lambda$. By continuity of μ, L is compact. Since $L \subseteq \mathord{\Uparrow}\varepsilon$, the fact that λ is Lebesgue gives $\delta \in E$ with $y \in \lambda_\delta(y) \subseteq \mathord{\Uparrow}\varepsilon$ for all $y \in L$. We have $x \in (\lambda\mu)_\delta(x) \subseteq U$ for all $x \in K$. □

We now revisit the technique introduced in Martin (2003a).

Lemma 11.5. Let X be a countable set. Then there is a measurement

$$|\cdot| : \mathscr{P}(X) \to [0,\infty)^*$$

with $\ker|\cdot| = \{X\}$.

Trivially, the measurement $|\cdot|$ is Lebesgue.

Theorem 11.6. Let D be an ω-continuous dcpo. Then the following are equivalent:

(i) The space $\max(D)$ is regular.
(ii) There is a Lebesgue measurement $\mu : D \to [0,\infty)^*$ that satisfies $\ker\mu = \max(D)$.
(iii) The space $\max(D)$ is Polish.

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Proof. The direction (ii) ⇒ (i) is covered by Theorem 11.2. The equivalence of (i) and (iii) will follow immediately from Theorem 11.1 once we have shown (i) ⇒ (ii).

For (i) ⇒ (ii), let $B \subseteq D$ be a countable basis for D. As in Martin (2003a), let

$$I = \{(a,b) \in B^2 : Cl_\sigma(\Uparrow b) \cap \max(D) \subseteq \Uparrow a \cap \max(D)\},$$

and notice that this is a countable set. Define $\lambda : D \to \mathscr{P}(I)$ by

$$\lambda(x) = \{(a,b) \in I : x \in U_{ab}\},$$

where $U_{ab} = (D \setminus Cl_\sigma(\Uparrow b)) \cup \Uparrow a$. In Martin (2003a), it is shown that λ is Scott continuous with $\lambda \to \sigma_{\max(D)}$ and $\ker\lambda = \max(D)$. What is new is that λ is actually Lebesgue.

Let $U \subseteq D$ be an open set containing a compact $K \subseteq \max(D)$. Using the regularity of $\max(D)$ followed by the compactness of K, there is a finite set $\varepsilon = \{(a_i, b_i) : 1 \leqslant i \leqslant n\} \subseteq I$ such that $K \subseteq \bigcup_{i=1}^n \Uparrow b_i \cap \max(D)$ and $a_i \ll b_i$ with $a_i \in U$ for $1 \leqslant i \leqslant n$. We claim $x \in \lambda_\varepsilon(x) \subseteq U$ for all $x \in K$.

Let $x \in K$ be arbitrary. First, ε is finite and $\varepsilon \sqsubseteq \lambda x$, so $\varepsilon \ll \lambda x$. This means $x \in \lambda_\varepsilon(x)$. For any other $y \in \lambda_\varepsilon(x)$, we have $y \sqsubseteq x$ and $\varepsilon \ll \lambda y$. By construction, there is $(a_i, b_i) \in \varepsilon$ with $x \in Cl_\sigma(\Uparrow b_i) \cap \max(D) \subseteq \Uparrow a_i \cap \max(D)$. Because $(a_i, b_i) \in \varepsilon \ll \lambda y$, we have $y \in (D \setminus Cl_\sigma(\Uparrow b_i)) \cup \Uparrow a_i$.

But $x \in Cl_\sigma(\Uparrow b_i)$ and $y \sqsubseteq x$, so $y \in Cl_\sigma(\Uparrow b_i)$. Then we must have $y \in \Uparrow a_i \subseteq U$. This proves $x \in \lambda_\varepsilon(x) \subseteq U$ for all $x \in K$. Thus, λ is Lebesgue.

Finally, by Lemma 11.5, there is a measurement $|\cdot| : \mathscr{P}I \to [0,\infty)^*$ with $\ker|\cdot| = \{I\}$. Then the composition

$$D \xrightarrow{\lambda} \mathscr{P}I \xrightarrow{|\cdot|} [0,\infty)^*$$

is a Lebesgue measurement $\mu : D \to [0,\infty)^*$ with $\ker\mu = \max(D)$ by Lemma 11.4. □

A quick glance at the preceding proof shows that it applies unchanged to establish the equivalence of (i) and (ii) for any ω-continuous poset whose maximal elements meet every non-empty compact $K = \uparrow K$. Notice that this means we have shown that $\max(D)$ is metrisable when regular for countably based domains *without* using Urysohn's lemma.

Corollary 11.7. Let D be an ω-continuous dcpo with $\max(D)$ regular. Then the Vietoris hyperspace of $\max(D)$ embeds in $\max(\mathbf{C}D)$ as the kernel of a measurement on $\mathbf{C}D$.

Proof. An explicit homeomorphism is given by $K \mapsto K^*$. □

A moment of reflection is in order here. As opposed to having shown that the convex powerdomain of *some* countably based model of a metric space can represent Vietoris hyperspace, we have shown that this is always the case. It is a phenomenon exhibited by all countably based models of metric spaces. This is not the first time a powerdomain construction has been shown to provide the domain theoretic analogue of a well-known classical consideration. The *normalised probabilistic powerdomain* $\mathbf{P}^1D$ is another example.

Theorem 11.8. Let D be an ω-continuous dcpo with $X = \max(D)$ regular in its relative Scott topology. Then the space of normalised Borel measures in their weak topology $\mathbf{M}^1X$ embeds into $\max(\mathbf{P}^1D)$.

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The way this result is proved in Martin (2003a) is as follows. First, in Edalat (1997) it is shown that the theorem is true if $\max(D)$ is regular and is a G_δ subset of D. Given this, a result like Theorem 11.6 gives a measurement μ such that $\ker \mu = \max(D)$. In particular, $\max(D)$ is a G_δ subset of D.

12. Closing remarks

In our study of the map $f \cup g : \mathscr{P}_{com}(\ker \mu) \to \mathscr{P}_{com}(\ker \mu)$ for two contractions f and g on a domain with a Lebesgue measurement (D, μ), two properties seem indispensable. The first is that $\ker \mu$ is always a Hausdorff space; the second is that $K^* \in \max(\mathbf{C}D)$ for $K \in \mathscr{P}_{com}(\ker \mu)$. Apart from these two, it would seem that extensions of Theorem 8.4 should be possible to a class of spaces beyond (but including) the completely metrisable. This idea has influenced our presentation; things are written so that anyone wishing to pursue such an extension will be able to clearly identify the main issues in need of resolution.

13. Ideas

(1) Characterise $\max(\mathbf{C}D)$. When do we have $\ker \bar{\mu} = \max(\mathbf{C}D)$?
(2) Prove that $\bar{\mu}$ is a Lebesgue measurement iff it is a measurement.
(3) What is a measurement on an abstract basis?

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