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Pre-image pressure and invariant measures

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Abstract. We define and study a new invariant called pre-image pressure and its relationship with invariant measures. More precisely, for a given dynamical system (X, f) (where X is a compact metric space and f is a continuous map from X to itself) and $\varphi \in C(X, R)$ (the space of real-valued continuous functions on X), we prove a variational principle for pre-image pressure $P_{\mathrm{pre}}(f, \varphi)$, $P_{\mathrm{pre}}(f, \varphi) = \sup_{\mu \in \mathcal{M}(f)}\{h_{\mathrm{pre},\mu}(f) + \int \varphi \, d\mu\}$, where $h_{\mathrm{pre},\mu}(f)$ is the pre-image entropy (W.-C. Cheng and S. Newhouse. *Ergod. Th. & Dynam. Sys.* **25** (2005), 1091–1113) and $\mathcal{M}(f)$ is the set of invariant measures of f. Moreover, we also prove that pre-image pressure determines the invariant measures and give some applications of pre-image pressure to equilibrium states.

1. *Introduction*

Entropies are fundamental to our current understanding of dynamical systems. There are two main entropies named topological entropy (see [**1**]) and measure-theoretic (or metric) entropy (see [**2, 3**]). Topological entropy measures the maximal exponential growth rate of orbits for arbitrary topological dynamical systems, and measure-theoretic (or metric) entropy measures the maximal loss of information for the iteration of finite partitions in a measure-preserving transformation . Topological pressure is a generalization to topological entropy for a dynamical system (see [**3**]).

Recently, the pre-image structure of maps has become deeply characterized via entropies (see [**4–9**]). Several important pre-image entropy invariants, such as pointwise pre-image entropy, pointwise branch entropy, partial pre-image entropy and bundle-like pre-image entropy, etc., have been introduced and their relationships with topological entropy have been established. Cheng and Newhouse defined a pre-image entropy and proved analogs of many known results for topological and measure-theoretic entropies (see [**10**]). In this paper we define and study a new invariant called pre-image pressure, which is a generalization of the Cheng–Newhouse pre-image entropy for a dynamical system. More precisely, in §2 we define and study the pre-image pressure and its properties, in §3 we prove a variational principle for pre-image pressure, in §4 we prove that pre-image pressure determines invariant measures and we give some applications of pre-image pressure to equilibrium states in §5.

2. *Pre-image pressure*

In this section, we define and study the pre-image pressure and its properties.

Let $\mathbb{N}$ be the set of all natural numbers. Let f be a continuous map of a compact metric space (X, d) to itself. We consider the Bowen–Dinaburg metrics generated by f,

$$d_n^f(x, y) := \max_{0 \le i \le n-1} d(f^i(x), f^i(y)).$$

For $\epsilon > 0$, $n \in \mathbb{N}$ and a compact subset $K \subset X$, a subset E of K is said to be (n, ϵ)-separated with respect to f if $x, y \in E$, $x \neq y$ implies $d_n^f(x, y) > \epsilon$. Let $s_n(\epsilon, K, f)$ denote the largest cardinality of any (n, ϵ)-separated set of K with respect to f.

Let $C(X, R)$ be the space of real-valued continuous functions of X. For $\varphi \in C(X, R)$ and $n \in \mathbb{N}$ we denote $\sum_{i=0}^{n-1} \varphi(f^i(x))$ by $(S_n\varphi)(x)$. For $\epsilon > 0$, $x \in X$ and $k \in \mathbb{N}$, we put

$$P_{\mathrm{pre},n}(f, \varphi, \epsilon, f^{-k}(x)) := \sup_E \sum_{y \in E} e^{(S_n\varphi)(y)},$$

where the supremum is taken over all (n, ϵ)-separated sets of $f^{-k}(x)$. Then we put

$$P_{\mathrm{pre}}(f, \varphi, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log P_{\mathrm{pre},n}(f, \varphi, \epsilon),$$

where $P_{\mathrm{pre},n}(f, \varphi, \epsilon) = \sup_{x \in X, k \ge n} P_{\mathrm{pre},n}(f, \varphi, \epsilon, f^{-k}(x))$, and we define the pre-image pressure of f with respect to φ as

$$P_{\mathrm{pre}}(f, \varphi) := \lim_{\epsilon \to 0} P_{\mathrm{pre}}(f, \varphi, \epsilon).$$

It is clear that $P_{\mathrm{pre}}(f, \varphi) \le P(f, \varphi)$ (topological pressure, see [**3**]) and $P_{\mathrm{pre}}(f, 0) = h_{\mathrm{pre}}(f)$ (pre-image entropy, see [**10, 11**]). $P_{\mathrm{pre}}(f, \varphi) \le \|\varphi\|$ (the supremum norm of φ taken over on X) if f is a homeomorphism.

A subset F of compact subset K is said to be an (n, ϵ)-spanning set with respect to f if, for each $x \in K$, there is a $y \in F$ such that $d_n^f(x, y) \le \epsilon$. For $\epsilon > 0$, $x \in X$ and $k \in \mathbb{N}$, we put

$$Q_{\mathrm{pre},n}(f, \varphi, \epsilon, f^{-k}(x)) := \inf_F \sum_{y \in F} e^{(S_n\varphi)(y)},$$

where the infimum is taken over all (n, ϵ)-spanning sets of $f^{-k}(x)$. We write

$$Q_{\mathrm{pre}}(f, \varphi, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log Q_{\mathrm{pre},n}(f, \varphi, \epsilon),$$

where $Q_{\mathrm{pre},n}(f, \varphi, \epsilon) := \sup_{x \in X,\, k \ge n} Q_{\mathrm{pre},n}(f, \varphi, \epsilon, f^{-k}(x))$.

Let α be an open cover of X. For $x \in X$ and $k \in \mathbb{N}$, we put

$$q_n(f, \varphi, \alpha, f^{-k}(x)) := \inf_\beta \sum_{B \in \beta} \inf_{y \in B} e^{(S_n\varphi)(y)},$$

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where the infimum is taken over all finite subcovers β of $\bigvee_{i=0}^{n-1} f^{-i}\alpha$ with respect to $f^{-k}(x)$, and put

$$p_n(f, \varphi, \alpha, f^{-k}(x)) := \inf_{\beta} \sum_{B\in\beta} \sup_{y\in B} e^{(S_n\varphi)(y)},$$

where the infimum is taken over all finite subcovers β of $\bigvee_{i=0}^{n-1} f^{-i}\alpha$ with respect to $f^{-k}(x)$. Write

$$q_{\mathrm{pre},n}(f, \varphi, \alpha) := \sup_{x\in X,\ k\geq n} q_n(f, \varphi, \alpha, f^{-k}(x)),$$

and

$$p_{\mathrm{pre},n}(f, \varphi, \alpha) := \sup_{x\in X,\ k\geq n} p_n(f, \varphi, \alpha, f^{-k}(x)).$$

Clearly $q_{\mathrm{pre},n}(f, \varphi, \alpha) \leq p_{\mathrm{pre},n}(f, \varphi, \alpha)$. In addition we have the following lemma.

LEMMA 2.1. *Let $f : X \to X$ be continuous and $\varphi \in C(X, R)$.*

(i) *If α is an open cover of X with Lebesgue number δ, then $q_{\mathrm{pre},n}(f, \varphi, \alpha) \leq Q_{\mathrm{pre},n}(f, \varphi, \delta/2)$.*

(ii) *If $\epsilon > 0$ and γ is an open cover with $\mathrm{diam}(\gamma) \leq \epsilon$, then $P_{\mathrm{pre},n}(f, \varphi, \epsilon) \leq p_{\mathrm{pre},n}(f, \varphi, \gamma)$.*

(iii) *If α is an open cover of X, then*

$$\lim_{n\to\infty} \frac{1}{n} \log p_{\mathrm{pre},n}(f, \varphi, \alpha)$$

exists and equals $\inf_n (1/n) \log p_{\mathrm{pre},n}(f, \varphi, \alpha)$.

(iv) *If α, γ are open covers of X and $\alpha \prec \gamma$ (i.e. for each $C \in \gamma$, there is an $A \in \alpha$ such that $C \subset A$), then $q_{\mathrm{pre},n}(f, \varphi, \alpha) \leq q_{\mathrm{pre},n}(f, \varphi, \gamma)$.*

(v) *If $d(x, y) \leq \mathrm{diam}(\alpha)$ implies $|\varphi(x) - \varphi(y)| \leq \delta$, then $p_{\mathrm{pre},n}(f, \varphi, \alpha) \leq e^{n\delta} q_{\mathrm{pre},n}(f, \varphi, \alpha)$.*

Proof. (i) Let $x \in X$ and $n, k \in \mathbb{N}$. If F is an $(n, \delta/2)$-spanning set of $f^{-k}(x)$, then

$$f^{-k}(x) \subset \bigcup_{y\in F} \bigcap_{i=0}^{n-1} f^{-i}\bar{B}(f^i(y), \delta/2),$$

where $\bar{B}(y, \epsilon) = \{z \in X : d(y, z) \leq \epsilon\}$. Since each $\bar{B}(f^i(y), \delta/2)$ is a subset of a member of α we have $q_n(f, \varphi, \alpha, f^{-k}(x)) \leq \sum_{y\in F} e^{(S_n\varphi)(y)}$ and hence $q_{\mathrm{pre},n}(f, \varphi, \alpha) \leq Q_{\mathrm{pre},n}(f, \varphi, \delta/2)$.

(ii) Let $x \in X, n, k \in \mathbb{N}$ and let E be an (n, ϵ)-separated set of $f^{-k}(x)$. Since no member of $\bigvee_{i=0}^{n-1} f^{-i}\gamma$ contains two elements of E we have $\sum_{y\in E} e^{(S_n\varphi)(y)} \leq p_n(f, \varphi, \gamma, f^{-k}(x))$ and hence $P_{\mathrm{pre},n}(f, \varphi, \epsilon) \leq p_{\mathrm{pre},n}(f, \varphi, \gamma)$.

(iii) It suffices to show that $p_{\mathrm{pre},n+m}(f, \varphi, \alpha) \leq p_{\mathrm{pre},n}(f, \varphi, \alpha) \cdot p_{\mathrm{pre},m}(f, \varphi, \alpha)$. Let $k \geq n + m$. If β is a finite subcover of $\bigvee_{i=0}^{n-1} f^{-i}\alpha$ with respect to $f^{-k}(x)$ and γ is a finite subcover of $\bigvee_{i=0}^{m-1} f^{-i}\alpha$ with respect to $f^{-k+n}(x)$, then $\beta \vee f^{-n}\gamma$, where $\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$, is a finite subcover of $\bigvee_{i=0}^{n+m-1} f^{-i}\alpha$ with respect to $f^{-k}(x)$. This implies

$$\sum_{D\in\beta\vee f^{-n}\gamma} \sup_{y\in D} e^{(S_{n+m}\varphi)(y)} \leq \bigg(\sum_{B\in\beta} \sup_{y\in B} e^{(S_n\varphi)(y)}\bigg)\bigg(\sum_{C\in\gamma} \sup_{y\in C} e^{(S_m\varphi)(y)}\bigg).$$

Hence, $p_{n+m}(f, \varphi, \alpha, f^{-k}(x)) \leq p_n(f, \varphi, \alpha, f^{-k}(x)) \cdot p_m(f, \varphi, \alpha, f^{-k+n}(x))$. Therefore, $p_{\mathrm{pre},n+m}(f, \varphi, \alpha) \leq p_{\mathrm{pre},n}(f, \varphi, \alpha) \cdot p_{\mathrm{pre},m}(f, \varphi, \alpha)$.

(iv) and (v) easily follow from the definitions. □

Now we investigate some properties of pre-image pressure.

PROPOSITION 2.1. (Spanning set, open covers and separated set define the same pre-image pressure) *We have the following.*

(i) $Q_{\mathrm{pre},n}(f, \varphi, \epsilon, f^{-k}(x)) \leq P_{\mathrm{pre},n}(f, \varphi, \epsilon, f^{-k}(x))$.

(ii) *If* $\delta > 0$ *is such that* $d(x, y) < \epsilon/2$ *implies* $|\varphi(x) - \varphi(y)| < \delta$*, then for* $n_1, n_2, l \in \mathbb{N}$ *and* $l \geq n_1$ *we have*

$$p_{\mathrm{pre},n_1+n_2}(f, \varphi, \epsilon, f^{-l}(x)) \leq e^{(n_1+n_2)\delta} Q_{\mathrm{pre},n_1}(f, \varphi, \epsilon/2, f^{-l}(x)) Q_{\mathrm{pre},n_2}(f, \varphi, \epsilon/2, f^{-l+n_1}(x)).$$

(iii) $P_{\mathrm{pre}}(f, \varphi) = \lim_{\epsilon\to 0} Q_{\mathrm{pre}}(f, \varphi, \epsilon)$.

(iv) $P_{\mathrm{pre}}(f, \varphi) = \lim_{k\to\infty}[\lim_{n\to\infty}(1/n) \log p_{\mathrm{pre},n}(f, \varphi, \alpha_k)]$ *if* $\{\alpha_k\}$ *is a sequence of open covers with* $\mathrm{diam}(\alpha_k) \to 0$.

(v) $P_{\mathrm{pre}}(f, \varphi) = \lim_{\epsilon\to 0} \liminf_{n\to\infty}(1/n) \log P_{\mathrm{pre},n}(f, \varphi, \epsilon)$.

(vi) $P_{\mathrm{pre}}(f, \varphi) = \lim_{\epsilon\to 0} \liminf_{n\to\infty}(1/n) \log Q_{\mathrm{pre},n}(f, \varphi, \epsilon)$.

Proof. (i) This follows from the fact that a (n, ϵ)-separated set of a compact subset K that cannot be enlarged to a (n, ϵ)-separated set must be a (n, ϵ)-spanning set for K.

(ii) Let E be an $(n_1 + n_2, \epsilon)$-separated subset of $f^{-l}(x)$, F_1 be an $(n_1, \epsilon/2)$-spanning subset of $f^{-l}(x)$ and F_2 be an $(n_2, \epsilon/2)$-spanning subset of $f^{-l+n_1}(x)$. Define $\phi : E \to F_1 \times F_2$ by choosing, for each $y \in E$, some point $\phi(y) = (y_1, y_2) \in F_1 \times F_2$ with $d^f_{n_1}(y, y_1) \leq \epsilon/2$ and $d^f_{n_2}(f^{n_1}(y), y_2) \leq \epsilon/2$, then ϕ is injective. Hence

$$\begin{aligned}\Bigg(\sum_{y_1\in F_1} e^{(S_{n_1}\varphi)(y_1)}\Bigg)\Bigg(\sum_{y_2\in F_2} e^{(S_{n_2}\varphi)(y_2)}\Bigg) &= \sum_{(y_1,y_2)\in F_1\times F_2} e^{(S_{n_1}\varphi)(y_1)+(S_{n_2}\varphi)(y_2)} \\ &\geq \sum_{(y_1,y_2)\in\phi(E)} e^{(S_{n_1}\varphi)(y_1)+(S_{n_2}\varphi)(y_2)} \\ &\geq e^{-(n_1+n_2)\delta} \sum_{y\in E} e^{(S_{n_1+n_2}\varphi)(y)}.\end{aligned}$$

Therefore, (ii) is correct.

(iii) Let $x \in X, \epsilon > 0$, $k \in \mathbb{N}$ and set $Q_{\mathrm{pre},0}(f, \varphi, \epsilon, f^{-k}(x)) = 1$. (iii) holds by (i) and (ii).

(iv) If $\delta > 0$ and γ is an open cover with $\mathrm{diam}(\gamma) \leq \delta$, then $P_{\mathrm{pre},n}(f, \varphi, \delta) \leq p_{\mathrm{pre},n}(f, \varphi, \gamma)$ by Lemma 2.1(ii). Using Lemma 2.1(iii) we have

$$P_{\mathrm{pre}}(f, \varphi, \delta) \leq \lim_{n\to\infty}(1/n) \log p_{\mathrm{pre},n}(f, \varphi, \gamma).$$

Therefore, $P_{\mathrm{pre}}(f, \varphi) \leq \lim_{k\to\infty}[\lim_{n\to\infty}(1/n) \log p_{\mathrm{pre},n}(f, \varphi, \alpha_k)]$.

If α is an open cover and δ is a Lebesgue number for α, then $q_{\mathrm{pre},n}(f,\varphi,\alpha) \le P_{\mathrm{pre},n}(f,\varphi,\delta/2)$ by Lemma 2.1(i) and part (i) of the proposition. Let $\tau_\alpha = \sup\{|\varphi(x) - \varphi(y)| : d(x,y) \le \mathrm{diam}(\alpha)\}$, then $p_{\mathrm{pre},n}(f,\varphi,\alpha) \le e^{n\tau_\alpha} q_{\mathrm{pre},n}(f,\varphi,\alpha)$ by Lemma 2.1(v). Thus $p_{\mathrm{pre},n}(f,\varphi,\alpha) \le e^{n\tau_\alpha} P_{\mathrm{pre},n}(f,\varphi,\delta/2)$. Hence,

$$\lim_{n\to\infty}(1/n)\log p_{\mathrm{pre},n}(f,\varphi,\alpha) \le \tau_\alpha + P_{\mathrm{pre}}(f,\varphi).$$

Therefore, $\lim_{k\to\infty}[\lim_{n\to\infty}(1/n)\log p_{\mathrm{pre},n}(f,\varphi,\alpha_k)] \le P_{\mathrm{pre}}(f,\varphi)$ and (iv) is proved.

(v) and (vi) Let α_ϵ denote the cover of X by all open balls of radius 2ϵ and γ_ϵ denote any cover by balls of radius $\epsilon/2$. By Lemma 2.1(i), (ii) and (v) and part (i) of the proposition, we have $e^{-n\tau_{4\epsilon}} p_{\mathrm{pre},n}(f,\varphi,\alpha_\epsilon) \le q_{\mathrm{pre},n}(f,\varphi,\alpha_\epsilon) \le Q_{\mathrm{pre},n}(f,\varphi,\epsilon) \le P_{\mathrm{pre},n}(f,\varphi,\epsilon) \le p_{\mathrm{pre},n}(f,\varphi,\gamma_\epsilon)$, where $\tau_{4\epsilon} = \sup\{|\varphi(x)-\varphi(y)| : d(x,y) \le 4\epsilon\}$.

Therefore, (v) and (vi) follow by (iv). □

PROPOSITION 2.2. (Pre-image pressure is a topologically conjugate invariant) *If $f_i : X_i \to X_i$ $(i = 1, 2)$ is a continuous map of a compact metric space (X_i, d_i) and $\phi : X_1 \to X_2$ is a homeomorphism with $\phi \circ f_1 = f_2 \circ \phi$, then $P_{\mathrm{pre}}(f_2,\varphi) = P_{\mathrm{pre}}(f_1,\varphi\circ\phi)$ for any $\varphi \in C(X_2, R)$.*

Proof. Let $\epsilon > 0$, then there is an $\delta > 0$ such that $d_1(x,y) < \delta$ implies $d_2(\phi(x),\phi(y)) < \epsilon$. Let $x \in X_2, k, n > 0$ and E be a (n,ϵ)-separated set of $f_2^{-k}(x)$, then $\phi^{-1}(E)$ is a (n,δ)-separated set of $f_1^{-k}(\phi^{-1}(x))$ and

$$\sum_{y\in E} e^{\varphi(y)+\varphi(f_2(y))+\cdots+\varphi(f_2^{n-1}(y))} = \sum_{z\in\phi^{-1}E} e^{\varphi(\phi z)+\varphi(\phi f_1(z))+\cdots+\varphi(\phi f_1^{n-1}(z))}.$$

Hence, $P_{\mathrm{pre}}(f_2,\varphi,\epsilon) \le P_{\mathrm{pre}}(f_1,\varphi\circ\phi,\delta)$. Therefore, $P_{\mathrm{pre}}(f_2,\varphi) \le P_{\mathrm{pre}}(f_1,\varphi\circ\phi)$. Similarly we have $P_{\mathrm{pre}}(f_1,\varphi\circ\phi) \le P_{\mathrm{pre}}(f_2,\varphi\circ\phi\circ\phi^{-1}) = P_{\mathrm{pre}}(f_2,\varphi)$. □

PROPOSITION 2.3. (Power rule for pre-image pressure) *Let $f : X \to X$ be a continuous map of the compact metric space (X,d) and $\varphi \in C(X,R)$, then $P_{\mathrm{pre}}(f^m, S_m\varphi) = mP_{\mathrm{pre}}(f,\varphi)$ for any $m > 0$ (here $(S_m\varphi)(x) = \sum_{i=0}^{m-1}\varphi(f^i(x))$).*

Proof. Write $g = f^m$. Let $n \in \mathbb{N}, k \ge n$ and $x \in X$. If E is an (n,ϵ)-separated subset of $g^{-k}(x)$ with respect to g, then E is also an (nm,ϵ)-separated subset of $f^{-mk}(x)$ with respect to f. Hence

$$P_{\mathrm{pre},n}(g, S_m\varphi, \epsilon, g^{-k}(x)) \le P_{\mathrm{pre},nm}(f,\varphi,\epsilon,f^{-mk}(x)).$$

So we have

$$\begin{aligned}
P_{\mathrm{pre}}(g, S_m\varphi,\epsilon) &= \limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X,k\ge n} P_{\mathrm{pre},n}(g,S_m\varphi,\epsilon,g^{-k}(x))\\
&\le \limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X,k\ge n} P_{\mathrm{pre},nm}(f,\varphi,\epsilon,f^{-mk}(x))\\
&\le \limsup_{n\to\infty}\frac{m}{nm}\log\sup_{x\in X,k\ge nm} P_{\mathrm{pre},nm}(f,\varphi,\epsilon,f^{-k}(x))\\
&\le m\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X,k\ge n} P_{\mathrm{pre},n}(f,\varphi,\epsilon,f^{-k}(x))\\
&= mP_{\mathrm{pre}}(f,\varphi,\epsilon).
\end{aligned}$$

Therefore, $P_{\mathrm{pre}}(f^m, S_m\varphi) \le mP_{\mathrm{pre}}(f,\varphi)$.

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Let $\delta > 0$, then there exists $\epsilon > 0$ such that if $d(x, y) < \epsilon/2$ then $|\varphi(x) - \varphi(y)| < \delta$. For $\epsilon > 0$ above, there are $\eta > 0$ such that $d(x, y) < \eta$ implies $d(f^j(x), f^j(y)) < \epsilon/4$ for all $0 \le j \le m - 1$. Let $n > 0$, $k \ge n$.

CLAIM. *If $l, s \in \mathbb{N}$ such that $l \ge ms$, then $P_{\mathrm{pre},ms}(f, \varphi, \epsilon/4, f^{-l}(x)) \le P_{\mathrm{pre},s}(g, S_m\varphi, \eta, f^{-l}(x))$.*

In fact, if E is an $(ms, \epsilon/4)$-separated subset of $f^{-l}(x)$ with respect to f, then E is also an (s, η)-separated subset of $f^{-l}(x)$ with respect to g. Hence,

$$\begin{aligned} P_{\mathrm{pre},ms}(f, \varphi, \epsilon/4, f^{-l}(x)) &= \sup_E \sum_{y\in E} e^{(S_{ms}\varphi)(y)} \\ &= \sup_E \sum_{y\in E} e^{(S_m\varphi)(y)+\cdots+(S_m\varphi)(g^{s-1}(y))} \\ &\le P_{\mathrm{pre},s}(g, S_m\varphi, \eta, f^{-l}(x)), \end{aligned}$$

and the claim is thus confirmed.

Write $k = mn_2 - l_2$ and $n - l_2 = mn_1 + l_1$, where $0 \le l_1, l_2 < m$. Let $C(j, \epsilon) = s_j(\epsilon, X, f)e^{j\|\varphi\|}$. By Proposition 2.1(i), (ii) and the previous claim, we have

$$\begin{aligned} &P_{\mathrm{pre},n}(f, \varphi, \epsilon, f^{-k}(x)) \\ &\quad\le e^{n\delta} Q_{\mathrm{pre},n-l_2}(f, \varphi, \epsilon/2, f^{-k}(x)) Q_{\mathrm{pre},l_2}(f, \varphi, \epsilon/2, f^{-k+n-l_2}(x)) \\ &\quad\le C(l_2, \epsilon/2)e^{n\delta} Q_{\mathrm{pre},mn_1+l_1}(f, \varphi, \epsilon/2, f^{-k}(x)) \\ &\quad\le C(l_2, \epsilon/2)e^{(2n-l_2)\delta} Q_{\mathrm{pre},mn_1}(f, \varphi, \epsilon/4, f^{-k}(x)) Q_{\mathrm{pre},l_1}(f, \varphi, \epsilon/4, f^{-k+mn_1}(x)) \\ &\quad\le C(l_2, \epsilon/2)C(l_1, \epsilon/4)e^{(2n-l_2)\delta} P_{\mathrm{pre},mn_1}(f, \varphi, \epsilon/4, f^{-k}(x)) \\ &\quad\le C(l_2, \epsilon/2)C(l_1, \epsilon/4)e^{(2n-l_2)\delta} P_{\mathrm{pre},n_1}(g, S_m\varphi, \eta, f^{-k}(x)) \\ &\quad= C(l_2, \epsilon/2)C(l_1, \epsilon/4)e^{(2n-l_2)\delta} P_{\mathrm{pre},n_1}(g, S_m\varphi, \eta, g^{-n_2}(f^{l_2}(x))). \end{aligned}$$

Hence,

$$\begin{aligned} P_{\mathrm{pre}}(f, \varphi, \epsilon) &= \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X, k\ge n} P_{\mathrm{pre},n}(f, \varphi, \epsilon, f^{-k}(x)) \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X, k\ge n_1} C(l_2, \epsilon/2)C(l_1, \epsilon/4)e^{(2n-l_2)\delta} \\ &\qquad \times P_{\mathrm{pre},n_1}(g, S_m\varphi, \eta, g^{-k}(x)) \\ &= 2\delta + \limsup_{n\to\infty} \frac{1}{mn_1 + l_1 + l_2} \log \sup_{x\in X, k\ge n_1} P_{\mathrm{pre},n_1}(g, S_m\varphi, \eta, g^{-k}(x)) \\ &= 2\delta + \frac{1}{m} P_{\mathrm{pre}}(g, S_m\varphi, \eta). \end{aligned}$$

Therefore, we can get $mP_{\mathrm{pre}}(f, \varphi) \le P_{\mathrm{pre}}(g, S_m\varphi)$. □

PROPOSITION 2.4. (Product rule of pre-image pressure) *If $f_i : X_i \to X_i$ $(i = 1, 2)$ is a continuous map of a compact metric space (X_i, d_i) and if $\varphi_i \in C(X_i, R)$, then $P_{\mathrm{pre}}(f_1 \times f_2, \varphi_1 \times \varphi_2) = P_{\mathrm{pre}}(f_1, \varphi_1) + P_{\mathrm{pre}}(f_2, \varphi_2)$, where $\varphi_1 \times \varphi_2 \in C(X_1 \times X_2, R)$ is defined by $(\varphi_1 \times \varphi_2)(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2)$.*

Proof. Consider the metric on $X_1 \times X_2$ given by

$$d((x_1, x_2), (x_2, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

For $x = (x_1, x_2) \in X_1 \times X_2$, $n, k \in \mathbb{N}$ and $k \geq n$. If F_i is a (n, ϵ)-spanning set for $f_i^{-k}(x_i)$ then $F_1 \times F_2$ is an (n, ϵ)-spanning set for $(f_1 \times f_2)^{-k}(x_1, x_2)$ with respect to $f_1 \times f_2$. Also

$$\sum_{(y_1,y_2)\in F_1\times F_2} \exp\bigg(\sum_{i=0}^{n-1}(\varphi_1 \times \varphi_2)(f_1 \times f_2)^i(y_1, y_2)\bigg)$$
$$= \bigg(\sum_{y_1\in F_1} \exp\bigg(\sum_{i=0}^{n-1}\varphi_1(f_1^i(y_1))\bigg)\bigg)\bigg(\sum_{y_2\in F_2} \exp\bigg(\sum_{i=0}^{n-1}\varphi_2(f_2^i(y_2))\bigg)\bigg).$$

Hence,

$$Q_{\mathrm{pre},n}(f_1 \times f_2, \varphi_1 \times \varphi_2, \epsilon, (f_1 \times f_2)^{-k}(x_1, x_2))$$
$$\leq Q_{\mathrm{pre},n}(f_1, \varphi_1, \epsilon, f_1^{-k}(x_1))Q_{\mathrm{pre},n}(f_2, \varphi_2, \epsilon, f_2^{-k}(x_2)).$$

Therefore, $P_{\mathrm{pre}}(f_1 \times f_2, \varphi_1 \times \varphi_2) \leq P_{\mathrm{pre}}(f_1, \varphi_1) + P_{\mathrm{pre}}(f_2, \varphi_2)$.

If E_i is an (n, ϵ)-separated set for $f_i^{-k}(x_i)$, then $E_1 \times E_2$ is an (n, ϵ)-separated set for $(f_1 \times f_2)^{-k}(x_1, x_2)$ with respect to $f_1 \times f_2$. So,

$$P_{\mathrm{pre},n}(f_1 \times f_2, \varphi_1 \times \varphi_2, \epsilon) \geq P_{\mathrm{pre},n}(f_1, \varphi_1, \epsilon) \cdot P_{\mathrm{pre},n}(f_2, \varphi_2, \epsilon).$$

Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log P_{\mathrm{pre},n}(f_1 \times f_2, \varphi_1 \times \varphi_2, \epsilon)$$
$$\geq \liminf_{n\to\infty} \frac{1}{n} \log P_{\mathrm{pre},n}(f_1, \varphi_1, \epsilon) + \limsup_{n\to\infty} \frac{1}{n} P_{\mathrm{pre},n}(f_2, \varphi_2, \epsilon).$$

Proposition 2.1(v) gives

$$P_{\mathrm{pre}}(f_1 \times f_2, \varphi_1 \times \varphi_2) \geq P_{\mathrm{pre}}(f_1, \varphi_1) + P_{\mathrm{pre}}(f_2, \varphi_2).$$ □

3. *Variational principle for pre-image pressure*

In this section we prove a variational principle for pre-image pressure.

LEMMA 3.1. [**3**] *Let $a_1, \ldots, a_k$ be given real numbers. If $p_i \geq 0$ and $\sum_{i=1}^k p_i = 1$, then*

$$\sum_{i=1}^{k} p_i(a_i - \log p_i) \leq \log\bigg(\sum_{i=1}^{k} e^{a_i}\bigg)$$

and equality holds if and only if

$$p_i = \frac{e^{a_i}}{\sum_{j=1}^k e^{a_j}}.$$ □

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In the following we denote by $\mathcal{B}(X)$ the collection of all Borel subsets and denote by $\mathcal{M}(X, f)$ (or $\mathcal{M}(f)$ for short) the set of f-invariant Borel probability measures for a continuous map f of a compact metric space X into itself. Set $\mathcal{B}^- = \bigcap_{n=0}^{\infty} f^{-n}\mathcal{B}(X)$. For finite partitions α, β, we set $\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. If $0 \le j \le n$ are positive integers, we let $\alpha_j^n = \bigvee_{i=j}^{n-1} f^{-i}\alpha$ and $\alpha^n = \alpha_0^{n-1}$. It is not hard to see that $H_\mu(\alpha^n|\mathcal{B}^-)$ (see [**3**]) is a non-negative sub-additive sequence for a partition α and $\mu \in \mathcal{M}(f)$. We define the measure-theoretic (or metric) pre-image entropy of α with respect to f as

$$h_\mu(\alpha|\mathcal{B}^-) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha^n|\mathcal{B}^-) = \inf_{n\ge 0} \frac{1}{n} H_\mu(\alpha^n|\mathcal{B}^-),$$

and define the measure-theoretic (or metric) pre-image entropy of f as

$$h_{\text{pre},\mu}(f) = \sup_\alpha h_\mu(\alpha|\mathcal{B}^-).$$

THEOREM 3.1. (Variational principle for pre-image pressure) *Let $f : X \to X$ be a continuous map of the compact metric space X and $\varphi \in C(X, R)$. Then*

$$P_{\text{pre}}(f, \varphi) = \sup_{\mu\in\mathcal{M}(f)} \left\{ h_{\text{pre},\mu}(f) + \int \varphi\, d\mu \right\}.$$

Proof. (1) Let $\mu \in \mathcal{M}(f)$. We shall show that

$$h_{\text{pre},\mu}(f) + \int \varphi\, d\mu \le P_{\text{pre}}(f, \varphi).$$

Let $\xi = \{A_1, \ldots, A_k\}$ be a partition of $(X, \mathcal{B})$. Let $a > 0$ be given and choose $\epsilon > 0$ such that $\epsilon k \log k < a$. Since μ is regular there are compact sets $B_j \subset A_j$ with $\mu(A_j \backslash B_j) < \epsilon, 1 \le j \le k$. Let α be the partition $\alpha = \{B_0, B_1, \ldots, B_k\}$ where $B_0 = X \big\backslash \bigcup_{j=1}^k B_j$. Then $H_\mu(\xi|\alpha) < \epsilon k \log k < a$. Let

$$b = \min_{1\le i\ne j\le k} d(B_i, B_j) > 0.$$

Pick $0 < \delta < b/8$ such that $d(x, y) < 4\delta$ implies $|\varphi(x) - \varphi(y)| < \epsilon$.

Let $\beta_1 \le \beta_2 \le \cdots$ be a non-decreasing sequence of finite partitions with diameters tending to zero. Thus, $\mathcal{B} = \bigvee_{j=1}^\infty \beta_j$ and for any $k > 0$ and $n > 0$

$$H_\mu(\alpha^n|f^{-k}\mathcal{B}) = \lim_{j\to\infty} H_\mu(\alpha^n|f^{-k}\beta_j).$$

Let $\epsilon_1 = \epsilon_1(n, \epsilon) > 0$ such that $d(x, y) < \epsilon_1$, then $d(f^i(x), f^i(y)) < \delta$ for $0 \le i < n$.

The collection $\{f^{-k}(x) : x \in f^k X\}$ is an upper semi-continuous decomposition of X. Hence for each $x \in f^k X$ there is an $\epsilon_2(x, k, \epsilon_1)$ such that if $d(x, y) < \epsilon_2(x, k, \epsilon_1)$, $y \in f^k X$ and $y_1 \in f^{-k}(y)$, then there is an $x_1 \in f^{-k}(x)$ such that $d(x_1, y_1) < \epsilon_1$. Let $\mathcal{U}$ be the collection of open $\epsilon_2(x, k, \epsilon_1)$ balls in $f^k X$ as x varies in $f^k X$ and let ϵ_3 be a Lebesgue number for $\mathcal{U}$.

Since $\text{diam}(\beta_j) \to 0$ as $j \to \infty$, we may choose j_0 such that if $j \ge j_0$ and $B \in \beta_j$, then $\text{diam}(\bar{B}) < \epsilon_3$. Let $j \ge j_0$. For a set $C \in f^{-k}\beta_j$, let μ_C denote the conditional

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measure of μ restricted to C and let $\alpha_C^n = \{A \cap C : A \in \alpha^n, A \cap C \neq \emptyset\}$. If $A \cap C \in \alpha_C^n$, let $\gamma(A, C) = \sup\{(S_n\varphi)(x) : x \in A \cap C\}$, then by Lemma 3.1,

$$\begin{aligned} &H_\mu(\alpha^n | f^{-k}\beta_j) + \int S_n\varphi \, d\mu \\ &\quad = \sum_{C \in f^{-k}\beta_j} \left[\mu(C) H_{\mu_C}(\alpha_C^n) + \int_C S_n\varphi \, d\mu \right] \\ &\quad \leq \sum_{C \in f^{-k}\beta_j} \mu(C) \sum_{A \cap C \in \alpha_C^n} \mu_C(A \cap C)[-\log \mu_C(A \cap C) + \gamma(A, C)] \\ &\quad \leq \max_{C \in f^{-k}\beta_j} \log \sum_{A \cap C \in \alpha_C^n} e^{\gamma(A,C)}. \end{aligned}$$

For each $A \cap C \in \alpha_C^n$ choose some $x_A \in \overline{A \cap C}$ such that $(S_n\varphi)(x_A) = \gamma(A, C)$. Let $B \in \beta_j$ such that $C = f^{-k}B$.

Since $f^k(x_A) \in \bar{B}$ and $\operatorname{diam}(\bar{B}) < \epsilon_3$, there is an $u_B \in f^k X$ such that if $y \in \bar{B} \cap f^k X$, then $d(u_B, y) < \epsilon_2(u_B, k, \epsilon_1)$. This implies $d(u_B, f^k(x_A)) < \epsilon_2(u_B, k, \epsilon_1)$. Hence, there is a point $\phi_1(A) \in f^{-k}(u_B)$ such that $d(x_A, \phi_1(A)) < \epsilon_1$. So $d(f^i(x_A), f^i(\phi_1(A))) < \delta$ for all $0 \leq i < n$.

Let E_C be a maximal (n, δ)-separated set in $f^{-k}(u_B)$. Since E_C spans $f^{-k}(u_B)$, there is a point $\phi_2(A) \in E_C$ such that $d(f^i(\phi_1(A)), f^i(\phi_2(A))) \leq \delta$ for all $0 \leq i < n$. Hence $d(f^i(x_A), f^i(\phi_2(A))) \leq 2\delta$ for all $0 \leq i < n$. Then $\gamma(A, C) \leq (S_n\varphi)(\phi_2(A)) + n\epsilon$.

CLAIM. *If* $y \in E_C$ *then* $\operatorname{card}(\{A \cap C \in \alpha_C^n : \phi_2(A) = y\}) \leq 2^n$.

In fact, let $A, \widetilde{A}$ be such that $\phi_2(A) = \phi_2(\widetilde{A})$. Then for all $0 \leq i < n$ we have $d(f^i(x_A), f^i(x_{\widetilde{A}})) \leq 4\delta$. Since each ball of radius 4δ meets at most the closures of two members of α, $\{A \cap C \in \alpha_C^n : \phi_2(A) = y\}$ has cardinality at most 2^n and the claim is thus confirmed.

Thus,

$$\sum_{A \cap C \in \alpha_C^n} e^{\gamma(A,C) - n\epsilon} \leq \sum_{A \cap C \in \alpha_C^n} e^{(S_n\varphi)(\phi_2(A))} \leq 2^n \sum_{y \in E_C} e^{(S_n\varphi)(y)}.$$

Hence,

$$H_\mu(\alpha^n | f^{-k}\beta_j) + \int S_n\varphi \, d\mu \leq (\epsilon + \log 2)n + \log \sup_{x \in X} P_{\mathrm{pre},n}(f, \varphi, \delta, f^{-k}(x)).$$

Let $j \to \infty$ and $k \to \infty$, we have

$$H_\mu(\alpha^n | \mathcal{B}^-) + \int S_n\varphi \, d\mu \leq (\epsilon + \log 2)n + \log \sup_{x \in X, k \geq n} P_{\mathrm{pre},n}(f, \varphi, \delta, f^{-k}(x)).$$

So

$$h_\mu(\alpha | \mathcal{B}^-) + \int \varphi \, d\mu \leq \epsilon + \log 2 + P_{\mathrm{pre}}(f, \varphi, \delta) \leq \epsilon + \log 2 + P_{\mathrm{pre}}(f, \varphi).$$

Now $h_\mu(\xi | \mathcal{B}^-) \leq h_\mu(\alpha | \mathcal{B}^-) + H_\mu(\xi | \alpha)$ by [**11**, Lemma 4.8], then

$$h_\mu(\xi | \mathcal{B}^-) + \int \varphi \, d\mu \leq 2a + \log 2 + P_{\mathrm{pre}}(f, \varphi),$$

and, therefore,

$$h_{\mathrm{pre},\mu}(f) + \int \varphi\, d\mu \le 2a + \log 2 + P_{\mathrm{pre}}(f, \varphi).$$

Replacing f with f^n and φ with $S_n\varphi \left(= \sum_{i=0}^{n-1} \varphi \circ f^i\right)$ in the above inequality, respectively, and by Proposition 2.3, we have

$$n\left[h_{\mathrm{pre},\mu}(f) + \int f\, d\mu\right] \le 2a + \log 2 + nP_{\mathrm{pre}}(f, \varphi).$$

So we have

$$h_{\mathrm{pre},\ \mu}(f) + \int \varphi\, d\mu \le P_{\mathrm{pre}}(f, \varphi).$$

(2) Let $\epsilon > 0$. We shall produce an f-invariant measure μ such that

$$h_{\mathrm{pre},\ \mu}(f) + \int f\, d\mu \ge P_{\mathrm{pre}}(f, \varphi, \epsilon).$$

Choose sequences $n_i \to \infty$, $k_i > n_i$ and $x_i \in X$ such that

$$P_{\mathrm{pre}}(f, \varphi, \epsilon) = \lim_{i\to\infty} \frac{1}{n_i} \log P_{\mathrm{pre},n_i}(f, \varphi, \epsilon, f^{-k_i}(x_i)).$$

Let E_i be an (n_i, ϵ)-separated set of $f^{-k_i}(x_i)$ such that

$$\log \sum_{y\in E_i} e^{(S_{n_i}\varphi)(y)} \ge \log P_{\mathrm{pre},n_i}(f, \varphi, \epsilon, f^{-k_i}(x_i)) - 1.$$

Letting δ_x denote the point mass at point $x \in X$, let

$$\sigma_i = \frac{\sum_{y\in E_i} e^{(S_{n_i}\varphi)(y)}\delta_y}{\sum_{z\in E_i} e^{(S_{n_i}\varphi)(z)}},$$

and let

$$\mu_i = \frac{1}{n_i} \sum_{j=0}^{n_i-1} \sigma_i \circ f^{-j}.$$

We may assume without loss of generality that $\mu = \lim_{i\to\infty} \mu_i$. We know that $\mu \in \mathcal{M}(f)$.

We choose a finite partition α of $(X, \mathcal{B})$ such that for each $A \in \alpha$, $\mu(\partial A) = 0$ and $\mathrm{diam}(A) < \epsilon$.

Let $\mathcal{C} = \{E \in \mathcal{B}^- : \mu(E) = 0\}$. For any σ-algebra $\mathcal{A}$ of subsets of X, there is an enlarged σ-algebra $\mathcal{A}_{\mathcal{C}}$ defined by $A \in \mathcal{A}_{\mathcal{C}}$ if and only if there are sets B, M, N such that $A = B \cup M$, $B \in \mathcal{A}$, $N \in \mathcal{C}$ and $M \subseteq N$. We consider the σ-algebra $\mathcal{B}^k = (f^{-k}\mathcal{B})_{\mathcal{C}}$ for $k \ge 1$. Letting $\mathcal{B}^\infty = \bigcap_{k\ge 1} \mathcal{B}^k$, we have $\mathcal{B}^- \subset \mathcal{B}^\infty \subset \mathcal{B}^k$ $(k \ge 1)$.

Now, each element $A \in \mathcal{B}^{k_i}$ can be expressed as the disjoint union $A = B \cup C$ with $B \in f^{-k_i}\mathcal{B}$ and $C \in \mathcal{C}$. Since σ_i is supported on $f^{-k_i}(x_i)$, we have $\sigma_i(C) = 0$. Hence, for any finite partition γ, we have

$$H_{\sigma_i}(\gamma|\mathcal{B}^{k_i}) = H_{\sigma_i}(\gamma|f^{-k_i}(x_i)).$$

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Since each element of $\alpha^{n_i}|f^{-k_i}(x_i)$ contains at most one element of E_i, by definition of σ_i and Lemma 3.1 we have

$$\begin{aligned}H_{\sigma_i}(\alpha^{n_i}|\mathcal{B}^{k_i}) + \int S_{n_i}\varphi\, d\sigma_i &= H_{\sigma_i}(\alpha^{n_i}|f^{-k_i}(x_i)) + \int S_{n_i}\varphi\, d\sigma_i \\ &= \sum_{y\in E_i} \sigma_i(\{y\})((S_{n_i}\varphi)(y) - \log\sigma_i(\{y\})) \\ &= \log \sum_{y\in E_i} e^{(S_{n_i}\varphi)(y)}.\end{aligned}$$

Fix $q \in \mathbb{N}$ with $1 \le q < n_i$. For $0 \le j \le q-1$ put $a(j) = [(n_i - j/q)]$. Here $[b]$ denotes the integer part of $b > 0$. Fix $0 \le j \le q-1$, so by [**3**, (ii) of Remark 2 in §8.2] we have

$$\alpha^{n_i} = \bigvee_{r=0}^{a(j)-1} f^{-(rq+j)}\alpha^q \vee \bigvee_{l\in S} f^{-l}\alpha,$$

and S has cardinality of at most $2q$. Therefore,

$$\begin{aligned}\log \sum_{y\in E_i} e^{(S_{n_i}\varphi)(y)} &= H_{\sigma_i}(\alpha^{n_i}|\mathcal{B}^{k_i}) + \int S_{n_i}\varphi\, d\sigma_i \\ &\le \sum_{r=0}^{a(j)-1} H_{\sigma_i}(f^{-(rq+j)}\alpha^q|\mathcal{B}^{k_i}) + H_{\sigma_i}\bigg(\bigvee_{l\in S} f^{-l}\alpha|\mathcal{B}^{k_i}\bigg) + \int S_{n_i}\varphi\, d\sigma_i \\ &\le \sum_{r=0}^{a(j)-1} H_{\sigma_i}(f^{-(rq+j)}\alpha^q|f^{-(rq+j)}(\mathcal{B}^{k_i})) + 2q\log k + \int S_{n_i}\varphi\, d\sigma_i \\ &= \sum_{r=0}^{a(j)-1} H_{\sigma_i\circ f^{-(rq+j)}}(\alpha^q|\mathcal{B}^{k_i}) + 2q\log k + \int S_{n_i}\varphi\, d\sigma_i.\end{aligned}$$

Summing up over j from 0 to $q-1$ and using [**3**, (iii) of Remark 2 in §8.2] we obtain

$$q\log \sum_{y\in E_i} e^{(S_{n_i}\varphi)(y)} \le \sum_{p=0}^{n_i-1} H_{\sigma_i\circ f^{-p}}(\alpha^q|\mathcal{B}^{k_i}) + 2q^2\log k + q\int S_{n_i}\varphi\, d\sigma_i.$$

Now divide by n_i and use [**10**, Lemma 6.1(35)] to obtain

$$\begin{aligned}\frac{q}{n_i}\log \sum_{y\in E_i} e^{(S_{n_i}\varphi)(y)} &\le H_{\mu_i}(\alpha^q|\mathcal{B}^{k_i}) + \frac{2q^2}{n_i}\log k + q\int \varphi\, d\mu_i \\ &\le H_{\mu_i}(\alpha^q|\mathcal{B}^{\infty}) + \frac{2q^2}{n_i}\log k + q\int \varphi\, d\mu_i.\end{aligned}$$

Using [**10**, Lemma 6.1(34)], we have

$$q P_{\text{pre}}(f,\varphi,\epsilon) \le H_\mu(\alpha^q|\mathcal{B}^-) + q\int \varphi\, d\mu \le H_\mu(\alpha^q|\mathcal{B}^-) + q\int\varphi\, d\mu.$$

Dividing by q and letting $q\to\infty$ we have

$$P_{\text{pre}}(f,\varphi,\epsilon) \le h_\mu(\alpha|\mathcal{B}^-) + \int\varphi\, d\mu \le h_{\text{pre},\mu}(f) + \int \varphi\, d\mu. \qquad \square$$

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A point $x \in X$ is said to be a non-wandering point if for each neighborhood U of x there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. Let $\Omega(f)$ denote the non-wandering set of f. It is well known that $\mu(\Omega(f)) = 1$ for each $\mu \in \mathcal{M}(f)$. From Theorem 3.1, the following corollaries are obvious.

COROLLARY 3.1.1. *Let $f : X \to X$ be a continuous map of a compact metric space and let $\varphi \in C(X, R)$. Then:*

(i) $P_{\text{pre}}(f, \varphi) = P_{\text{pre}}(f|_{\Omega(f)}, \varphi|_{\Omega(f)})$;

(ii) $P_{\text{pre}}(f, \varphi) = P_{\text{pre}}(f|_{\bigcap_{n=0}^{\infty} f^n X}, \varphi|_{\bigcap_{n=0}^{\infty} f^n X})$.

COROLLARY 3.1.2. *If $f : X \to X$ is uniquely ergodic and $\mathcal{M}(f) = \{\mu\}$ then*

$$P_{\text{pre}}(f, \varphi) = h_{\text{pre},\mu}(f) + \int \varphi \, d\mu.$$

Remark. Applying Theorem 3.1, we can give another proof of $P_{\text{pre}}(f_1 \times f_2, \varphi_1 \times \varphi_2) \geq P_{\text{pre}}(f_1, \varphi_1) + P_{\text{pre}}(f_2, \varphi_2)$ in the product rule of pre-image pressure. Let $\epsilon > 0$. Theorem 3.1 implies there are invariant measures μ, ν such that $h_{\text{pre},\mu}(f_1) + \int \varphi_1 \, d\mu > P_{\text{pre}}(f_1, \varphi_1) - \epsilon$ and $h_{\text{pre},\nu}(f_2) + \int \varphi_2 \, d\nu > P_{\text{pre}}(f_2, \varphi_2) - \epsilon$. Then

$$\begin{aligned} P_{\text{pre}}(f_1 \times f_2, \varphi_1 \times \varphi_2) &\geq h_{\text{pre},\, \mu\times\nu}(f_1 \times f_2) + \int \varphi_1 \times \varphi_2 \, d(\mu \times \nu) \\ &= h_{\text{pre},\, \mu}(f_1) + h_{\text{pre},\nu}(f_2) + \int \varphi_1 \, d\mu + \int \varphi_2 \, d\nu \\ &> P_{\text{pre}}(f_1, \varphi_1) + P_{\text{pre}}(f_2, \varphi_2) - 2\epsilon. \end{aligned}$$

Therefore, $P_{\text{pre}}(f_1 \times f_2, \varphi_1 \times \varphi_2) \geq P_{\text{pre}}(f_1, \varphi_1) + P_{\text{pre}}(f_2, \varphi_2)$.

4. *Pre-image pressure determines invariant measures*

In this section we shall show how $P_{\text{pre}}(f, \cdot)$ determines the invariant measures of f when $f : X \to X$ is a continuous map of a compact metric space X. Recall that a finite signed measure on X is a map $\mu : \mathcal{B} \to R$, which is countably additive.

LEMMA 4.1. *Let $f : X \to X$ be a continuous transformation of a compact metric space X. If $\varphi, \psi \in C(X, R)$ and $c \in R$, then the following are true.*

(i) *$\varphi \leq \psi$ implies $P_{\text{pre}}(f, \varphi) \leq P_{\text{pre}}(f, \psi)$. In particular, $h_{\text{pre}}(f) + \inf \varphi \leq P_{\text{pre}}(f, \varphi) \leq h_{\text{pre}}(f) + \sup \varphi$.*

(ii) *$P_{\text{pre}}(f, \varphi + c) = P_{\text{pre}}(f, \varphi) + c$.*

(iii) *$|P_{\text{pre}}(f, \varphi) - P_{\text{pre}}(f, \psi)| \leq \|\varphi - \psi\|$.*

(iv) *$P_{\text{pre}}(f, \cdot)$ is convex.*

(v) *$P_{\text{pre}}(f, \varphi + \psi \circ f - \psi) = P_{\text{pre}}(f, \varphi)$.*

(vi) *$P_{\text{pre}}(f, c\varphi) \leq c P_{\text{pre}}(f, \varphi)$ if $c \geq 1$ and $P_{\text{pre}}(f, c\varphi) \geq c P_{\text{pre}}(f, \varphi)$ if $c \leq 1$.*

(vii) *$|P_{\text{pre}}(f, \varphi)| \leq P_{\text{pre}}(f, |\varphi|)$.*

(viii) *$P_{\text{pre}}(f, \varphi + \psi) \leq P_{\text{pre}}(f, \varphi) + P_{\text{pre}}(f, \psi)$.*

Proof. (i), (ii) easily follow from the definition of pre-pressure.

(iii) Let $\epsilon > 0$. By Theorem 3.1 there exists a $\mu \in \mathcal{M}(f)$ such that

$$P_{\text{pre}}(f, \varphi) < h_{\text{pre},\mu}(f) + \int \varphi \, d\mu + \epsilon.$$

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Hence

$$P_{\text{pre}}(f,\varphi)-P_{\text{pre}}(f,\psi)<\left(h_{\text{pre},\mu}(f)+\int\varphi\,d\mu\right)-\left(h_{\text{pre},\mu}(f)+\int\psi\,d\mu\right)+\epsilon$$
$$=\int(\varphi-\psi)\,d\mu+\epsilon\le\|\varphi-\psi\|+\epsilon.$$

Therefore, $P_{\text{pre}}(f,\varphi)-P_{\text{pre}}(f,\psi)\le\|\varphi-\psi\|$.

Similarly, we have $P_{\text{pre}}(f,\psi)-P_{\text{pre}}(f,\varphi)\le\|\varphi-\psi\|$. This proves (iii).

(iv) Let $a\in[0,1]$ and $\epsilon>0$. By Theorem 3.1 there is a $\mu\in\mathcal{M}(f)$ such that

$$P_{\text{pre}}(f,a\varphi+(1-a)\psi)<h_{\text{pre},\mu}(f)+\int(a\varphi+(1-a)\psi)\,d\mu+\epsilon.$$

Hence

$$\begin{aligned}&P_{\text{pre}}(f,a\varphi+(1-a)\psi)\\&\quad<h_{\text{pre},\mu}(f)+\int(a\varphi+(1-a)\psi)\,d\mu+\epsilon\\&\quad=a\left(h_{\text{pre},\mu}(f)+\int\varphi\,d\mu\right)+(1-a)\left(h_{\text{pre},\mu}(f)+\int\psi\,d\mu\right)+\epsilon\\&\quad\le aP_{\text{pre}}(f,\varphi)+(1-a)P_{\text{pre}}(f,\psi)+\epsilon.\end{aligned}$$

Therefore, $P_{\text{pre}}(f,a\varphi+(1-a)\psi)\le aP_{\text{pre}}(f,\varphi)+(1-a)P_{\text{pre}}(f,\psi)$.

(v) Note that $\int(\psi\circ f-\psi)\,d\mu=0$ for each $\mu\in\mathcal{M}(f)$. This implies

$$P_{\text{pre}}(f,\varphi+\psi\circ f-\psi)=\sup_{\mu\in\mathcal{M}(f)}\left\{h_{\text{pre},\mu}(f)+\int(\varphi+\psi\circ f-\psi)\,d\mu\right\}$$
$$=\sup_{\mu\in\mathcal{M}(f)}\left\{h_{\text{pre},\mu}(f)+\int\varphi\,d\mu\right\}=P_{\text{pre}}(f,\varphi).$$

The proofs of (vi), (vii) and (viii) are obtained in a similar manner as above by applying Theorem 3.1 and are thus omitted. □

THEOREM 4.1. (Pre-image pressure determines invariant measures) *Let $f:X\to X$ be a continuous map of a compact metric space with $h_{\text{pre}}(f)<\infty$. Let $\mu:\mathcal{B}\to R$ be a finite signed measure. Then $\mu\in\mathcal{M}(f)$ if and only if*

$$\int\varphi\,d\mu\le P_{\text{pre}}(f,\varphi)\quad\textit{for all }\varphi\in C(X,R).$$

Proof. The proof follows the idea of the proof of [**3**, Theorem 9.11] and is omitted. □

The pre-image entropy map of the continuous transformation $f:X\to X$ is the map $\mu\to h_{\text{pre},\mu}(f)$, which is defined on $\mathcal{M}(f)$ and has values in $[0,\infty]$. We denote by $h_{(pre,\cdot)}(f)$ the pre-image entropy map. It is said that $h_{(pre,\cdot)}(f)$ is upper semi-continuous at $\mu_0\in\mathcal{M}(f)$ if

$$\limsup_{\mu\to\mu_0}h_{\text{pre},\mu}(f)\le h_{\text{pre},\mu_0}(f),$$

i.e. for $\epsilon>0$, there is a neighborhood U of μ_0 in $\mathcal{M}(f)$ such that $\mu\in U$ implies $h_{\text{pre},\mu}(f)<h_{\text{pre},\mu_0}(f)+\epsilon$.

THEOREM 4.2. *Let $f : X \to X$ be a continuous map of a compact metric space with $h_{\text{pre}}(f) < \infty$ and let $\mu_0 \in \mathcal{M}(f)$. Then*

$$h_{\text{pre},\mu_0}(f) = \inf\left\{ P_{\text{pre}}(f,\varphi) - \int \varphi\, d\mu_0 : \varphi \in C(X,R) \right\}$$

if and only if $h_{(pre,\cdot)}(f)$ is upper semi-continuous at μ_0.

Proof. The proof follows completely from that of [**3**, Theorem 9.12] and is omitted. □

5. *Equilibrium states*

In this section, we give some applications of pre-image pressure $P_{\text{pre}}(f,\cdot)$ to equilibrium states.

Given $\varphi \in C(X,R)$. A member μ of $\mathcal{M}(f)$ is called an *equilibrium state* for φ if

$$P_{\text{pre}}(f,\varphi) = h_{\text{pre},\mu}(f) + \int \varphi\, d\mu.$$

Let $\mathcal{M}_\varphi(f)$ denote the collection of all equilibrium states for φ.

A tangent functional to the convex function $P_{\text{pre}}(f,\cdot)$ at φ is a finite signed Borel measure μ on X such that

$$P_{\text{pre}}(f,\varphi+\psi) - P_{\text{pre}}(f,\varphi) \geq \int \psi\, d\mu \quad \text{for all } \psi \in C(X,R).$$

We let $\mathcal{T}_\varphi(f)$ denote the collection of all tangent functionals to $P_{\text{pre}}(f,\cdot)$ at φ.

THEOREM 5.1. *Let $f : X \to X$ be a continuous map of a compact metric space and let $\varphi \in C(X,R)$. Then:*

(i) *$\mathcal{M}_\varphi(f)$ is convex;*
(ii) *the extreme points of $\mathcal{M}_\varphi(f)$ are precisely the ergodic members of $\mathcal{M}_\varphi(f)$;*
(iii) *if the pre-image entropy map is upper semi-continuous then $\mathcal{M}_\varphi(f)$ is compact and non-empty;*
(iv) *if $\varphi,\psi \in C(X,R)$ and if there exists $c \in R$ such that $\varphi - \psi - c$ belongs to the closure of the set $\{\varphi\circ f - \varphi : \varphi \in C(X,R)\}$ in $C(X,R)$, then $\mathcal{M}_\varphi(f) = \mathcal{M}_\psi(f)$.*

Proof. For each $\nu \in \mathcal{M}(f)$, we let

$$L(\varphi,\nu) = h_{\text{pre},\nu}(f) + \int \varphi\, d\nu.$$

(i) This follows from the fact that the pre-image entropy map is affine [**10**, Theorem 2.3].

(ii) Let μ be an extreme point of $\mathcal{M}_\varphi(f)$. To show that μ is ergodic, it is sufficient to show that μ is an extreme point of $\mathcal{M}(f)$. Let $\mu_1,\mu_2 \in \mathcal{M}(f)$ and $p \in (0,1)$ such that $\mu = p\mu_1 + (1-p)\mu_2$. Then $pL(\varphi,\mu_1) + (1-p)L(\varphi,\mu_2) = L(\varphi,\mu) = P_{\text{pre}}(f,\varphi)$. It follows from Theorem 3.1 that $L(\varphi,\mu_1) = L(\varphi,\mu_2) = P_{\text{pre}}(f,\varphi)$. Hence $\mu_1,\mu_2 \in \mathcal{M}_\varphi(f)$. Since μ is an extreme point of $\mathcal{M}_\varphi(f)$, $\mu_1 = \mu_2 = \mu$. Therefore μ is an extreme point of $\mathcal{M}(f)$.

(iii) By the upper semi-continuity of the pre-image entropy map, $M_\varphi(f)$ is non-empty and compact.

(iv) Note that

$$\int \varphi \, d\mu = \int \psi \, d\mu + c \quad \text{for all } \mu \in \mathcal{M}(f).$$

Therefore,

$$h_{\mathrm{pre},\mu}(f) + \int \varphi \, d\mu = h_{\mathrm{pre},\mu}(f) + \int \psi \, d\mu + c,$$

and $P_{\mathrm{pre}}(f, \varphi) = P_{\mathrm{pre}}(f, \psi) + c$. Hence $\mathcal{M}_\varphi(f) = \mathcal{M}_\psi(f)$. □

However, the following example shows that the set $\mathcal{M}_{\varphi(f)}$ may be empty if the pre-image entropy map is not upper semi-continuous.

Example 5.1. Choose numbers β_n such that $1 < \beta_n < 2$ but $\beta_n \to 2$. Let $T_n : X_n \to X_n$ denote the one-sided β_n-shift [**3**, §7.3]. We know $h(T_n) = \log \beta_n$. By [**8**, Proposition 2.2] we have $h_{\mathrm{pre}}(T_n) = \log \beta_n$, where $h_{\mathrm{pre}}(T_n)$ denotes the Cheng–Newhouse pre-image entropy of T_n. Suppose d_n is a metric on X_n and suppose $d_n(x, y) \le 1$, for all $x, y \in X_n$. We define a new space X which will be the disjoint union of the X_n together with a 'compactification' point x_∞.

Define the metric ρ on X by $\rho(x, y) = (1/n^2) d_n(x, y)$ if $x, y \in X_n$, $\rho(x, y) = \sum_{i=n}^{p} 1/i^2$ if $x \in X_n$, $y \in X_p$ and $n < p$, and $\rho(x, x_\infty) = \sum_{i=n}^{\infty} 1/i^2$ if $x \in X_n$.

Then (X, ρ) is a compact metric space and the subsets X_n converge to x_∞. The transformation $T : X \to X$ with $T(x) = T_n(x)$ if $x \in X_n$ and $T(x_\infty) = x_\infty$ is a continuous transformation. If $\mu \in \mathcal{M}(T)$ then $\mu = \sum_{n=1}^{\infty} p_n \mu_n + \big(1 - \sum_{n=1}^{\infty} p_n\big) \delta_{x_\infty}$, where $\mu_n \in \mathcal{M}(X_n, T_n)$ and $p_n \ge 0$, $\sum_{n=1}^{\infty} p_n \le 1$. Let $\mathcal{E}(X, T)$ denote the set of extreme points of $\mathcal{M}(T)$. Hence if $\mu \in \mathcal{E}(X, T)$ then either $\mu \in \mathcal{E}(X_n, T_n)$ for some n or $\mu = \delta_{x_\infty}$. Therefore, $h_{\mathrm{pre}}(T) = \sup\{h_{\mathrm{pre},\mu}(T) : \mu \in \mathcal{E}(X, T)\} = \sup_{n \ge 1} \sup\{h_{\mathrm{pre},\mu_n}(T_n) : \mu_n \in \mathcal{E}(X_n, T_n)\} = \sup_{n \ge 1} h_{\mathrm{pre}}(T_n) = \log 2$. If $\mathcal{M}_0(T) \ne \emptyset$, then by Theorem 5.1(ii) $\mathcal{M}_0(T)$ contains some ergodic measure μ. Then $\mu \in M(X_n, T_n)$ for some n, so $h_{\mathrm{pre},\mu}(T) = \log \beta_n$. This is a contradiction. Therefore $\mathcal{M}_0(T) = \emptyset$.

Let $\mathcal{M}^u(f) = \{\mu \in \mathcal{M}(f) : h_{\{\mathrm{pre},\cdot\}}(f)$ be upper semi-continuous at $\mu\}$.

THEOREM 5.2. *Let $f : X \to X$ be a continuous map of a compact metric space with $h_{\mathrm{pre}}(f) < \infty$ and let $\varphi \in C(X, R)$. Then:*

(i) $\mathcal{M}_\varphi(f) \subset \mathcal{T}_\varphi(f) \subset \mathcal{M}(f)$;

(ii) $\mathcal{T}_\varphi(f) = \bigcap_{n=1}^{\infty} \overline{\{\mu \in \mathcal{M}(f) : h_{\mathrm{pre},\mu}(f) + \int \varphi \, d\mu > P_{\mathrm{pre}}(f, \varphi) - 1/n\}}$;

(iii) $\mathcal{M}_\varphi(f) = \mathcal{T}_\varphi(f) \cap \mathcal{M}^u(f)$.

Proof. The proofs of (i) and (ii) follow [**3**, Theorem 9.14] and the remark of [**3**, Theorem 9.15], respectively, and are omitted.

(iii) Using (ii) we have that $\mathcal{T}_\varphi(f) \cap \mathcal{M}^u(f) \subset \mathcal{M}_\varphi(f)$. Now let $\mu \in \mathcal{M}_\varphi(f)$, i.e.

$$h_{\mathrm{pre},\mu}(f) + \int \varphi \, d\mu = P_{\mathrm{pre}}(f, \varphi).$$

If $\mu_n \in \mathcal{M}(f)$, $\mu_n \to \mu$, then

$$h_{\mathrm{pre},\mu_n}(f) + \int \varphi \, d\mu_n \le P_{\mathrm{pre}}(f, \varphi),$$

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i.e.

$$h_{\mathrm{pre},\mu_n}(f) \leq h_{\mathrm{pre},\mu}(f) + \left(\int \varphi\, d\mu - \int \varphi\, d\mu_n\right).$$

Hence, $\limsup_{n\to\infty} h_{\mathrm{pre},\mu_n}(f) \leq h_{\mathrm{pre},\mu}(f)$, i.e. the pre-image entropy map $h_{\{\mathrm{pre},\cdot\}}(f)$ is upper semi-continuous at μ. Therefore $\mu \in \mathcal{T}_\varphi(f) \cap \mathcal{M}^u(f)$. □

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