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ASSOUAD DIMENSION OF RANDOM PROCESSES

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Abstract In this paper we study the Assouad dimension of graphs of certain Lévy processes and functions defined by stochastic integrals. We do this by introducing a convenient condition which guarantees a graph to have full Assouad dimension and then show that graphs of our studied processes satisfy this condition.

Keywords: Brownian motion; Wiener process; stochastic integral; Assouad dimension; graph

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1. Definitions and motivations

Studying the dimension of various random processes has been of interest for some time. In this paper we will consider the Assouad dimension of graphs of certain random processes, notably Lévy processes and functions defined by stochastic integrals.

Lévy processes $X(t)$ were first introduced by Paul Lévy in 1934 [**8**] and are defined as the stochastic processes satisfying the following conditions.

1. $X(0) = 0$ almost surely.
2. For all $t, h > 0$, $X(t + h) - X(t)$ is equal to $X(h)$ in distribution (stationary increments).
3. For all $0 < t_1 < t_2 < \cdots < t_k$, the random variables $X(t_i) - X(t_{i-1})$ are independent (independence of increments).
4. For all $t > 0$, $\lim_{h\to 0} X(t + h) - X(t) = 0$ in probability (continuity).

We can construct $X(t)$ such that it is almost surely right continuous with left limits (denoted càdlàg). Such processes are standard tools in many areas of modern mathematics and their applications. A common example of a Lévy process is the *Wiener process*

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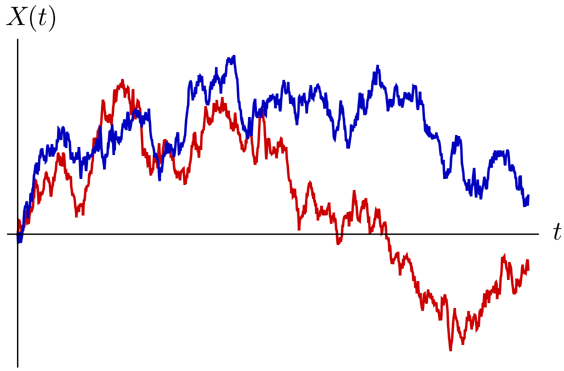

Figure 1. Two graphs of one-dimensional Brownian motion.

(or Brownian motion) where property 2 (stationary increments) is replaced by Gaussian increments, so $X(t+h) - X(t)$ is normally distributed with mean 0 and variance h, see Figure 1 for examples. One can similarly define d-dimensional Brownian motion by considering the vector-valued stochastic process $(W_1, \dots, W_d)$ where the W_i are independent Weiner processes.

The geometric properties, such as dimension, of Wiener processes have been a particularly well-studied area. This includes studying the *graphs*, level sets and *trails* of such processes, which can often be thought of as fractals as they often display some *statistical self-affinity*. For any left continuous function $X : \mathbb{R} \to \mathbb{R}$, we define the graph of the function by:

$$G_X^{I \subset \mathbb{R}} = \{(t, y) \mid y = X(t), t \in I\} \cup J,$$

where J is the union of vertical segments joining the discontinuities. J is well defined because X is right continuous. It is clear that if X is continuous then J is empty. Taylor [**13**] first calculated the Hausdorff dimension of d-dimensional Brownian motion $B_d : \mathbb{R} \to \mathbb{R}^d$, where he showed that almost surely

$$\dim_{\mathrm{H}} G_{B_1}^{[0,1]} = \tfrac{3}{2}$$

and for any $d \geq 2$

$$\dim_{\mathrm{H}} B_d([0,1]) = 2.$$

Another generalization of Brownian motion is *fractional Brownian motion*, first introduced by Mandelbrot and Van Ness [**9**]. Index-h fractional Brownian motion (fBm) on $\mathbb{R}$ with $0 < h < 1$ is defined as the stochastic integral

$$B_h(t) = c(h)^{-1} \int_{-\infty}^{\infty} ((t-x)_+^{h-1/2} - (-x)_+^{h-1/2}) W(\mathrm{d}x)$$

where W is the Wiener measure, $(x)_+ = \max\{0, x\}$ and $c(h) = \Gamma(h + 1/2)$, where Γ denotes the gamma function, see Figure 2 for examples. Equivalently, this is a Gaussian random process $B_h(t)$ where the following hold.

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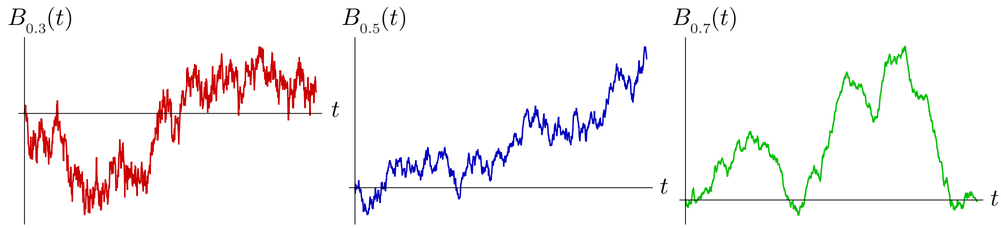

(A) Graph of fBm with h=0.3 (B) Graph of fBm with h=0.5 (C) Graph of fBm with h=0.7

Figure 2. Graphs of Index-h fBm for different values of h.

1. $B_h(0) = 0$ almost surely.
2. For all $t, u > 0$, $B_h(t+u) - B_h(t)$ has normal distribution with mean 0 and variance u^{2h} (Gaussian increments).
3. For all $t > 0$, $\lim_{u \to 0} B_h(t+u) - B_h(t) = 0$ in probability (continuity).

One can see that when $h = 1/2$, $B_{1/2}(t) = B(t)$ is simply Brownian motion. Much progress has been made on the properties of fBm, see for instance [**1**, **4**, **7**]. Notably, it was shown that, almost surely, the graph over the unit interval of index-h fBm has Hausdorff dimension $2 - h$.

Studying the Assouad dimension of various fractals and its properties is an increasingly popular area of research. In this paper, we are interested in calculating the Assouad dimension of the graph $G_X^{[0,1]}$ for β-scaling (or Lévy β-stable) processes and stochastic integrals X. These results will be compared with the previously obtained Hausdorff dimensions.

We say that $X(t)$ satisfies a β-scaling property if, for any $t, a > 0$:

$$a^{-(1/\beta)} X(at) =^d X(t),$$

where '$=^d$' denotes 'equal in distribution'. For example, the Wiener process has the 2-scaling property.

A non-empty compact bounded set F is said to be *s-homogeneous* if there exists a constant $C > 0$ such that for all $0 < R$, $r \in (0, R]$ and $x \in F$

$$N(B(x,R) \cap F, r) \leq C\left(\frac{R}{r}\right)^s$$

where $B(x,R)$ denotes the closed ball of centre x and radius R, and $N(E,r)$ is the number of squares of the $r \times r$ grid that intersect the compact set E.

The *Assouad dimension* of a non-empty compact bounded set F is then defined to be

$$\dim_A F = \inf \{s \geq 0 \colon F \text{ is } s\text{-homogeneous}\}.$$

This dimension provides information on the extremal local scaling of a set; in this setting, it will tell us about the maximal fluctuations of a random process. For a more detailed introduction to this dimension, see [**5**, **12**].

This paper will be split into three parts. First we will define a condition, Definition 2.1, which guarantees that a graph will have full Assouad dimension. Then in § 3 we show that graphs of β-scaling Lévy processes satisfy this condition and, combining these results, we prove that graphs of functions defined by certain stochastic integrals also have full Assouad dimension. Finally in § 4 we remark that our results extend to higher dimensions.

2. Assouad dimension of graphs

In this section we will state a convenient condition to check whether a graph of a function $f : [0,1] \to \mathbb{R}$ has full Assouad dimension.

We begin with a definition.

Definition 2.1. Let $R_1, R_2 > 0$ be positive numbers and let $n_1, n_2 > 0$ be integers. Given a point $a \in \mathbb{R}^2$, we define $W_{n_1\times n_2}^{R_1\times R_2}(a)$ as the following collection of sets:

$$x\bigg\{D_{i,j}+a \mid D_{i,j} = \left[\frac{i}{n_1}R_1, \frac{i+1}{n_1}R_1\right] \times \left[\frac{j}{n_2}R_2, \frac{j+1}{n_2}R_2\right], \\ i \in \{0,\dots,n_1-1\}, j \in \{0,\dots,n_2-1\}\bigg\}.$$

We see that $W_{n_1\times n_2}^{R_1\times R_2}(a)$ is the collection of rectangles with disjoint interiors which partitions the $R_1 \times R_2$ rectangle whose bottom left vertex is a.

Then let $N_{n_1\times n_2}^{R_1\times R_2}(a, G_f^{[0,1]}) = \#\{W_{n_1\times n_2}^{R_1\times R_2}(a) \cap G_f^{[0,1]}\}$; this is the number of rectangles which intersect the graph.

The following theorem is a direct consequence of the definition of Assouad dimension, and we omit the proof.

Theorem 2.2. *If there exists an $A > 0$ and sequences*

$$a_i \in \mathbb{R}^2, \qquad R_i \in (0,1), \quad n_i \in \mathbb{N}\ (\forall i \in \mathbb{N})$$

with $n_i \to \infty$ such that for all $i \in \mathbb{N}$

$$N_{n_i\times n_i}^{R_i\times R_i}(a_i, G_f^{[0,1]}) \geq A n_i^2,$$

then

$$\dim_{\mathrm{A}} G_f^{[0,1]} = 2.$$

While this might seem a restrictive condition to ask a general function to satisfy, it is quite natural in the setting of Wiener processes, owing to the almost sure unbounded variation and β-scaling property of the process. Considering squares instead of balls in the definition of Assouad dimension is similar to the definition of the Furstenberg star dimension, which is in fact equivalent to the Assouad dimension, see [**2**].

Remark 2.3. Note that one could replace the inequality with the following equality

$$N_{n_i \times n_i}^{R_i \times R_i}(a_i, G_f^{[0,1]}) = n_i^2.$$

This follows from [**6**, Theorem 2.4], where it is shown that a set has full Assouad dimension if and only if it has the unit ball as a weak tangent. This means that any cover of our set is also a cover of a ball and so all smaller squares are needed in the cover.

3. Applications to β-scaling Lévy processes

Let $X(t)$ be a Lévy process. We assume that $X(1)$ is non-vanishing almost everywhere on $\mathbb{R}$ as a random variable, that is, the distribution function of $X(1)$ is 0 only on a set of measure 0.

Then for any β-scaling Lévy process X we can compute the probability of the following event '$N_{n\times n}^{1\times 1}(0, G_X^{[0,1]}) = n^2$'. This is a positive number depending only on n, and we use $P(n)$ to denote this number. The event 'X hits a rectangle' is measurable when X is continuous; when it is discontinuous we join the graph with a vertical line and the process is càdlàg, so the event 'X hits a rectangle' is still measurable. Thus our event is measurable as the union of measurable events.

For a β-scaling random process X, we can decompose the graph into countably many disjoint parts:

$$G_X^{[0,1]} = \bigcup_{i=0}^{\infty} G_X^{I_i},$$

where I_i are closed intervals with disjoint interiors such that their union is the unit interval. In our case, one could think of this as partitioning the unit interval by intervals of length $1/2^i$. For example, take $a_1 = 0$ and for all $i \geq 1$ let $a_{i+1} = a_i + 1/2^i$ and $I_i = [a_i, a_i + 1/2^i]$.

Denote by $|I_i|$ the length of interval $I_i = [a_i, b_i]$. As we can take X as a left continuous function, $X(a_i) \in \mathbb{R}$ is defined for all i. For each i we can apply a linear map $T_i : G_X^{I_i} \to [0,1]^2$:

$$T_i(x, y) = \left(\frac{1}{|I_i|}(x - a_i), \frac{1}{|I_i|^{1/\beta}}(y - X(a_i))\right).$$

By definition, it is easy to see that $T_i(G_X^{I_i})$ are independent β-scaling Lévy processes with the same, original distribution. For convenience we identify $T_i(G_X^{I_i}) = G_{X_i}^{[0,1]}$, where X_i are independent, identically distributed β-scaling Lévy processes.

Let n_i be a sequence of integers such that $\lim_{i\to\infty} n_i = \infty$. We can compute the probability of the event $A_i =$ '$N_{n_i\times n_i}^{1\times 1}((0,0), G_{X_i}^{[0,1]}) = n_i^2$'. From the discussions above, we see that the probability is $P(n_i)$. In fact denote $t_k = k/n_i^2$ for all $k \in \{1, \ldots, n_i^2\}$ and $D(j) = [j/n_i, (j+1)/n_i]$ for all $j \in \{0, \ldots, n_i - 1\}$. Then we can see that:

$$P(n_i) \geq P\left(\forall k \in [1, n_i^2], k \in \mathbb{N}, X(t_k) \in D\left(\left\{\frac{k}{n_i}\right\} n_i\right)\right) > 0.$$

Here $\{\cdot\}$ denotes the fractional part function. The last inequality follows from our assumption that $X(1)$ is non-vanishing almost everywhere on $\mathbb{R}$. It is clear that this restraint

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could be relaxed to non-vanishing on some interval without much effort. The rest follows from the property of independent increments.

We can choose n_i to grow slowly enough that $\sum_i P(n_i) = \infty$. Note that the R_i can be chosen so that each square is disjoint and, as Lévy processes are Markov, the events A_i are all independent. Then by the Borel–Cantelli lemma we see that with probability 1, infinitely many events A_i occur. Now if A_i happens, then:

$$N^{1\times 1}_{n_i\times n_i}((0,0), G^{[0,1]}_{X_i}) = n_i^2;$$

applying the function T_i^{-1} to the graph, we see that (remember $I_i = [a_i, b_i]$):

$$N^{|I_i|\times|I_i|^{1/\beta}}_{n_i\times n_i}((a_i, X(a_i)), G^{I_i}_X) = n_i^2.$$

Since $\beta \geq 1$, we see that $|I_i| \leq |I_i|^{1/\beta}$. Also remember that X can be taken to be a right continuous function, and we also include the vertical segments of the jumps in $G^{[0,1]}_X$. Therefore it is clear that there exists an absolute constant $C > 0$:

$$N\left(B((a_i, X(a_i)), |I_i|) \cap G^{[0,1]}_X, \frac{|I_i|}{n_i}\right) \geq Cn_i^2.$$

As infinitely many A_i occur, using Theorem 2.2, we see that:

$$\dim_{\mathrm{A}} G^{[0,1]}_X = 2.$$

We conclude the above argument as the following theorem.

Theorem 3.1. *Let X be a β-scaling Lévy process with $\beta \geq 1$, such that $X(1)$ is a random variable whose distribution function is non-vanishing almost everywhere. Then almost surely:*

$$\dim_{\mathrm{A}} G^{[0,1]}_X = 2.$$

We know that the Wiener process $W(t)$ is a 2-scaling Lévy process. Therefore, we see the following.

Corollary 3.2.

$$\dim_{\mathrm{A}} G^{[0,1]}_W = 2.$$

Ville Suomala and Changhao Chen, in a personal communication, kindly remarked that this result follows from the graph of Brownian motion having full lower porosity dimension. This approach is inspired by [3], where it was shown that the graph of Brownian motion has full upper porosity dimension. However, this porosity dimension technique does not extend to our following, more general, result, which relies upon this one.

In fact, we can say more about the Assouad dimension of random processes which are functions defined as stochastic integrals, such as fractal Brownian motion.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is zero only finitely often, continuous on some interval and has continuous derivative on that same interval. Then we define $B_f(t)$ as the function defined by the stochastic integral:*

$$B_f(t) = \int_0^t f(x) W(\mathrm{d}x).$$

We have that almost surely:

$$\dim_{\mathrm{A}} G_{B_f}^{[0,1]} = 2.$$

Remark 3.4. In particular, graphs of fractional Brownian motions with indices $0 < h < 1$ have full Assouad dimension almost surely.

Proof. Given a function f which is zero only finitely often, continuous and has continuous derivative on some interval, say J, we can simply focus on the function restricted to J, normalizing to obtain a function which is C^1 and zero only finitely often on the unit interval. We may then assume that $f(t) > 1$ for $t \in [0, 1]$ by again restricting our function to an interval where the function is bounded away from zero and normalizing.

As the Assouad dimension provides local information, if the dimension of the graph of the function defined by the stochastic integral of this new function is full then the dimension of the original graph is also full. Thus we assume for the rest of this proof that f is a C^1 function which is greater than 1.

Ideally, we would wish to integrate by parts in the standard Riemann–Stieltjes sense

$$\int_0^t f(x) W(\mathrm{d}x) = f(t)W(t) - \int_0^t W(x) f'(x)\, \mathrm{d}x. \qquad (*)$$

The problem is that the integral on the left-hand side of the above equation is interpreted as the Itô integral, for which regular integration by parts does not hold. There is, however, a generalization of this formula for stochastic integrals which holds as f and W are both semimartingales; see [**11**, Chapter 2, § 6] for further details. To be precise we should write the following equation

$$\int_0^t f(x) W(\mathrm{d}x) = f(t)W(t) - \int_0^t W(x) f'(x)\, \mathrm{d}x - [f, W]_t.$$

Here $[f, W]_t$ is the quadratic covariation between f and W.

Let $0 = t_1 < t_2 < \cdots < t_n = t$ be a partition P of $[0, t]$ and let $|P|$ be the maximum of $t_{k+1} - t_k, k \in \{0, \ldots, n-1\}$:

$$[f, W]_t = \lim_{|P| \to 0} \sum_{i=1}^{n-1} (f(t_{i+1}) - f(t_i))(W(t_{i+1}) - W(t_i)).$$

The above convergence is taken in the sense of probability. By using Cauchy–Schwarz we see that:

$$[f, W]_t \leq [f, f]_t^{1/2} [W, W]_t^{1/2}.$$

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However, it is standard that $[f,f]_t = 0$ and $[W,W]_t = t$ as f is C^1. So we see that the integral by parts formula (*) is indeed correct for this situation. The integral

$$\int_0^t W(x)f'(x)\,\mathrm{d}x$$

is defined as a random process whose sample space is that of the Wiener process, where fixing a sample path of the Wiener process will determine the integral. We are interested in almost sure properties of this process and will do so by considering almost sure properties of the Wiener process.

The strategy for the rest of this proof is to choose carefully a typical path of the Wiener process. We denote the sample space of the Wiener process as Ω.

First, we see that for almost all $\omega \in \Omega$, $W(t,\omega)$ is càdlàg in t, and therefore there is a constant C_ω such that:

$$|W(t,\omega)| \leq C_\omega$$

for all $t \in [0,1]$.

The second almost sure property is described in the proof of Theorem 3.1: that there are infinitely many intervals $I_i = [a_i, b_i] \subset [0,1]$ and a sequence $n_i \to \infty$ such that for $k \in \{0, 1, \ldots, n_i - 1\}$

$$\left| W\left(a_i + (k+1)\frac{|I_i|}{n_i}, \omega\right) - W\left(a_i + k\frac{|I_i|}{n_i}, \omega\right)\right| \geq |I_i|^{1/2} \geq |I_i|.$$

In the following discussion we shall fix a typical ω such that $W(t,\omega)$ satisfies the above two almost sure properties, in particular, we think of $C_\omega > 0$ as a fixed constant.

Then we see that:

$$\begin{aligned}
&\left| B_f\left(a_i + (k+1)\frac{|I_i|}{n_i}, \omega\right) - B_f\left(a_i + k\frac{|I_i|}{n_i}, \omega\right)\right| \\
&\quad = \left| \int_{a_i + k(|I_i|/n_i)}^{a_i+(k+1)(|I_i|/n_i)} f(x)W(\mathrm{d}x)\right| \\
&\quad = \left| f(a_i + k(|I_i|/n_i))W(x,\omega)\Big|_{a_i+k(|I_i|/n_i)}^{a_i+(k+1)(|I_i|/n_i)} + \int_{a_i+k(|I_i|/n_i)}^{a_i+(k+1)(|I_i|/n_i)} W(x)f'(x)\,\mathrm{d}x\right|.
\end{aligned}$$

Since f is C^1, we see that there is a constant C_f (which does not depend on i) such that for all $x \in [a_i + k(|I_i|/n_i), a_i + (k+1)(|I_i|/n_i)]$:

$$|W(x)f'(x)| \leq C_f C_\omega.$$

Then we have the following inequalities:

$$\left| f\left(a_i + k\frac{|I_i|}{n_i}\right)W(x,\omega)\Big|_{a_i+k(|I_i|/n_i)}^{a_i+(k+1)(|I_i|/n_i)}\right| \geq \left| f\left(a_i + k\frac{|I_i|}{n_i}\right)\right| |I_i|^{1/2} \geq |I_i|^{1/2} \qquad (**)$$

and

$$\left| \int_{a_i+k(|I_i|/n_i)}^{a_i+(k+1)(|I_i|/n_i)} W(x)f'(x)\,\mathrm{d}x\right| \leq C_f C_\omega \frac{|I_i|}{n_i}.$$

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Since $|I_i| \to 0$, we see that for a constant C' which depends only on f and ω:

$$\left| B_f\left(a_i + (k+1)\frac{|I_i|}{n_i}, \omega\right) - B_f\left(a_i + k\frac{|I_i|}{n_i}, \omega\right) \right| \geq C'|I_i|. \qquad (***)$$

The above inequality holds for all $k \in \{0, \dots, n_i - 1\}$.

Moreover, W has a 'zigzag' property. For even integers k we have

$$W\left(a_i + (k+1)\frac{|I_i|}{n_i}, \omega\right) - W\left(a_i + k\frac{|I_i|}{n_i}, \omega\right) > 0,$$

and for odd integers k we have

$$W\left(a_i + (k+1)\frac{|I_i|}{n_i}, \omega\right) - W\left(a_i + k\frac{|I_i|}{n_i}, \omega\right) < 0.$$

Heuristically this says that the process increases on the first interval, decreases on the second and so forth, zigzagging from top to bottom. We can see that the expressions inside the absolute values in (**) and (***) also satisfy similar 'zigzag' properties. Therefore, there is a constant $A = A(\omega, f) > 0$ such that:

$$N\left(B((a_i, B_f(a_i)), |I_i|) \cap G_{B_f}^{[0,1]}, \frac{|I_i|}{n_i}\right) \geq An_i^2.$$

This concludes the proof because the above argument holds for a set of full probability $\omega \in \Omega$. □

4. A remark on higher-dimensional Brownian motion

Definition 2.1 has a natural generalization in $\mathbb{R}^d$.

Definition 4.1. Let $R_1, \dots, R_d > 0$ be positive numbers and $n_1, \dots, n_d$ be integers. Given a point $a \in \mathbb{R}^d$, we define $W_{n_1 \times \cdots \times n_d}^{R_1 \times \cdots \times R_d}(a)$ as the following collection of sets:

$$\left\{ D_{i_1,\dots,i_d} + a \mid D_{i_1,\dots,i_d} = \left[\frac{i_1}{n_1}R_1, \frac{i_1+1}{n_1}R_1\right] \times \cdots \times \left[\frac{i_d}{n_d}R_d, \frac{i_d+1}{n_d}R_d\right], \right.$$
$$\left. i_j \in \{0, \dots, n_j - 1\}, j \in \{1, \dots, d\} \right\}.$$

We see that $W_{n_1 \times n_2}^{R_1 \times R_2}(a)$ is a collection of rectangles with disjoint interiors.

Using the above definition and a similar argument as the one in § 3, Theorem 2.2 also extends to higher dimensions. We can show the following result, for which we omit the proof.

Theorem 4.2. *Let $B_d(t)$ be the d-dimensional Brownian motion, from $\mathbb{R}$ to $\mathbb{R}^d$. Then almost surely:*

$$\dim_A B_d([0, 1]) = d.$$

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We can compare this result with the well-known one

$$\dim_{\mathrm{H}} B_d([0,1]) = 2$$

for $d \geq 2$. The Hausdorff dimension being 2 here can be thought of as reflecting that higher-dimensional Brownian motion is transient, while the Assouad dimension shows that there are still areas of maximal fluctuation.

Brownian motion also provides examples of Salem sets that can have different Hausdorff and Assouad dimensions; we refer the reader to [7] for further discussion on the links between random processes and Salem sets.

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References

1. R. J. Adler, Hausdorff dimension and Gaussian fields, *Ann. Probab.* **5** (1977), 145–151.
2. C. Chen, M. Wu and W. Wu, *Accessible values of Assouad and the lower dimensions of subsets*, Preprint, (2016), available at: https://arxiv.org/abs/1602.02180.
3. J. T. Cox and P. S. Griffin, How porous is the graph of Brownian motion, *Trans. Amer. Math. Soc.* **325**(1) (1991), 19–140.
4. K. Falconer, From fractional Brownian motion to multifractional and multistable motion, in *Benoit Mandelbrot – a life in many dimensions*) (eds M. Frame and N. Cohen), Volume 4 (World Scientific Publishing, 2015).
5. J. M. Fraser, Assouad type dimensions and homogeneity of fractals, *Trans. Amer. Math. Soc.* **366** (2014), 6687–6733.
6. J. M. Fraser and H. Yu, *Arithmetic patches, weak tangents, and dimension*, Preprint, (2016), available at: http://arxiv.org/abs/1611.06960.
7. J.-P. Kahane, *Some random series of functions*, 2nd edn, Cambridge Studies in Advanced Mathematics (Cambridge University Press, 1985).
8. P. Lévy, Sur les intégrales dont les éléments sont des variables aléatoires indépendentes, *Annali della Scuola Normale Superiore di Pisa* **3** (1934), 337–366.
9. B. B. Mandelbrot and J. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.* **10**(4) (1968), 422–437.
10. P. Möters and Y. Peres, *Brownian motion*, Cambridge Series in Statistical and Probabilistic Mathematics (Cambridge University Press, 2010).
11. P. Protter, *Stochastic integration and differential equations*, 2nd edn, Stochastic Modelling and Applied Probability (Springer-Verlag, Berlin Heidelberg, 2005).
12. J. C. Robinson, *Dimensions, embeddings and attractors* (Cambridge University Press, Cambridge, 2011).
13. S. J. Taylor, The Hausdorff α-dimensional measure of Brownian paths in n-space, *Math. Proc. Camb. Philos. Soc.* **49**(2) (1953), 31–39.

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