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ON LARGE ORBITS OF FINITE SOLVABLE GROUPS ON CHARACTERS

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Abstract

We prove that if a solvable group A acts coprimely on a solvable group G, then A has a relatively 'large' orbit in its corresponding action on the set of ordinary complex irreducible characters of G. This improves an earlier result of Keller and Yang ['Orbits of finite solvable groups on characters', *Israel J. Math.* **199** (2014), 933–940].

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1. Introduction

Let a finite group A act (via automorphisms) on a finite group G. Such an action induces an action of A on the set $\mathrm{Irr}(G)$ in an obvious way (where $\mathrm{Irr}(G)$ denotes the set of complex irreducible characters of G). When G is elementary abelian, we are back to studying linear group actions. However, for nonabelian G, not much is known about this interesting action and we are only aware of a few major results on the action of A on $\mathrm{Irr}(G)$.

One such result is due to Moretó [3] who proved the existence of a 'large' orbit on $\mathrm{Irr}(G)$ when A is a p-group for some prime p and G is solvable such that $(|A|, |G|) = 1$. Keller and Yang [1] extended this result and established the existence of a 'large' orbit on $\mathrm{Irr}(G)$ whenever both A and G are solvable with $(|A|, |G|) = 1$. Yang also studied the special situation where A is nilpotent in [6]. The main result of [1] is the following theorem.

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THEOREM 1.1. *Let A and G be finite solvable groups such that A acts faithfully and coprimely on G. Let b be an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \mathrm{Irr}(G)$. Then $|A| \leq b^{49}$.*

As discussed in [1], it seems that the bound 49 is far from the best possible. For example, it was proved in [1] that if $2, 3 \notin \pi = \pi(A)$, then $|A| \leq b^4$. It was also remarked that the best bound is probably close to b^2. It would be interesting to construct nontrivial examples in GAP but this seems challenging.

The main purpose of this note is to provide a modest improvement on the bound. The main idea is to restructure the group decomposition and estimate the bound from a different perspective. We prove the following result.

THEOREM 1.2. *Let A and G be finite solvable groups such that A acts faithfully and coprimely on G. Let b be an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \mathrm{Irr}(G)$. Then $|A| \leq b^{27.41}$.*

2. Notation and preliminary results

We first fix some notation. In this paper, we use $\mathbf{F}(G)$ to denote the Fitting subgroup of G. Let $\mathbf{F}_0(G) \leq \mathbf{F}_1(G) \leq \mathbf{F}_2(G) \leq \cdots \leq \mathbf{F}_n(G) = G$ denote the ascending Fitting series, that is, $\mathbf{F}_0(G) = 1$, $\mathbf{F}_1(G) = \mathbf{F}(G)$ and $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G) = \mathbf{F}(G/\mathbf{F}_i(G))$. Here, $\mathbf{F}_i(G)$ is the ith ascending Fitting subgroup of G. We use $\mathrm{fl}(G)$ to denote the Fitting length of the group G. We use $\Phi(G)$ to denote the Frattini subgroup of G.

PROPOSITION 2.1 [2, Theorem 3.5(a)]. *Let G be a finite solvable group and let $V \neq 0$ be a finite, faithful, completely reducible G-module. Then $|G| \leq |V|^{\alpha}/\lambda$, where $\alpha = \ln((24)^{1/3} \cdot 48)/\ln 9$ and $\lambda = 24^{1/3}$.*

PROPOSITION 2.2. *Let G be a finite solvable group and let $V \neq 0$ be a finite, faithful, completely reducible G-module. Suppose $\mathrm{fl}(G) \leq 2$. Then $|G| \leq |V|^{\gamma}/\eta$, where $\gamma = \ln((6)^{1/2} \cdot 24)/\ln 9$ and $\eta = 6^{1/2}$.*

PROOF. One can mimic the proof of [2, Theorem 3.5(a)]. Note that one has to avoid S_4 and $\mathrm{GL}(2, 3)$ in the group structure since $\mathrm{fl}(S_4) = 3$ and $\mathrm{fl}(\mathrm{GL}(2, 3)) = 3$. □

PROPOSITION 2.3 [2, Theorem 3.3(a)]. *Let G be a finite nilpotent group and let $V \neq 0$ be a finite, faithful, completely reducible G-module. Then $|G| \leq |V|^{\beta}/2$, where $\beta = \ln 32/\ln 9$.*

PROPOSITION 2.4 [1, Theorem 3.1]. *Assume that a solvable π-group A acts faithfully on a solvable π'-group G. Let b be an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \mathrm{Irr}(G)$. Let $\Gamma = AG$ be the semidirect product. Let $K_{i+1} = \mathbf{F}_{i+1}(\Gamma)/\mathbf{F}_i(\Gamma)$ and let $K_{i+1,\pi}$ be the Hall π-subgroup of K_{i+1} for all $i \geq 1$. Let $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma)) = V_{i1} + V_{i2}$, where V_{i1} is the π part of $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$ and V_{i2} is the π' part of $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$ for all $i \geq 1$. Let $K \vartriangleleft \Gamma$ such that $\mathbf{F}_i(\Gamma) \vartriangleleft K$. Let $L_{i+1,\pi} = K_{i+1,\pi} \cap K$. Then $|\mathbf{C}_{L_{i+1,\pi}}(V_{i1})| \leq b^2$ and $|\mathbf{C}_{L_{i+1,\pi}}(V_{i1})| \leq b$ if $L_{i+1,\pi}$ is abelian. The order of the maximum abelian quotient of $\mathbf{C}_{L_{i+1,\pi}}(V_{i1})$ is less than or equal to b for all $i \geq 1$.*

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3. Main results

Now we are ready to prove Theorem 1.2, which we restate here.

THEOREM 3.1. *Let A be a solvable π-group that acts faithfully on a solvable π'-group G. Let b be an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \mathrm{Irr}(G)$. Then $|A| \leq b^{27.41}$.*

PROOF. Let $\Gamma = AG$ be the semidirect product of A and G. By Gaschutz's theorem, $\Gamma/\mathbf{F}(\Gamma)$ acts faithfully and completely reducibly on $\mathrm{Irr}(\mathbf{F}(\Gamma)/\Phi(\Gamma))$. It follows from [5, Theorem 3.3] that there exists $\lambda \in \mathrm{Irr}(\mathbf{F}(\Gamma)/\Phi(\Gamma))$ such that $T = \mathbf{C}_\Gamma(\lambda) \leq \mathbf{F}_8(\Gamma)$.

Let $K_2 = \mathbf{F}_2(\Gamma)/\mathbf{F}_1(\Gamma)$ and let $K_{2,\pi}$ be the Hall π-subgroup of K_2. Then $K_{2,\pi}$ acts faithfully and completely reducibly on $K_1/\Phi(\Gamma/\mathbf{F}_0(\Gamma))$. It is clear that we may write $K_1/\Phi(\Gamma/\mathbf{F}_0(\Gamma)) = V_{11} + V_{12}$, where V_{11} is the π part of $K_1/\Phi(\Gamma/\mathbf{F}_0(\Gamma))$ and V_{12} is the π' part of $K_1/\Phi(\Gamma/\mathbf{F}_0(\Gamma))$.

It is also clear that $K_1 = \mathbf{F}(\Gamma)$ is a π'-group and $V_{11} = 0$. Thus, $K_{2,\pi} = \mathbf{C}_{K_{2,\pi}}(V_{11})$ acts faithfully and completely reducibly on V_{12}. Proposition 2.4 shows that $|K_{2,\pi}| \leq b^2$ and the order of the maximum abelian quotient of $K_{2,\pi}$ is bounded above by b (and thus $|V_{22}| \leq b$).

Set $G_2 = \mathbf{F}_8(\Gamma)/\mathbf{F}(\Gamma)$ and $G_3 = \mathbf{C}_{G_2/\mathbf{F}(G_2)}(V_{21})$. Thus, $|G_2/\mathbf{F}(G_2)/\mathbf{C}_{G_2/\mathbf{F}(G_2)}(V_{21})| \leq b^\alpha$ by Proposition 2.1. We note that G_3 acts faithfully and completely reducibly on V_{22} and $\mathrm{fl}(G_3) \leq 6$.

Let $\mathbf{F}(G_3)/\Phi(G_3) = V_{31} + V_{32}$, where V_{31} is the π part of $\mathbf{F}(G_3)/\Phi(G_3)$ and V_{32} is the π' part of $\mathbf{F}(G_3)/\Phi(G_3)$. Proposition 2.4 shows that the order of the π part of $\mathbf{F}(G_3)$ is bounded by b^2 and the order of the abelian quotient of the π part of $\mathbf{F}(G_3)$ is bounded by b (and thus $|V_{32}| \leq b$).

Set $G_4 = \mathbf{C}_{G_3/\mathbf{F}(G_3)}(V_{31})$. Thus, $|G_3/\mathbf{F}(G_3)/\mathbf{C}_{G_3/\mathbf{F}(G_3)}(V_{31})| \leq b^\alpha$ by Proposition 2.1. We note that G_4 acts faithfully and completely reducibly on V_{32} and $\mathrm{fl}(G_4) \leq 5$.

Let $\mathbf{F}(G_4)/\Phi(G_4) = V_{41} + V_{42}$, where V_{41} is the π part of $\mathbf{F}(G_4)/\Phi(G_4)$ and V_{42} is the π' part of $\mathbf{F}(G_4)/\Phi(G_4)$. Proposition 2.4 shows that the order of the π part of $\mathbf{F}(G_4)$ is bounded by b^2 and the order of the abelian quotient of the π part of $\mathbf{F}(G_4)$ is bounded by b (and thus $|V_{42}| \leq b$).

Set $G_5 = \mathbf{C}_{G_4/\mathbf{F}(G_4)}(V_{41})$. Thus, $|G_4/\mathbf{F}(G_4)/\mathbf{C}_{G_4/\mathbf{F}(G_4)}(V_{41})| \leq b^\alpha$ by Proposition 2.1. We note that G_5 acts faithfully and completely reducibly on V_{42} and $\mathrm{fl}(G_5) \leq 4$.

Let $\mathbf{F}(G_5)/\Phi(G_5) = V_{51} + V_{52}$, where V_{51} is the π part of $\mathbf{F}(G_5)/\Phi(G_5)$ and V_{52} is the π' part of $\mathbf{F}(G_5)/\Phi(G_5)$. Proposition 2.4 shows that the order of the π part of $\mathbf{F}(G_5)$ is bounded by b^2 and the order of the abelian quotient of the π part of $\mathbf{F}(G_5)$ is bounded by b (and thus $|V_{52}| \leq b$).

Set $G_6 = \mathbf{C}_{G_5/\mathbf{F}(G_5)}(V_{51})$. Thus, $|G_5/\mathbf{F}(G_5)/\mathbf{C}_{G_5/\mathbf{F}(G_5)}(V_{51})| \leq b^\alpha$ by Proposition 2.1. We note that G_6 acts faithfully and completely reducibly on V_{52} and $\mathrm{fl}(G_6) \leq 3$.

Let $\mathbf{F}(G_6)/\Phi(G_6) = V_{61} + V_{62}$, where V_{61} is the π part of $\mathbf{F}(G_6)/\Phi(G_6)$ and V_{62} is the π' part of $\mathbf{F}(G_6)/\Phi(G_6)$. Proposition 2.4 shows that the order of the π part of $\mathbf{F}(G_6)$ is bounded by b^2 and the order of the abelian quotient of the π part of $\mathbf{F}(G_6)$ is bounded by b (and thus $|V_{62}| \leq b$).

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Set $G_7 = \mathbf{C}_{G_6/\mathbf{F}(G_6)}(V_{61})$. Thus, $|G_6/\mathbf{F}(G_6)/\mathbf{C}_{G_6/\mathbf{F}(G_6)}(V_{61})| \leq b^{\gamma}$ by Proposition 2.2. We note that G_7 acts faithfully and completely reducibly on V_{62} and $\mathrm{fl}(G_7) \leq 2$.

Let $\mathbf{F}(G_7)/\Phi(G_7) = V_{71} + V_{72}$, where V_{71} is the π part of $\mathbf{F}(G_7)/\Phi(G_7)$ and V_{72} is the π' part of $\mathbf{F}(G_7)/\Phi(G_7)$. Proposition 2.4 shows that the order of the π part of $\mathbf{F}(G_7)$ is bounded by b^2 and the order of the abelian quotient of the π part of $\mathbf{F}(G_7)$ is bounded by b (and thus $|V_{72}| \leq b$).

Set $G_8 = \mathbf{C}_{G_7/\mathbf{F}(G_7)}(V_{71})$. Thus, $|G_7/\mathbf{F}(G_7)/\mathbf{C}_{G_7/\mathbf{F}(G_7)}(V_{71})| \leq b^{\beta}$ by Proposition 2.3. We note that G_8 acts faithfully and completely reducibly on V_{72} and $\mathrm{fl}(G_8) \leq 1$. Proposition 2.4 shows that the order of the π part of $G_8 = \mathbf{F}(G_8)$ is bounded by b^2.

Next, we show that $|\Gamma : T|_{\pi} \leq b$.

Let χ be any irreducible character of G lying over λ. Then every irreducible character of Γ that lies over χ also lies over λ and hence has degree divisible by $|\Gamma : T|$. However, χ extends to its stabiliser in Γ and thus some irreducible character of Γ lying over χ has degree $\chi(1)|A : C_A(\chi)|$. Therefore, the π-part of $|\Gamma : T|$ divides $|A : \mathbf{C}_A(\chi)|$ which is at most b. This gives

$$|A| \leq b^{2\cdot 7} \cdot b^{\alpha\cdot 4} \cdot b^{\gamma} \cdot b^{\beta} \cdot b \leq b^{27.41},$$

and the result follows. □

When $(|A|, |G|) = 1$, the orbit sizes of A on $\mathrm{Irr}(G)$ are the same as the orbit sizes in the natural action of A on the conjugacy classes of G. The following result follows immediately from Theorem 1.2.

THEOREM 3.2. *Let A be a solvable π-group that acts faithfully on a solvable π'-group G. Let b be an integer such that $|A : \mathbf{C}_A(C)| \leq b$ for all $C \in \mathrm{cl}(G)$. Then $|A| \leq b^{27.41}$.*

We now give an application of our main result. Take a chief series

$$\Delta : 1 = G_0 < G_1 < \cdots < G_n = G$$

of a finite group G. Let $\mathrm{Ord}_{\mathcal{S}}(G)$ denote the product of the orders of all solvable chief factors G_i/G_{i-1} in Δ. Let $\mu(G)$ be the number of nonabelian chief factors in Δ. Clearly, the constants $\mathrm{Ord}_{\mathcal{S}}(G)$ and $\mu(G)$ are independent of the choice of chief series Δ of G. As an application of Theorem 3.1, we can strengthen the solvable case of [4, Theorem 4.7].

THEOREM 3.3. *Let a finite group A act faithfully on a finite group G with $(|A|, |G|) = 1$. Assume G is solvable. If b is an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \mathrm{Irr}(G)$, then $2^{\mu(G)} \cdot \mathrm{Ord}_{\mathcal{S}}(A) \leq b^{27.41}$.*

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