Bull. Aust. Math. Soc. **101** (2020), 401–414
doi:10.1017/S000497271900100X

A SHIFTED CONVOLUTION SUM OF d_3 AND THE FOURIER COEFFICIENTS OF HECKE–MAASS FORMS II

HENGCAI TANG

(Received 13 June 2019; accepted 6 August 2019; first published online 26 September 2019)

Abstract

Let $d_3(n)$ be the divisor function of order three. Let g be a Hecke–Maass form for Γ with $\Delta g = (1/4 + t^2)g$. Suppose that $\lambda_g(n)$ is the nth Hecke eigenvalue of g. Using the Voronoi summation formula for $\lambda_g(n)$ and the Kuznetsov trace formula, we estimate a shifted convolution sum of $d_3(n)$ and $\lambda_g(n)$ and show that

$$\sum_{n\le x} d_3(n)\lambda_g(n-1) \ll_{t,\varepsilon} x^{8/9+\varepsilon}.$$

This corrects and improves the result of the author ['Shifted convolution sum of d_3 and the Fourier coefficients of Hecke–Maass forms', *Bull. Aust. Math. Soc.* **92** (2015), 195–204].

2010 *Mathematics subject classification*: primary 11F30; secondary 11F11, 11F66.

Keywords and phrases: divisor function, Hecke–Maass form, trace formula.

1. Introduction

Let $d_3(n)$ be the divisor function of order three, which gives the coefficients of the Dirichlet series for $\zeta^3(s)$. Let f be a holomorphic Hecke eigenform of weight k for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ whose Fourier expansion at infinity is

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz)$$

with $e(x) = e^{2\pi i x}$. In 1995, Pitt [10] first considered the shifted convolution sum

$$\Psi(f, x) = \sum_{n\le x} d_3(n)\lambda_f(n-1).$$

By analytical continuation of the Dirichlet series $\sum_{n=1}^{\infty} d_3(n)\lambda_f(n-1)/n^s$, he found that

$$\Psi(f, x) \ll x^{71/72+\varepsilon}.$$

This project is supported by the National Natural Science Foundation of China (No. 11871193) and the Foundation of Henan University (No. CX3071A0780001).

In 2013, with the help of the idea on the shifted convolution sums for GL(3) × GL(2) from [9], Munshi [8] improved the upper bound to $x^{34/35+\varepsilon}$. In 2015, the author [11] considered the Maass case and further improved the bound to $x^{29/30+\varepsilon}$. These two results rely on the estimates of exponential sums from [9, Lemma 11]. Recently, Xi [13] pointed out that the estimate should have the form

$$\mathcal{T}(n, q_1m', h; q_1, q_1, q_2) \ll q_1^{7/2} q_2^{5/2} \sqrt{(m', q_1q_2)}.$$

The reason is that in the sum $\mathcal{T}^{q_1=\tilde{q}_1}$ on [9, page 2359], α runs though a complete residue system modulo q_1^2, rather than modulo q_1. Inserting the corrected estimate into the argument of [11, page 203] gives the bound $x^{37/38+\varepsilon}$. The work of Pitt [10], Munshi [8] and the author [11] rely on the Voronoi summation formula for the divisor function and the circle method of Jutila. In 2016, by means of a smooth decomposition of $d_3(n)$, Topacogullari [12] found a new approach in the holomorphic case using the Kuznetsov trace formula and obtained

$$\Psi(f, x) \ll x^{8/9+\varepsilon},$$

which is still the best result. We adapt the idea of Topacogullari [12] to the Maass case and prove the following result.

Theorem 1.1. *Let g be a Hecke–Maass form for Γ with $\Delta g = (1/4 + t^2)g$ and Fourier expansion*

$$g(z) = y^{1/2} \sum_{n\neq 0} \lambda_g(n) K_{it}(2\pi|n|y) e(nx).$$

Normalise g by setting $\lambda_g(1) = 1$ so that $\lambda_g(n)$ is the nth Hecke eigenvalue of g. Let

$$\Phi(w) = \sum_n d_3(n)\lambda_g(n-1) w\Big(\frac{n}{x}\Big),$$

where $w : \mathbb{R} \to [0, \infty)$ is a smooth function with compact support in $[1, 2]$ such that

$$w^{(\nu)} \ll y^{\nu} \quad \text{for } \nu \geq 0 \quad \text{and} \quad \int |w^{(\nu)}(\xi)|\, d\xi \ll y^{\nu-1} \quad \text{for } \nu \geq 1,$$

where $y = x^{\eta}$ with a parameter $0 < \eta < 1$. Then

$$\Phi(w) \ll x^{5/6+\varepsilon}(x^{7/192} + x^{\eta/2}).$$

By the Rankin–Selberg theory, it is well known that

$$\sum_{n\leq x} |\lambda_g(n)|^2 = C_t x + O_t(x^{3/5}). \tag{1.1}$$

Taking $w = 0$ in the intervals $(0, 1 - 1/y] \cup [2 + 1/y, \infty)$,

$$\sum_{x<n\leq 2x} d_3(n)\lambda_g(n-1) = \Phi(w) + O\Big(\sum_{x<n<x+x/y} |d_3(n)\lambda_g(n-1)|\Big).$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

For the error term, Cauchy's inequality and (1.1) give the bound

$$\sum_{x<n<x+x/y} |d_3(n)\lambda_g(n-1)| \ll \Big(\sum_{x<n<x+x/y} d_3^2(n)\Big)^{1/2}\Big(\sum_{x<n<x+x/y} |\lambda_g(n-1)|^2\Big)^{1/2}$$
$$\ll x^{1-\eta+\varepsilon} + x^{4/5-\eta/2+\varepsilon}.$$

Combining this with the bound of $\Phi(w)$ and taking $\eta = \frac{1}{9}$,

$$\sum_{n\le x} d_3(n)\lambda_g(n-1) \ll x^{8/9+\varepsilon},$$

where the implied constant depends on t and ε.

Remark 1.2. Note that $\frac{8}{9}$ is less than all the earlier exponents. Compared with the work of the author [11] and Topacogullari [12], there are several differences and new difficulties in this short note. Firstly, the circle method of Jutila, which plays an important role in [11], disappears. Secondly, the Kuznetsov trace formula and the large-sieve inequality are used to handle the Kloosterman sums. Finally, estimation of Bessel functions in the Voronoi summation formula of $\lambda_j(n)$ is more complicated than in the case of the holomorphic Hecke eigenforms in [12].

2. Preliminaries

In this section, we introduce some lemmas. The first is the Voronoi summation formula for $\lambda_g(n)$, which is a variation of the result proved by Kowalski *et al.* [6].

Lemma 2.1. *Let f be a compactly supported smooth function on $(0,\infty)$. Then*

$$\sum_{\substack{n=1\\ n\equiv b\,(\mathrm{mod}\,c)}}^{\infty} \lambda_g(n)f(n) = \frac{1}{c}\sum_{q|c}\frac{1}{q}\sum_{m=1}^{\infty}\lambda_g(m)S(-b,-m;q)G_1\Big(\frac{m}{q^2}\Big)$$
$$+\frac{1}{c}\sum_{q|c}\frac{1}{q}\sum_{m=1}^{\infty}\lambda_g(m)S(-b,m;q)G_2\Big(\frac{m}{q^2}\Big),$$

where $S(b,m;c)$ is the classical Kloosterman sum and

$$G_1(y) = \int_0^\infty f(x)J_g(4\pi\sqrt{xy})\,dx, \quad G_2(y) = \int_0^\infty f(x)K_g(4\pi\sqrt{xy})\,dx \tag{2.1}$$

with

$$J_g(x) = -\frac{\pi}{\sin \pi it}(J_{2it}(x) - J_{-2it}(x)), \quad K_g(x) = 4\varepsilon_g\cos(\pi t)K_{2it}(x)$$

and $a\bar{a}\equiv 1\,(\mathrm{mod}\,q)$, $\varepsilon_g = 1$ or -1 according as g is even or odd.

Proof. Kowalski *et al.* [6] showed that

$$\sum_{m=1}^{\infty}\lambda_g(m)e\Big(\frac{am}{q}\Big)f(m) = \frac{1}{q}\sum_{m=1}^{\infty}\lambda_g(m)e\Big(-\frac{\bar{a}m}{q}\Big)G_1\Big(\frac{m}{q^2}\Big) + \frac{1}{q}\sum_{m=1}^{\infty}\lambda_g(m)e\Big(\frac{\bar{a}m}{q}\Big)G_2\Big(\frac{m}{q^2}\Big), \tag{2.2}$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

where a, q are positive integers with $(a, q) = 1$. The property of additive characters implies that

$$\sum_{\substack{n=1 \\ n\equiv b \,(\mathrm{mod}\, c)}}^{\infty} \lambda_g(n)f(n) = \frac{1}{c}\sum_{q|c}\sum_{n=1}^{\infty}\sideset{}{^*}\sum_{u\,(\mathrm{mod}\, q)} e\Big(\frac{(n-b)u}{q}\Big)\lambda_g(n)f(n)$$

$$= \frac{1}{c}\sum_{q|c}\sideset{}{^*}\sum_{u\,(\mathrm{mod}\, q)} e\Big(\frac{-bu}{q}\Big)\sum_{n=1}^{\infty} e\Big(\frac{nu}{q}\Big)\lambda_g(n)f(n).$$

Therefore, (2.2) gives the final result. □

By the power series expansion of the J-Bessel function and the K-Bessel function,

$$J_g^{(\nu)}(x), K_g^{(\nu)}(x) \ll x^{-\nu} \quad \text{for } x \le 1.$$

For large x, [1, (4.17)] implies that

$$J_g(x) = \Re\Big(e\Big(\frac{w(x,t)}{2\pi}\Big)f(x,t)\Big). \tag{2.3}$$

Here,

$$w(x,t) = t\,\mathrm{arcsinh}\frac{x}{t} - \sqrt{t^2+x^2} \ll x, \qquad x^j\frac{\partial^j f(x,t)}{\partial x^j} \ll_t \frac{1}{x^{1/2}}$$

for any integer $j \ge 0$. Furthermore,

$$K_g(x) = 4\varepsilon_g\cos(\pi t)\sqrt{\frac{\pi}{2x}}e^{-x}\Big(1 + O\Big(\frac{1}{x}\Big)\Big) \ll_t e^{-x}. \tag{2.4}$$

Next, we introduce the Kuznetsov trace formula. Let $\Gamma_0(q)$ be the Hecke congruence subgroup of level q. Denote by $M_k(\Gamma_0(q))$ the space of holomorphic cusp forms of weight k with dimension $\theta_k(q)$ and $f_{j,k}$ $(1 \le j \le \theta_k(q))$ the orthonormal Hecke eigenbasis. The Fourier expansion of $f_{j,k}$ around ∞ is

$$f_{j,k}(z) = \sum_{m\ge1}\psi_{j,k}(m)e(mz).$$

For $\Gamma_0(q)$, we have the spectral decomposition

$$L^2(\Gamma_0(q)/\mathbb{H}) = \mathbb{C} \oplus L^2_{\mathrm{cusp}}(\Gamma_0(q)\backslash\mathbb{H}) \oplus L^2_{\mathrm{Eis}}(\Gamma_0(q)\backslash\mathbb{H}),$$

where $L^2_{\mathrm{cusp}}(\Gamma_0(q)\backslash\mathbb{H})$ is the space spanned by the cusp forms and $L^2_{\mathrm{Eis}}(\Gamma_0(q)\backslash\mathbb{H})$ is a continuous direct sum spanned by Eisenstein series (see below).

Let u_0 be the constant function and let $\{u_j\}_{j\ge1}$ run over an orthonormal Hecke eigenbasis of $L^2_{\mathrm{cusp}}(\Gamma_0(q)\backslash\mathbb{H})$. Denote the real eigenvalues corresponding to these functions by $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$. Each u_j has a Fourier expansion

$$u_j(z) = \sqrt{y}\sum_{n\ne0}\rho_j(n)K_{i\kappa_j}(2\pi n|y|)e(nx),$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

where $K_s(y)$ is the K-Bessel function and $\kappa_j^2 = \lambda_j - 1/4$. We choose the sign of κ_j so that $i\kappa_j \geq 0$ if $\lambda_j < 1/4$ and $\kappa_j \geq 0$ if $\lambda_j \geq 1/4$. The Selberg eigenvalue conjecture that $\lambda_1 \geq 1/4$ remains open. The eigenvalues with $0 < \lambda_j < 1/4$ as well as the corresponding κ_j are called exceptional. Let $\theta \in \mathbb{R}^+$ be such that $i\kappa_j \leq \theta$ for all exceptional κ_j uniformly for all level q. The work of Kim and Sarnak [5] allows us to take $\theta = 7/64$.

The Eisenstein series is given by

$$E_{\mathfrak{c}}(z; s) = \sum_{\tau \in \Gamma_{\mathfrak{c}} \backslash \Gamma_0(q)} \Im(\sigma_{\mathfrak{c}} \tau z)^s \quad \text{for } z \in \mathbb{H},\ \Re s > 1.$$

Here, $\mathfrak{c}$ is a cusp of $\Gamma_0(q)$, $\Gamma_{\mathfrak{c}}$ is the stabiliser of $\mathfrak{c}$ and $\sigma_{\mathfrak{c}}\infty = \mathfrak{c}$, $\sigma_{\mathfrak{c}}^{-1}\Gamma_{\mathfrak{c}}\sigma_{\mathfrak{c}} = \Gamma_\infty$. Note that the space $L^2_{\mathrm{Eis}}(\Gamma_0(q)\backslash\mathbb{H})$ is the continuous direct sum spanned by $E_{\mathfrak{c}}(z; 1/2 + ir)$ ($r \in \mathbb{R}$). The Fourier expansion of $E_{\mathfrak{c}}(z; s)$ around ∞ is

$$\begin{aligned} E_{\mathfrak{c}}(z; s) = \delta_{\mathfrak{c}\infty} y^s + \sqrt{\pi}\frac{\Gamma(s - 1/2)}{\Gamma(s)} \varphi_{\mathfrak{c},0}(s) y^{1-s} \\ + \frac{2\pi^s \sqrt{y}}{\Gamma(s)} \sum_{n \neq 0} |n|^{s-1/2} \varphi_{\mathfrak{c},n}(s) K_{s-1/2}(2\pi|n|y) e(nx), \end{aligned}$$

where $\delta_{\mathfrak{c}\infty} = 1$ if $\mathfrak{c} = \infty$ and 0 otherwise. For a smooth function ϕ, we define the Bessel transforms

$$\begin{aligned} \hat{\phi}(r) &= \frac{\pi}{\sinh(\pi r)} \int_0^\infty \frac{J_{2ir}(x) - J_{-2ir}(x)}{2i} \phi(x) \frac{dx}{x}, \\ \tilde{\phi}(k) &= \int_0^\infty J_k(x)\phi(x) \frac{dx}{x}, \\ \check{\phi}(r) &= \frac{4}{\pi} \cosh(\pi r) \int_0^\infty K_{2ir}(x)\phi(x) \frac{dx}{x}. \end{aligned}$$

With this notation, the Kuznetsov trace formula can be stated as follows.

Lemma 2.2 [3, Theorem 1]. *Let $m, n \in \mathbb{N}$ and let ϕ be a compactly supported smooth function on $(0, \infty)$. Then*

$$\begin{aligned} &\sum_{c \equiv 0 \,(\mathrm{mod}\, q)} \frac{S(m, n; c)}{c} \phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big) \\ &= \sum_{j=1}^\infty \frac{\hat{\phi}(\kappa_j)}{\cosh(\pi\kappa_j)} \overline{\rho_j(m)} \rho_j(n) \\ &\quad + \frac{1}{\pi} \sum_{\mathfrak{c}} \int_{-\infty}^\infty \Big(\frac{m}{n}\Big)^{-ir} \overline{\varphi_{\mathfrak{c},m}\Big(\frac{1}{2} + ir\Big)} \varphi_{\mathfrak{c},n}\Big(\frac{1}{2} + ir\Big) \hat{\phi}(r)\, dr \\ &\quad + \frac{1}{2\pi} \sum_{k \equiv 0 \,(\mathrm{mod}\, 2)} \tilde{\phi}(k - 1) \frac{i^k (k-1)!}{4\pi(\sqrt{mn})^{k-1}} \sum_{1 \leq j \leq \theta_k(q)} \overline{\psi_{j,k}(m)} \psi_{j,k}(n) \end{aligned}$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

and

$$\sum_{c\equiv 0\,(\mathrm{mod}\,q)} \frac{S(m,-n;c)}{c}\phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big)$$
$$= \sum_{j=1}^{\infty} \frac{\check{\phi}(\kappa_j)}{\cosh(\pi\kappa_j)}\overline{\rho_j(m)}\rho_j(n)$$
$$+ \frac{1}{\pi}\sum_{\mathfrak{c}}\int_{-\infty}^{\infty}(mn)^{ir}\varphi_{\mathfrak{c},m}\Big(\frac{1}{2}+ir\Big)\varphi_{\mathfrak{c},n}\Big(\frac{1}{2}+ir\Big)\check{\phi}(r)\,dr.$$

To bound $\hat{\phi}$, $\tilde{\phi}$, $\check{\phi}$, we use the following results from [2, Lemma 2.1].

Lemma 2.3. *Let $f:(0,\infty)\to\mathbb{C}$ be a smooth compactly supported function such that*

$$\operatorname{supp} f \asymp X \quad \textit{and} \quad f^{(\nu)} \ll \frac{1}{Y^{\nu}}, \quad \nu = 0,1,2,$$

for positive X, Y with $X \gg Y$. Then

$$\hat{f}(ir), \check{f}(ir) \ll \frac{1+Y^{-2r}}{1+Y} \quad \textit{for } 0\le r<\frac{1}{4},$$
$$\hat{f}(r), \check{f}(r), \tilde{f}(r) \ll \frac{1+|\log Y|}{1+Y} \quad \textit{for } r\ge 0,$$
$$\hat{f}(r), \check{f}(r), \tilde{f}(r) \ll \Big(\frac{X}{Y}\Big)^2\Big(\frac{1}{r^{5/2}}+\frac{X}{r^3}\Big) \quad \textit{for } r \gg \max(X,1).$$

Assume that $w:(0,\infty)\to\mathbb{C}$ is a smooth compactly supported function such that

$$\operatorname{supp} w \asymp X \quad \text{and} \quad w^{(\nu)} \ll \frac{1}{X^{\nu}}, \quad \nu\ge 0.$$

For $\alpha>0$, define

$$f(\xi) := e\Big(\xi\frac{\alpha}{2\pi}\Big)w(\xi).$$

Then we have the following bounds for the Bessel transforms of f.

Lemma 2.4 [12, Lemma 2.6]. *Assume that $X \ll 1$, $\alpha X \gg 1$. Then, for $\nu, \mu \ge 0$,*

$$\hat{f}(ir), \check{f}(ir) \ll X^{-2r+\varepsilon}\Big(X^{\mu}+\frac{1}{(\alpha X)^{\nu}}\Big) \quad \textit{for } 0\le r<\frac{1}{4},$$
$$\hat{f}(r), \check{f}(r), \tilde{f}(r) \ll \frac{\alpha^{\varepsilon}}{\alpha X}\Big(\frac{\alpha X}{r}\Big)^{\nu} \quad \textit{for } r\ge 0.$$

We also use the large-sieve inequalities for Fourier coefficients of cusp forms.

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

Lemma 2.5 [3, Theorem 2]. *Let $K, N \geq 1$ and (a_n) be a sequence of complex numbers. Then*

$$\left.\begin{array}{c}\displaystyle\sum_{|\kappa_j|\leq K}\frac{1}{\cosh(\pi\kappa_j)}\Big|\sum_{n\sim N}a_n\rho_j(n)\Big|^2\\ \displaystyle\sum_{\substack{2\leq k\leq K\\ 2|k}}\frac{(k-1)!}{(4\pi)^{k-1}}\sum_{1\leq j\leq\theta_k(q)}\Big|\sum_{n\sim N}a_n n^{-(k-1)/2}\psi_{j,k}(n)\Big|^2\\ \displaystyle\sum_{\mathfrak{c}}\int_{-K}^{K}\Big|\sum_{n\sim N}a_n n^{ir}\varphi_{\mathfrak{c},n}\Big(\frac{1}{2}+ir\Big)\Big|^2\,dr\end{array}\right\}\ll\Big(K^2+\frac{N^{1+\varepsilon}}{q}\Big)\sum_{n\sim N}|a_n|^2.$$

Without averaging over n, we have the following result.

Lemma 2.6 [12, Lemmas 2.9 and 2.10]. *For $K, n \geq 1$,*

$$\left.\begin{array}{c}\displaystyle\sum_{|\kappa_j|\leq K}\frac{|\rho_j(n)|^2}{\cosh(\pi\kappa_j)}\\ \displaystyle\sum_{\substack{2\leq k\leq K\\ 2|k}}\frac{(k-1)!}{(4\pi)^{k-1}}\sum_{1\leq j\leq\theta_k(q)}|\psi_{j,k}(n)|^2\\ \displaystyle\sum_{\mathfrak{c}}\int_{-K}^{K}\Big|\varphi_{\mathfrak{c},n}\Big(\frac{1}{2}+ir\Big)\Big|^2\,dr\end{array}\right\}\ll K^2+(qKn)^{\varepsilon}(q,n)^{1/2}\frac{n^{1/2}}{q}.$$

For the exceptional eigenvalues,

$$\sum_{\kappa_j\text{ exc.}}|\rho_j(h)|^2X^{4i\kappa_j}\ll(Xh)^{\varepsilon}\frac{(hX^2)^{2\theta}}{q^{4\theta}}(h,q)^{1/2}\Big(1+\frac{h^{1/2}}{q}\Big)$$

for $X, q, h \geq 1$ with $h^{1/2}X \geq q$.

3. Proof of Theorem 1.1

Firstly, we need a smooth decomposition of $d_3(n)$, given by Meurman [7], in order to apply the Kuznetsov trace formula. This is

$$d_3(n)=\sum_{abc=n}[\nu_1(a)\nu_1(b)\nu_1(c)-3\nu_1(a)\nu_1(b)+3\nu_1(a)\nu_2(b)(2-\nu_2(c))],\tag{3.1}$$

where

$$\nu_1(\xi)=\nu\Big(\frac{\xi}{x^{1/3}}\Big),\quad \nu_2(\xi)=\nu\Big(\frac{\xi}{\sqrt{x/a}}\Big)$$

and $\nu:\mathbb{R}\to[0,\infty)$ a smooth function satisfying

$$\operatorname{supp}\nu\subset[-2,2],\quad \nu(\xi)=1\quad\text{for }\xi\in[-1,1],\quad \nu^{(n)}(\xi)\ll|\xi|^{-n}.$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

Let h_X be smooth compactly supported functions such that

$$\sum_X h_X = 1, \quad \operatorname{supp} h_X \subset \Big[\frac{X}{2}, X\Big], \quad h_X^{(\nu)} \ll \frac{1}{X^\nu},$$

where the summation runs over powers of two. Denote by $h(a, b, c)$ the term in the summation (3.1) and set

$$h_{ABC}(a, b, c) := h(a, b, c)h_A(a)h_B(b)h_C(c)$$

and

$$\Phi(w, A, B, C) := \sum_{a,b,c} h_{ABC}(a, b, c)\lambda_g(abc - 1)w\Big(\frac{abc}{x}\Big).$$

Thus,

$$\Phi(w) = \sum_{A,B,C} \Phi(w, A, B, C).$$

We can assume that

$$ABC \asymp x, \quad A \ll B \ll C$$

because, otherwise, $\Phi(w, A, B, C) = 0$.

Set $m = abc - 1$. Then

$$\begin{aligned}\Phi(w, A, B, C) &= \sum_{a,b} \sum_{m\equiv 1 \,(\mathrm{mod}\, ab)} \lambda_g(m)w\Big(\frac{m+1}{x}\Big)h_{ABC}\Big(a, b, \frac{m+1}{ab}\Big)\\ &= \sum_{a,b} \sum_{m\equiv 1 \,(\mathrm{mod}\, ab)} \lambda_g(m)f(m; a, b),\end{aligned}$$

where

$$f(\xi; a, b) := w\Big(\frac{\xi+1}{x}\Big)h_{ABC}\Big(a, b, \frac{\xi+1}{ab}\Big).$$

Note that

$$\operatorname{supp} f(*; a, b) \asymp x, \quad \frac{\partial^{\nu_1+\nu_2}}{\partial\xi^{\nu_1}\partial b^{\nu_2}} f(\xi; a, b) \ll \frac{y^{\nu_1}}{x^{\nu_1}B^{\nu_2}}. \tag{3.2}$$

For the inner sum, we apply the Voronoi summation formula (Lemma 2.1). Thus,

$$\begin{aligned}\Phi(w, A, B, C) &= \sum_{a,b} \sum_{c|ab} \sum_{m=1}^{\infty} \lambda_g(m)\frac{S(1, -m; c)}{c} G_1\Big(\frac{m}{c^2}; a, b\Big)\\ &\quad + \sum_{a,b} \sum_{c|ab} \sum_{m=1}^{\infty} \lambda_g(m)\frac{S(1, m; c)}{c} G_2\Big(\frac{m}{c^2}; a, b\Big)\\ &:= \Phi_1(w, A, B, C) + \Phi_2(w, A, B, C),\end{aligned}$$

where G_1, G_2 are given by (2.1) with $f(x)$ replaced by $f(x; a, b)$. In what follows, we only consider Φ_1 because Φ_2 can be estimated similarly. By an elementary argument with $ab = ct$, $s = (a, t)$, $a = a_1 s$, $t = t_1 s$ and the property of the Möbius function,

$$\Phi_1(w, A, B, C) = \sum_{r,s,t} \frac{\mu(r)}{r^2 st} \sum_a \sum_m \lambda_g(m) \sum_{ar|c} \frac{S(1, -m; c)}{c} \frac{F(c, m; a, r, s, t)}{a}.$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

Here,

$$a \asymp \frac{A}{rs}, \quad c \asymp \frac{AB}{rst}, \quad r, s, t \ll A,$$

and

$$F(c, m; a, r, s, t) := \frac{ar}{c} \int_0^\infty J_g\Big(\frac{4\pi}{c}\sqrt{mx}\Big) f\Big(x; ars, \frac{ct}{a}\Big)\, dx.$$

Trivially,

$$F(c, m; a, r, s, t) \ll x^{1+\varepsilon}\frac{ar}{c}.$$

By the properties of Bessel functions $(z^\nu J_\nu(z))' = z^\nu J_{\nu-1}(z)$ (see [4, Section 7.13.2 (17)]), (3.2) and partial integration,

$$F(c, m; a, r, s, t) \ll \frac{y^{\nu-1}}{x^{\nu/2-3/4}} \frac{c^{\nu+1/2}}{m^{\nu/2+1/4}} \frac{ar}{c}$$

for $m \gg c^2/x$. Define

$$M_0 = \frac{x^\varepsilon y^2}{x}\Big(\frac{AB}{rst}\Big)^2.$$

For $m > M_0$, the contribution to Φ_1 is negligible, where $\lambda_g(n) \le n^{7/64+\varepsilon}$. So, it suffices to consider the sums

$$R(M; a, r, s, t) := \sum_{m\sim M} \lambda_g(m) \sum_{ar|c} \frac{S(1, -m; c)}{c} F(c, m; a, r, s, t),$$

where $M \le M_0$. In order to use the Kuznetsov trace formula (Lemma 2.2), we define

$$\tilde{F}(c, m; a, r, s, t) := h(m)\frac{arc}{4\pi\sqrt{m}} \int_0^\infty J_g(c\sqrt{x}) f\Big(x; ars, \frac{4\pi\sqrt{m}}{c}\frac{t}{a}\Big)\, dx,$$

where $h(m)$ is a smooth compactly supported function such that

$$h(m) = 1 \quad \text{for } m \in [M, 2M], \quad \operatorname{supp} f \asymp M \quad \text{and} \quad h^{(\nu)}(m) \ll \frac{1}{M^\nu}.$$

Then

$$F(c, m; a, r, s, t) = \tilde{F}\Big(\frac{4\pi\sqrt{m}}{c}, m; a, r, s, t\Big)$$

for $m \in [M, 2M]$. Next, let

$$G_0(\lambda) := x^{1+\varepsilon}\frac{rt}{B} \min\Big(M, \frac{1}{M\lambda^2}\Big)$$

for $0 < \lambda < 1$. We have

$$\tilde{F}(c, m; a, r, s, t) = \int_0^1 G_0(\lambda) G_\lambda(c; a, r, s, t) e(m\lambda)\, d\lambda$$

with

$$G_\lambda(c; a, r, s, t) := \frac{1}{G_0(\lambda)} \lim_{Y\to+\infty} \int_{-Y}^{Y} \tilde{F}(c, y; a, r, s, t) e(-\lambda y)\, dy.$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

Hence,

$$R(M;a,r,s,t)=\int_0^1 G_0(\lambda)\sum_{m\sim M}\lambda_g(m)e(m\lambda)\sum_{ar|c}\frac{S(1,-m;c)}{c}G_\lambda\Big(\frac{4\pi\sqrt{m}}{c};a,r,s,t\Big)\,d\lambda. \tag{3.3}$$

For the inner sum over c, we apply the Kuznetsov trace formula (the second formula in Lemma 2.2) and get

$$\sum_{ar|c}\frac{S(1,-m;c)}{c}G_\lambda\Big(\frac{4\pi\sqrt{m}}{c};a,r,s,t\Big) \tag{3.4}$$

$$=\sum_{j=1}^{\infty}\frac{\check{G}_\lambda(\kappa_j;a,r,s,t)}{\cosh(\pi\kappa_j)}\overline{\rho_j(1)}\rho_j(m)$$

$$+\frac{1}{\pi}\sum_{\mathfrak{c}}\int_{-\infty}^{\infty}m^{iy}\varphi_{\mathfrak{c},1}\Big(\frac{1}{2}+iy\Big)\varphi_{\mathfrak{c},m}\Big(\frac{1}{2}+iy\Big)\check{G}_\lambda(y;a,r,s,t)\,dy. \tag{3.5}$$

For $\check{G}_\lambda$, we use the following result, which will be proved in Section 4.

Lemma 3.1. *Let*

$$M_1:=\frac{x^\varepsilon}{x}\Big(\frac{AB}{rst}\Big)^2,\quad \Xi:=\sqrt{M}\frac{rst}{AB},\quad Z:=\sqrt{xM}\frac{rst}{AB}.$$

For $M\ll M_1$,

$$\check{G}_\lambda(i\kappa;a,r,s,t)\ll \Xi^{-2\kappa},\quad 0<\kappa\le \tfrac{1}{4}, \tag{3.6}$$

$$\check{G}_\lambda(\kappa;a,r,s,t)\ll \frac{x^\varepsilon}{1+\kappa^{5/2}},\quad \kappa\ge 0.$$

For $M_1\ll M\ll M_0$ *and any* $\nu\ge 0$,

$$\check{G}_\lambda(i\kappa;a,r,s,t)\ll x^{-\nu},\quad 0<\kappa\le \tfrac{1}{4}, \tag{3.7}$$

$$\check{G}_\lambda(\kappa;a,r,s,t)\ll \frac{x^\varepsilon}{Z^{5/2}}\Big(\frac{Z}{\kappa}\Big)^\nu,\quad \kappa>0. \tag{3.8}$$

Putting (3.5) into (3.3),

$$R(M;a,r,s,t)=\int_0^1 G_0(\lambda)\sum_{l=1}^{3}E_l(M;a,r,s,t)\,d\lambda,$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

where

$$E_1(M;a,r,s,t) = \sum_{\kappa_j\ge 0} \check{G}_\lambda(\kappa_j;a,r,s,t)\frac{\overline{\rho_j(1)}}{\cosh(\pi\kappa_j)} \sum_{m\sim M} \lambda_g(m)e(m\lambda)\rho_j(m),$$

$$E_2(M;a,r,s,t) = \sum_{\kappa_j \text{ exc.}} \check{G}_\lambda(\kappa_j;a,r,s,t)\frac{\overline{\rho_j(1)}}{\cosh(\pi\kappa_j)} \sum_{m\sim M} \lambda_g(m)e(m\lambda)\rho_j(m),$$

$$E_3(M;a,r,s,t) = \frac{1}{\pi}\sum_{\mathfrak{c}} \int_{-\infty}^{\infty} \check{G}_\lambda(y;a,r,s,t)\varphi_{\mathfrak{c},1}\Big(\frac{1}{2}+iy\Big)m^{iy} \times \sum_{m\sim M} \lambda_g(m)e(m\lambda)\varphi_{\mathfrak{c},m}\Big(\frac{1}{2}+iy\Big)\,dy.$$

Firstly, we consider $E_1(M;a,r,s,t)$. For $M \ll M_1$, we split the sum into two parts $E_{11}(M;a,r,s,t) = \sum_{0\le\kappa_j\le 1}(\cdots)$, $E_{12}(M;a,r,s,t) = \sum_{\kappa_j>1}(\cdots)$, according as $\kappa_j \le 1$ or not. By Cauchy's inequality,

$$E_{11}(M;a,r,s,t) \ll \max_{0\le\kappa_j\le 1} |\check{G}_\lambda(\kappa_j;a,r,s,t)| \Big(\sum_{\kappa_j\le 1}\frac{|\rho_j(1)|^2}{\cosh(\pi\kappa_j)}\Big)^{1/2} \times \Big(\sum_{\kappa_j\le 1}\frac{1}{\cosh(\pi\kappa_j)}\Big|\sum_{m\sim M}\lambda_g(m)e(m\lambda)\rho_j(m)\Big|^2\Big)^{1/2}.$$

Lemmas 3.1, 2.5 and 2.6 and (1.1) give the bound

$$E_{11}(M;a,r,s,t) \ll x^{\varepsilon}\Big(1+\frac{1}{ar}\Big)^{1/2}\Big(1+\frac{M}{ar}\Big)^{1/2}M^{1/2} \ll \frac{x^{\varepsilon}}{rt}\Big(\frac{AB}{x^{1/2}}+\frac{A^{3/2}B^2}{x}\Big).$$

Thus,

$$\int_0^1 G_0(\lambda)E_{11}(M;a,r,s,t)\,d\lambda \ll \int_0^{1/M} x^{1+\varepsilon}\frac{rt}{B}M\frac{1}{rt}\Big(\frac{AB}{x^{1/2}}+\frac{A^{3/2}B^2}{x}\Big)\,d\lambda + \int_{1/M}^1 x^{1+\varepsilon}\frac{rt}{B}\frac{1}{M\lambda^2}\frac{1}{rt}\Big(\frac{AB}{x^{1/2}}+\frac{A^{3/2}B^2}{x}\Big)\,d\lambda \ll x^{\varepsilon}(x^{1/2}A + A^{3/2}B) \ll x^{5/6+\varepsilon}. \tag{3.9}$$

For E_{12}, we use the dyadic subdivision. Setting $K=2^k$,

$$E_{12}(M;a,r,s,t) \ll \sum_{k=0}^{\infty} \max_{K<\kappa_j\le 2K} |\check{G}_\lambda(\kappa_j;a,r,s,t)| \Big(\sum_{K<\kappa_j\le 2K}\frac{|\rho_j(1)|^2}{\cosh(\pi\kappa_j)}\Big)^{1/2} \times \Big(\sum_{K<\kappa_j\le 2K}\frac{1}{\cosh(\pi\kappa_j)}\Big|\sum_{m\sim M}\lambda_g(m)e(m\lambda)\rho_j(m)\Big|^2\Big)^{1/2}.$$

Applying Lemmas 3.1, 2.5 and 2.6 and (1.1) again,

$$E_{12}(M;a,r,s,t) \ll x^{\varepsilon}\sum_{k=1}^{\infty}\frac{1}{K^{5/2}}\Big(K^2+\frac{1}{ar}\Big)^{1/2}\Big(K^2+\frac{M}{ar}\Big)^{1/2}M^{1/2} \ll \frac{x^{\varepsilon}}{rt}\Big(\frac{AB}{x^{1/2}}+\frac{A^{3/2}B^2}{x}\Big).$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

Hence, as before,

$$\int_0^1 G_0(\lambda)E_{12}(M;a,r,s,t)\,d\lambda \ll x^{5/6+\varepsilon}. \tag{3.10}$$

Secondly, for $M_1 \ll M \ll M_0$, write

$$E_1(M;a,r,s,t) = E_{13}(M;a,r,s,t) + E_{14}(M;a,r,s,t) := \sum_{\kappa_j \le Z}(\cdots) + \sum_{\kappa_j > Z}(\cdots).$$

Similarly to the first case,

$$E_{13}(M;a,r,s,t) \ll \frac{x^\varepsilon}{Z^{3/2}}\Big(Z^2 + \frac{M^{1/2}}{(ar)^{1/2}}\Big)^{1/2} M^{1/2} \ll \frac{x^\varepsilon y^{1/2}}{rt}\Big(\frac{AB}{x^{1/2}} + \frac{A^{3/2}B^2}{x}\Big)$$

and

$$E_{14}(M;a,r,s,t) \ll \frac{x^\varepsilon y^{1/2}}{rt}\Big(\frac{AB}{x^{1/2}} + \frac{A^{3/2}B^2}{x}\Big).$$

Then, for $l = 3, 4$,

$$\int_0^1 G_0(\lambda)E_{1l}(M;a,r,s,t)\,d\lambda \ll x^{5/6+\eta/2+\varepsilon}. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11),

$$\int_0^1 G_0(\lambda)E_{1}(M;a,r,s,t)\,d\lambda \ll x^{5/6+\eta/2+\varepsilon}. \tag{3.12}$$

We estimate $E_3(M;a,r,s,t)$ in the same way. If $M \ll M_1$, divide the integral into $[-1,1]$ and $(-\infty,-1)\cup(1,\infty)$ and, if $M \gg M_1$, divide the integral into $[-Z,Z]$ and $(-\infty,-Z)\cup(Z,\infty)$. Define $E_{3l}(M;a,r,s,t)$ $(1 \le l \le 4)$ as above. Similarly to $E_{1l}(M;a,r,s,t)$, we obtain the bound

$$\int_0^1 G_0(\lambda)E_{3}(M;a,r,s,t)\,d\lambda \ll x^{5/6+\eta/2+\varepsilon}. \tag{3.13}$$

The treatment of the exceptional eigenvalues, that is, $E_2(M;a,r,s,t)$, is as follows. If $M \gg M_1$, trivially, the bound for $\check{G}_\lambda$ in (3.7) implies that the contribution is negligible. But, for $M \ll M_1$, (3.6) is not sufficient for our purpose. Instead, Cauchy's inequality and Lemma 2.6 give the estimate

$$E_2(M;a,r,s,t) \ll \Big(\sum_{\kappa_j\,\mathrm{exc.}}\Big(\frac{1}{\Xi}\Big)^{4i\kappa_j}|\rho_j(1)|^2\Big)^{1/2}\Big(\sum_{\kappa_j\,\mathrm{exc.}}\frac{1}{\cosh(\pi\kappa_j)}\Big|\sum_{m\sim M}\lambda_g(m)e(m\lambda)\rho_j(m)\Big|^2\Big)^{1/2}$$
$$\ll x^\varepsilon\Big(\frac{1}{\sqrt{M}}\frac{AB}{rst}\frac{1}{ar}\Big)^{2\theta}\Big(1+\frac{M}{ar}\Big)^{1/2}M^{1/2} \ll x^\varepsilon\frac{1}{rt}\frac{x^\theta}{A^{2\theta}}\Big(\frac{AB}{x^{1/2}} + \frac{A^{3/2B^2}}{x}\Big),$$

which implies that

$$\int_0^1 G_0(\lambda)E_{2}(M;a,r,s,t)\,d\lambda \ll x^{5/6+\theta/3+\varepsilon}. \tag{3.14}$$

Combining (3.12), (3.13) and (3.14), we complete the proof.

For $\Phi_2(w,A,B,C)$, the only difference is that J_g is replaced by K_g. If $M \ll M_1$, this case is basically the same as for the J-Bessel functions, while, if $M \gg M_1$, it is simple because of (2.4) and can be trivially estimated.

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

4. Proof of Lemma 3.1

Recall that

$$G_\lambda(\kappa; a,r,s,t) = \frac{1}{G_0(\lambda)} \lim_{Y\to\infty} \int_{-Y}^{Y} \tilde{F}(\kappa, y; a,r,s,t) e(-\lambda y)\, dy,$$

$$\check{G}_\lambda(\kappa; a,r,s,t) = \frac{1}{G_0(\lambda)} \lim_{Y\to\infty} \int_{-Y}^{Y} h(y) e(-\lambda y) \frac{ar}{4\pi\sqrt{y}} \int_{-\infty}^{\infty} \check{H}_1(\kappa, y; \xi, a,r,s,t)\, d\xi\, dy,$$

where

$$H_1(\kappa, y; \xi, a,r,s,t) := \kappa J_g(\kappa\sqrt{\xi}) f\Big(\xi; ars, \frac{4\pi\sqrt{y}}{\kappa}\frac{t}{a}\Big).$$

By the properties of J_g and f, as a function of κ,

$$\operatorname{supp} H_1 \asymp \Xi, \qquad \frac{\partial^{(\nu)} H_1}{\partial \kappa^\nu} \ll x^\varepsilon \Xi \Big(\frac{x^\varepsilon}{\Xi}\Big)^\nu.$$

Lemma 2.4 implies that

$$\check{H}_1(i\kappa, y; \xi, a,r,s,t) \ll \Xi^{1-2\kappa}, \quad 0 \le \kappa < \tfrac{1}{4};$$

$$\check{H}_1(\kappa, y; \xi, a,r,s,t) \ll \frac{x^\varepsilon \Xi}{1+\kappa^{5/2}}, \quad \kappa \ge 0.$$

Trivially,

$$\check{G}_\lambda(i\kappa; a,r,s,t) \ll \frac{arx\sqrt{M}}{G_0(\lambda)} \Xi^{1-2\kappa} \ll \frac{M}{\min(M, \frac{1}{M\lambda^2})} \Xi^{-2\kappa}, \quad 0 \le \kappa < \frac{1}{4};$$

$$\check{G}_\lambda(\kappa; a,r,s,t) \ll \frac{arx\sqrt{M}}{G_0(\lambda)} \frac{x^\varepsilon \Xi}{1+\kappa^{5/2}} \ll \frac{M}{\min(M, \frac{1}{M\lambda^2})} \frac{x^\varepsilon \Xi}{1+\kappa^{5/2}}, \quad \kappa \ge 0.$$

By partial integration twice,

$$\check{G}_\lambda(i\kappa; a,r,s,t) \ll \frac{1}{M\lambda^2 G_0(\lambda)} \Xi^{-2\kappa}, \quad 0 \le \kappa < \frac{1}{4};$$

$$\check{G}_\lambda(\kappa; a,r,s,t) \ll \frac{1}{M\lambda^2 G_0(\lambda)} \frac{x^\varepsilon \Xi}{1+\kappa^{5/2}}, \quad \kappa \ge 0.$$

This gives the first statement of Lemma 3.1.

For $M \gg M_1$, by partial integration and (2.3),

$$\tilde{F}(\kappa, y; a,r,s,t) = h(y) \frac{ar}{2\pi i\sqrt{y}} \Re\Big(\int_0^\infty e\Big(\frac{w(\kappa\sqrt{\xi}, t)}{2\pi}\Big) \tilde{w}(\kappa, y; \xi, a,r,s,t)\, d\xi\Big)$$

with

$$\tilde{w}(\kappa, y; \xi, a,r,s,t) := \frac{\partial}{\partial \xi}\Big(\frac{\sqrt{t^2+\kappa^2\xi}\sqrt{\xi}}{t-\kappa\sqrt{\xi}} f(\kappa\sqrt{\xi}, t) f\Big(\xi; ars, \frac{4\pi t\sqrt{y}}{ac}\Big)\Big).$$

Note that the bounds

$$\operatorname{supp} \tilde{w} \asymp \Xi, \qquad \tilde{w}^{(\nu)} \ll \frac{\Xi^{1-\nu}}{Z^{3/2}}$$

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press

hold. Let

$$H_2(\kappa, y; \xi, a, r, s, t) := e\Big(\frac{w(\kappa\sqrt{\xi}, t)}{2\pi}\Big)\tilde{w}(\kappa, y; \xi, a, r, s, t).$$

Taking $\alpha = w(\kappa\sqrt{\xi}, t) \asymp \sqrt{\xi}$ and $X = \Xi$ in Lemma 2.4,

$$\check{H}_2(i\kappa, y; \xi, a, r, s, t) \ll x^{-\nu}, \quad 0 \le \kappa < \tfrac{1}{4};$$
$$\check{H}_2(\kappa, y; \xi, a, r, s, t) \ll x^{\varepsilon}\frac{\Xi}{Z^{5/2}}\Big(\frac{Z}{\kappa}\Big)^{\nu}, \quad \kappa > 0,$$

which imply the bounds (3.7) and (3.8).

Acknowledgement

The author expresses his sincere gratitude to the referee for a careful reading and several helpful suggestions.

References

[1] V. Blomer and J. Buttcane, 'On the subconvexity problem for L-functions on GL(3)', *Ann. Sci. ENS.* (to appear), arXiv:1504.02667.
[2] V. Blomer, G. Harcos and P. Michel, 'A Burgess-like subconvex bound for twisted L-functions', *Forum Math.* **19** (2007), 61–105.
[3] J.-M. Deshouillers and H. Iwaniec, 'Kloosterman sums and Fourier coefficients of cusp forms', *Invent. Math.* **70** (1982), 219–288.
[4] A. Erdélyi, W. Magnus, F. Obergettinger and F. Tricomi, *Higher Transcendental Functions II* (McGraw-Hill, New York, 1953).
[5] H. H. Kim and P. Sarnak, 'Appendix 2: refined estimates towards the Ramanujan and Selberg conjectures', *J. Amer. Math. Soc.* **16** (2003), 175–181.
[6] E. Kowalski, P. Michel and J. Vanderkam, 'Rankin–Selberg L-functions in the level aspect', *Duke Math. J.* **114** (2002), 123–191.
[7] T. Meurman, 'On the binary additive divisor problem', in: *Turku Sympos. Number Theory, Turku, 1999* (de Gruyter, Berlin, 2001), 223–246.
[8] R. Munshi, 'Shifted convolution of divisor function d_3 and Ramanujan τ-function', in: *The Legacy of Srinivasa Ramanujan*, Ramanujan Mathematical Society Lecture Notes Series, 20 (Ramanujan Mathematical Society, Mysore, 2013), 251–260.
[9] R. Munshi, 'Shifted convolution sums for GL(3) × GL(2)', *Duke Math. J.* **162** (2013), 2345–2362.
[10] N. J. E. Pitt, 'On shifted convolutions of $\zeta^3(s)$ with automorphic L-functions', *Duke Math. J.* **77** (1995), 383–406.
[11] H. C. Tang, 'Shifted convolution sum of d_3 and the Fourier coefficients of Hecke–Maass forms', *Bull. Aust. Math. Soc.* **92** (2015), 195–204.
[12] B. Topacogullari, 'The shifted convolution sum of divisor functions', *Q. J. Math.* **67** (2016), 331–363.
[13] P. Xi, 'A shifted convolution sum for GL(3) × GL(2)', *Forum Math.* **30** (2018), 1013–1027.

HENGCAI TANG, School of Mathematics and Statistics,
Institute of Modern Mathematics, Henan University, Kaifeng,
Henan 475004, PR China
e-mail: hctang@henu.edu.cn

https://doi.org/10.1017/S000497271900100X Published online by Cambridge University Press