Adv. Appl. Prob. **50,** 726–742 (2018)
*doi:*10.1017/apr.2018.33
© *Applied Probability Trust* 2018

WINDINGS OF PLANAR PROCESSES, EXPONENTIAL FUNCTIONALS AND ASIAN OPTIONS

WISSEM JEDIDI,* *King Saud University and Université de Tunis El Manar*
STAVROS VAKEROUDIS,** *University of the Aegean*

Abstract

Motivated by a common mathematical finance topic, we discuss the reciprocal of the exit time from a cone of planar Brownian motion which also corresponds to the exponential functional of Brownian motion in the framework of planar Brownian motion. We prove a conjecture of Vakeroudis and Yor (2012) concerning infinite divisibility properties of this random variable and present a novel simple proof of the result of DeBlassie (1987), (1988) concerning the asymptotic behavior of the distribution of the Bessel clock appearing in the skew-product representation of planar Brownian motion, as $t \to \infty$. We use the results of the windings approach in order to obtain results for quantities associated to the pricing of Asian options.

Keywords: Planar Brownian motion; Lévy process; stable process; winding; skew-product representation; Bessel clock; Bougerol's identity; infinite divisibility; Bernstein function; Lévy measure; Asian option

2010 Mathematics Subject Classification: Primary 60G51; 60G52; 60J65
Secondary 60F05; 60G44; 91G80

1. Introduction

Windings of two-dimensional processes, and especially of planar Brownian motion, have several applications in financial mathematics, when the exponential functionals of Brownian motion are of special interest. A fundamental example is the pricing of Asian options (see, e.g. [13], [19], [20], and [36]–[38]), where the payout of an Asian call option is defined as

$$\mathbb{E}\bigg[\bigg(\frac{1}{t}\int_0^t \mathrm{d}s \exp(\beta_s + \nu s) - K\bigg)^+\bigg],$$

where $(\beta_u,\ u \geq 0)$ is a real Brownian motion, $\nu \in \mathbb{R}$, and the nonnegative number K is the strike price. It is easy to show (see, e.g. [38] for further details) that the computation of this expectation simplifies to the computation of

$$\mathbb{E}\bigg[\bigg(\int_0^t \mathrm{d}s \exp(\beta_s + \nu s) - K\bigg)^+\bigg],$$

which follows from

$$\mathbb{E}\bigg[\int_0^t \mathrm{d}s \exp(\beta_s + \nu s)\bigg].$$

Received 22 June 2017; revision received 3 July 2018.
* Postal address: Department of Statistics & OR, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia. Email address: wissem_jedidi@yahoo.fr
** Postal address: Department of Mathematics, University of the Aegean, Vourliotis Building, Office Y5, 83200 Karlovasi, Samos, Greece. Email address: stavros.vakeroudis@gmail.com

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

In particular, Yor [36] contains a more detailed discussion for the distribution of the exponential functional

$$A_t^{(\nu)} := \int_0^t \mathrm{d}s \exp(\beta_s + \nu s)$$

taken up to a random time T_λ which follows the exponential distribution with parameter $\lambda > 0$ and is independent from β. More precisely, Yor [35] obtained

$$2A_{T_\lambda}^{(\nu)} \overset{\mathrm{D}}{=} \frac{Q_{1,a}}{2G_b} \overset{\mathrm{D}}{=} \frac{1-\mathcal{U}^{1/a}}{2G_b},$$

where $Q_{1,a} \sim \text{beta}(1, a)$, $G_b \sim \text{gamma}(b)$, $\mathcal{U} \sim U[0, 1]$, $a = (\frac{1}{2}\nu) + \frac{1}{2}\sqrt{2\lambda + \nu^2}$, $b = a - \nu$, and the random variables are assumed to be independent. The class of generalized gamma convolution distributions (GGC) is an important subclass of infinitely divisible distributions. Bondesson [6] devoted much attention to the study of this class. At this stage, we observe that $G_b \sim$ GGC and that $1/Q_{1,a} \sim$ GGC (see [8]) for the last result. From the stability property by product (see [7]), we easily conclude that $1/A_{T_\lambda}^{(\nu)} \sim$ GGC. This result gives a flavor of what we will obtain in Subsection 2.2.

We mainly deal with such exponential functionals and we investigate their distribution properties. The key observation of our approach is the fact that exponential functionals are strongly related to the windings of associated processes (see, e.g. Proposition 2.1) and this offers a possible direction to describe them. Indeed, instead of using an independent random time such as T_λ above, one may study the exponential functional up to the first hitting time of a specific level by another independent real Brownian motion, and this is related to the windings of planar Brownian motion as we demonstrate. The results of the exponential functional in the framework of windings may be used to study the asymptotic behavior of the exponential functional of interest. The goal of this paper is first to explore the distribution of exponential functionals in terms of planar processes and then to use this in order to discuss the exponential that we meet in the pricing of Asian options. Note that windings of different process types (e.g. jump processes) are related to different kinds of exponential functionals (e.g. exponential functionals of Lévy processes), hence, to different types of Asian options.

We first study exponential functionals in terms of planar Brownian motion, i.e. taken up to an independent random time different from the case mentioned above, i.e. (we suppose here that $\nu = 0$ but at the end of the paper we will also discuss the case where $\nu \neq 0$)

$$\int_0^{T_c^\gamma} \exp(2\beta_s)\,\mathrm{d}s,$$

where $(\gamma_u,\ u \geq 0)$ is another real Brownian motion independent from β and the exit time T^γ is given by (1.2). For more precise connection with the windings, see Proposition 2.1.

We also discuss the exponential functional associated to jump processes, i.e.

$$\int_0^t \exp(\alpha\xi_s)\,\mathrm{d}s,$$

where $(\xi_u,\ u \geq 0)$ is a (nonsymmetric) Lévy process and $\alpha \in (0, 2)$ is a constant. As we will see, this exponential functional is again related to the study of an associated planar stable process with index of stability α.

We consider the following processes:

- $Z = (Z_t,\ t \geq 0)$ a planar Brownian motion (BM);
- $U = (U_t,\ t \geq 0)$ a planar stable process of index $\alpha \in (0, 2)$,

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

both starting from a point different from 0 and, without loss of generality, we may consider that they are both issued from 1. For $W = Z$ or U and $\alpha \in (0, 2]$, we formally define the clock, its inverse, and the winding process, respectively, for all $t \geq 0$, by

$$H_t^W = \int_0^t \frac{\mathrm{d}s}{|W_s|^\alpha}, \quad A^W(u) = \inf\{t \geq 0, H^W(t) > u\} \quad \text{and} \quad \theta_t^W = \mathrm{Im}\biggl(\int_0^t \frac{\mathrm{d}W_s}{W_s}\biggr). \tag{1.1}$$

Note that the planar BM case, i.e. $W = Z$, corresponds to $\alpha = 2$. We also define the exit times from a cone of single and of double border of a process V (V will be W or a functional of W in the sequel) by

$$T_c^V = \inf\{t \geq 0\colon V_t \geq c\} \quad \text{and} \quad T_c^{|V|} = \inf\{t \geq 0\colon |V_t| \geq c\}, \qquad c > 0. \tag{1.2}$$

For the planar BM case ($W = Z$), it is well known (see, e.g. [16]) that since $Z_0 \neq 0$, the process Z does not visit almost surely the point 0 but keeps winding around it infinitely often. Hence, its continuous winding process θ_t^Z is well defined. We also recall the skew-product representation of planar BM (see, e.g. [24]), i.e.

$$\log|Z_t| + \mathrm{i}\theta_t^Z = \int_0^t \frac{\mathrm{d}Z_s}{Z_s} = (\beta_u + \mathrm{i}\gamma_u)\,\Big|_{u=H_t^Z}, \tag{1.3}$$

where $(\beta_u + \mathrm{i}\gamma_u,\ u \geq 0)$ denotes another planar BM starting from $\log 1 + \mathrm{i}0 = 0$ and H^Z is given by (1.1) with $\alpha = 2$ (for the Bessel clock H^Z, see also [34]). It is straightforward that the two σ-fields $\sigma\{|Z_t|,\ t \geq 0\}$ and $\sigma\{\beta_u,\ u \geq 0\}$ are identical, whereas $(\gamma_u,\ u \geq 0)$ is independent from $(|Z_t|,\ t \geq 0)$. Note that the inverse of H^Z is represented by the functional

$$A^Z(t) = \inf\{u \geq 0, H^Z(u) > t\} = \int_0^t \mathrm{d}s\, \exp(2\beta_s). \tag{1.4}$$

We refer the interested reader to, e.g. [22] for more details about planar BM.

Unlike planar BM, it is not possible to define the winding number directly for the isotropic planar stable process U; see [3], [12], and [17]. However, we can consider a path on a finite-time interval $[0, t]$ and 'fill in' the gaps with line segments in order to obtain the curve of a continuous function $f\colon [0, 1] \to \mathbb{C}$ such that $f(0) = 1$. The origin 0 is polar and U has no jumps across 0 almost surely, thus, we have $f(u) \neq 0$ for every $u \in [0, 1]$ and the process of the winding number of U around 0, $\theta^U = (\theta_t^U,\ t \geq 0)$, is well defined, has càdlàg paths of absolute length greater than π, and is given by

$$\exp(\mathrm{i}\theta_t^U) = \frac{U_t}{|U_t|}, \qquad t \geq 0.$$

For $W = Z$ or U, we study the exit times from a cone of single and of double border, i.e. the stopping times T^{θ^W} and $T^{|\theta^W|}$ given by (1.2), and also the asymptotic behavior of the associated winding process.

The rest of the paper is organized as follows. We start by discussing windings and the associated version of Spitzer's asymptotic theorem (that corresponds to the large-time asymptotics) first for planar BM, and then for isotropic planar stable processes (note that for the latter it is not exactly an analogue of Spitzer's asymptotic theorem but mostly a large-time asymptotics result). In Section 2 we characterize the distribution of the exit times from a single and from a double border cone which corresponds also to the exponential functional of BM in the framework of

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

planar BM. Then we turn our interest to infinite divisibility properties of this quantity and we prove a conjecture of Vakeroudis and Yor [30, Remark 3.2]. In Subsection 2.3 we present a new simple proof of DeBlassie's result (see [10] and [11]) stating that if $R(s)$ denotes a Bessel process then, for every $u > 0$ and for every $\lambda > 0$,

$$\mathbb{P}\biggl(\int_0^t R_s^{-2}\,\mathrm{d}s \le u\biggr) = O(t^{-\lambda}) \quad \text{as } t \to \infty.$$

Recall that

$$H_t^Z := \int_0^t \frac{1}{|Z_s|^2}\,\mathrm{d}s = \int_0^t R_s^{-2}\,\mathrm{d}s,$$

hence, the last result corresponds to the asymptotic behavior of the Bessel clock associated to planar BM H^Z as $t \to \infty$. The initial proof of DeBlassie used results of Burkholder together with a theorem taken from Port and Stone [23]. Here, we propose a novel elementary self-contained proof.

In Section 3 we focus on the windings of isotropic planar stable processes where a large-time asymptotics result due to Bertoin and Werner [3] is presented for the sake of completeness for use in the following section. Finally, in Section 4 we deal with application of the previous results to the pricing of Asian options. More precisely, we discuss separately the case of exponential functionals of BM and the one of Lévy processes.

To recapitulate, the main results of the paper are the following:

- we prove a conjecture of Vakeroudis and Yor [30] concerning infinite divisibility properties of the inverse of the exponential functional in terms of planar BM;
- we propose a novel simple self-contained proof of DeBlassie's result [10], [11] concerning the distribution of the Bessel clock appearing in the skew-product representation of planar BM for $t \to \infty$;
- we use results concerning exponential functionals in terms of windings in order to study exponential functionals needed for the pricing of Asian options by invoking Williams' 'pinching method'.

Our approach has intrinsic theoretical interest since it provides further characterization of the exponential functional associated to different stochastic processes, including analytic properties (e.g. infinite divisibility properties). On the other hand, it provides a new direction and perspectives in order to proceed to the pricing of different types of Asian options, such as the ones associated to jump (Lévy) processes.

2. Planar BM

2.1. Windings and exponential functionals

We first recall our main tool, which is Bougerol's celebrated identity in law; see [9]. It states that if $(\beta_u,\ u \ge 0)$ and $(\hat{\beta}_u,\ u \ge 0)$ are two independent linear BMs both started from 0, then we have the identity

$$\sinh(\beta_t) \overset{\mathrm{D}}{=} \hat{\beta}_{A_t^Z(\beta)=\int_0^t \mathrm{d}s\, \exp(2\beta_s)} \quad \text{for every fixed } t \ge 0. \tag{2.1}$$

For the proof and other developments of this identity, see [28] and the references therein. We study Bougerol's identity in law in terms of planar BM, which is strongly related to

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

exponential functionals of BM as we will demonstrate. To that end, we recall that the exit times T_c^{γ} and $T_c^{|\gamma|}$ for the BM γ associated to θ^Z are given by (1.2). We are now ready to state our first result; see also [27] and [29].

Proposition 2.1. *It holds that*

$$T_c^{\theta^Z} = A^Z_{T_c^{\gamma}} \quad \text{and} \quad T_c^{|\theta^Z|} = A^Z_{T_c^{|\gamma|}}.$$

Proof. The proof follows by the skew-product representation ($\theta_t^Z = \gamma_{H_t^Z}$) and using the fact that A^Z is the inverse of H^Z (see also (1.4)), i.e.

$$T_c^{\theta^Z} = \inf\{t : \theta_t^Z = c\} = \inf\{t : \gamma_{H_t^Z} = c\} = \inf\{A_s^Z : \gamma_s = c\} = A^Z_{T_c^{\gamma}}$$

when $s = H_t^Z$. The second relation follows similarly. □

From now on, all the results may be stated either for $A^Z_{T_c^{\gamma}}$ (respectively, $A^Z_{T_c^{|\gamma|}}$) or for $T_c^{\theta^Z}$ (respectively, $T_c^{|\theta^Z|}$). For applications in the mathematical finance framework, we will mostly use the former notation.

We recall Spitzer's celebrated asymptotic theorem for planar BM; see [26]. For other proofs, see, e.g. [31] and the references therein.

Theorem 2.1. (Spitzer's asymptotic theorem [26].) *The following convergence in law holds:*

$$\frac{2}{\log t}\,\theta_t^Z \xrightarrow{\text{D}} C_1, \qquad t \to \infty,$$

where C_1 denotes a standard Cauchy distributed random variable.

We introduce the function

$$\varphi(x) = \arg\sinh^2(\sqrt{x}) = \log^2(\sqrt{x} + \sqrt{1+x}) \tag{2.2}$$

which plays a key role in the rest of the paper. The next proposition is essentially from [27] and [29] and we state its proof for the sake of completeness.

Proposition 2.2. *The distributions of $A^Z_{T_c^{\gamma}}$ and $A^Z_{T_c^{|\gamma|}}$ are characterized by the following Gauss–Laplace transforms: for all $x \geq 0$ and $m = \pi/2c$, we have*

$$c\mathbb{E}\bigg[\sqrt{\frac{\pi}{2A^Z_{T_c^{\gamma}}}}\exp\bigg(-\frac{x}{2A^Z_{T_c^{\gamma}}}\bigg)\bigg] = \frac{1}{\sqrt{1+x}}\,\frac{c^2}{c^2+\varphi(x)}, \tag{2.3}$$

$$c\mathbb{E}\bigg[\sqrt{\frac{2}{\pi A^Z_{T_c^{|\gamma|}}}}\exp\bigg(-\frac{x}{2A^Z_{T_c^{|\gamma|}}}\bigg)\bigg] = \frac{1}{\sqrt{1+x}}\,f_m(x), \tag{2.4}$$

where φ is given by (2.2) and

$$f_m(x) = \frac{2}{(\sqrt{1+x}+\sqrt{x})^m + (\sqrt{1+x}-\sqrt{x})^m} = \frac{1}{\cosh(\sqrt{m^2\varphi(x)})}. \tag{2.5}$$

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

Remark 2.1. These Gauss–Laplace transforms fully characterize the distributions of $A^Z_{T_c^\gamma}$ and $A^Z_{T_c^{|\gamma|}}$. By some analytic computations, we can explore further the distributional properties of these random variables. From, e.g. (2.4), we can obtain the density function of $A^Z_{T_c^{|\gamma|}}$. For further details, see, e.g. [27] and [29].

Proof of Proposition 2.2. Let N denote a random variable following the distribution $\mathcal{N}(0,1)$. Bougerol's identity (2.1) applied for $t = T_c^\gamma$ yields the identities

$$\sinh(\beta_{T_c^\gamma}) \stackrel{\mathrm{D}}{=} \hat{\beta}_{A^Z_{T_c^\gamma}} \stackrel{\mathrm{D}}{=} \sqrt{A^Z_{T_c^\gamma}}\, N,$$

which, in turn, lead to, for every fixed $c > 0$,

$$\sinh(C_c) \stackrel{\mathrm{D}}{=} \hat{\beta}_{A^Z_{T_c^\gamma}}, \tag{2.6}$$

where $(C_c,\ c \geq 0)$ is a standard Cauchy process. We denote by h_c the probability density function of C_c, i.e.

$$h_c(y) = \frac{c}{\pi(c^2 + y^2)}, \qquad y \in \mathbb{R}.$$

Recalling, from (2.2), that $\arg\sinh(y) = \sqrt{\varphi(y^2)} = \log(y + \sqrt{1 + y^2})$ and identifying the densities of the two variables involved in (2.6), we obtain

$$\frac{1}{\sqrt{1 + y^2}} h_c(\arg\sinh y) = \frac{1}{\sqrt{1 + x^2}} h_c(\sqrt{\varphi(y^2)})$$

for the left-hand side and

$$\mathbb{E}\bigg[\frac{1}{\sqrt{2\pi A^Z_{T_c^\gamma}}} \exp\bigg(-\frac{y^2}{2A^Z_{T_c^\gamma}}\bigg)\bigg]$$

for the right-hand side. Performing the change of variable $x = y^2$, we obtain (2.3). Equation (2.4) follows by Bougerol's identity in law applied for $T_c^{|\gamma|}$ and by the same arguments as previously stated, since the density of $\beta_{T_c^{|\gamma|}}$ is given by (see, e.g. [5])

$$\frac{1}{2c}\frac{1}{\cosh(my)} = \frac{1}{c}\frac{1}{\mathrm{e}^{my} + \mathrm{e}^{-my}}. \qquad \square$$

2.2. Infinite divisibility properties

We will see that (2.3) and (2.4) yield infinite divisibility properties for the inverse of $A^Z_{T_c^\gamma}$ and $A^Z_{T_c^{|\gamma|}}$. For this purpose, we need some preparation.

Let BF denote the class of Bernstein functions, CBF the subclass of complete Bernstein functions, and TBF the sub-subclass of Thorin Bernstein functions; see [25] for more details. The class of infinitely divisible (ID) distributions (respectively, class of Bondesson distributions (BO) and class of generalized gamma convolution (GGC) distributions) corresponds to the distribution of a positive random variable X whose Laplace transform is such that

$$\mathbb{E}[\mathrm{e}^{-xX}] = \mathrm{e}^{-\phi(x)}, \qquad x \geq 0, \qquad \phi \in \mathrm{BF} \quad \text{(respectively, CBF, TBF).}$$

Thus, ϕ is represented by

$$\phi(x) = \mathrm{d}x + \int_{(0,\infty)} (1 - \mathrm{e}^{-xu})\, \nu(\mathrm{d}u),$$

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

where $d \geq 0$, the associated Lévy measure ν satisfies $\int_{(0,\infty)} \min(1, u)\nu(\mathrm{d}u) < \infty$, and the subclass CBF (respectively, TBF) corresponds to the case where ν is absolutely continuous with density function l such that

$$u \mapsto l(u) \text{ is completely monotone} \quad (\text{respectively, } u \mapsto ul(u)). \tag{2.7}$$

The inclusions TBF $\subset$ CBF $\subset$ BF $\subset$ justify the fact that GGC $\subset$ BO $\subset$ ID. Note that the classes BF and CBF are both convex cones stable by composition, whereas TBF is only a convex cone. Nevertheless, the subclass TBF enjoys the following property (see [25, Theorem 8.4, p. 112]): for a function $\phi \in$ TBF, we have

$$\psi \circ \phi \in \mathrm{TBF} \quad \text{for every } \psi \in \mathrm{TBF} \quad \Longleftrightarrow \quad \frac{\phi'}{\phi} \quad \text{is a Stieltjes function;} \tag{2.8}$$

see [25] for the definition of Stieltjes functions. Recall the function φ introduced in (2.2). Then

$$x \mapsto \varphi(x) = \arg\sinh^2(\sqrt{x}) = \log^2(\sqrt{x} + \sqrt{1+x}) \in \mathrm{TBF}. \tag{2.9}$$

Remark 2.2. Equation (2.9) was not observed in [25], but it could be obtained through the trivial equality $\sqrt{\varphi(x)} = \log(\sqrt{x} + \sqrt{x+1}) = \frac{1}{2}\arg\cosh(2x+1)$ and [25, entry 78, p. 338]. Further, some computations stemming from [25, entry 80, p. 338] mean that the logarithmic derivative of φ can be represented, for every $x > 0$, by

$$\frac{\varphi'(x)}{\varphi(x)} = \int_0^\infty \mathrm{e}^{-xu} f(x)\,\mathrm{d}x, \qquad f(x) = \mathrm{e}^{-x/2}\biggl(\cosh\Bigl(\frac{x}{2}\Bigr) + \int_0^\infty I_\nu\Bigl(\frac{x}{2}\Bigr)\,\mathrm{d}\nu\biggr), \tag{2.10}$$

where I_ν is the modified Bessel function of the first kind. Showing that the logarithmic derivative of φ is a Stieltjes function amounts to showing that the function f in (2.10) is completely monotone; however, this is not achieved here. Note that if the latter holds then we will have the following property which can be exploited in (2.17) and (2.18):

$$\log(1 + t\,\varphi) \in \mathrm{TBF} \quad \text{for all } t > 0. \tag{2.11}$$

For a positive random variable X, we denote by $X_{[u]}$ a version of the induced length-biased law of order u, i.e.

$$X_{[u]} \text{ is a realization of the distribution } \frac{x^u}{\mathbb{E}[X^u]}\mathbb{P}(X \in \mathrm{d}x), \tag{2.12}$$

whenever $\mathbb{E}[X^u] < \infty$. From [6, Theorem 6.2.4], we have

$$\text{if } X \sim \mathrm{GGC} \quad \text{and} \quad \mathbb{E}[X^u] < \infty \text{ for } u < 0 \quad \text{then } X_{[u]} \sim \mathrm{GGC}. \tag{2.13}$$

For the rest of the paper, we adopt the following notation: $G_{1/2}$ has the gamma distribution with shape parameter $\frac{1}{2}$ and scale parameter 1, and $\boldsymbol{e}_k$, $1 \leq k \leq n$, denotes n independent exponentially distributed random variables with parameter 1, independent of $G_{1/2}$ and the length-biased random variables

$$X_{1,c} := \biggl(\frac{1}{2\,A^Z_{T_c^\gamma}}\biggr)_{[1/2]} \quad \text{and} \quad X_{2,c} := \biggl(\frac{1}{2A^Z_{T_c^{|\gamma|}}}\biggr)_{[1/2]}. \tag{2.14}$$

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

Equations (2.3) and (2.4) become

$$\mathbb{E}_x[\mathrm{e}^{-X_{1,c}}] = \frac{1}{\sqrt{1+x}}\frac{c^2}{c^2+\varphi(x)}, \qquad \mathbb{E}_x[\mathrm{e}^{-X_{2,c}}] = \frac{1}{\sqrt{1+x}} f_m(x), \tag{2.15}$$

where φ and f_m are given by (2.9) and (2.5), respectively. The next proposition essentially comes from [30] and its proof requires the use of Chebyshev's polynomials. The second statement follows from (2.7) and (2.13).

Proposition 2.3. *For every integer m, we consider two situations.*

(i) *The function* $x \mapsto f_m(x)$ *is the Laplace transform of a positive random variable* $\boldsymbol{K} \sim$ GGC *which has the representation:*

- *for* $m = 2n+1$,

$$\boldsymbol{K} = G_{1/2} + \sum_{k=1}^{n} \frac{1}{a_k}\boldsymbol{e}_k, \qquad a_k = \sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n+1}\right), \quad k = 1, 2, \ldots, n;$$

- *for* $m = 2n$,

$$\boldsymbol{K} = \sum_{k=1}^{n} \frac{1}{b_k}\boldsymbol{e}_k, \qquad b_k = \sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n}\right), \quad k = 1, 2, \ldots, n.$$

The associated Lévy measures are:

- *for* $m = 2n+1$,

$$\nu(\mathrm{d}z) = \frac{\mathrm{d}z}{z}\sum_{k=1}^{n} \mathrm{e}^{-a_k z};$$

- *for* $m = 2n$,

$$\nu(\mathrm{d}z) = \frac{\mathrm{d}z}{z}\sum_{k=1}^{n} \mathrm{e}^{-b_k z}.$$

(ii) *The random variables* $X_{2,c}$ *given in (2.14) satisfy the identity in law* $X_{2,c} \stackrel{\mathrm{D}}{=} G'_{1/2} + \boldsymbol{K} \sim$ GGC, *where* $G'_{1/2}$ *is a copy of* $G_{1/2}$, *independent of* $\boldsymbol{K}$. *We also have* $1/(A^Z_{T_c^\gamma})$ *and* $1/(A^Z_{T_c^{|\gamma|}}) \sim$ GGC.

For other results and variants concerning properties of $A^Z_{T_c^\gamma}$ and $A^Z_{T_c^{|\gamma|}}$, we refer the interested reader to [30] and [31], and the references therein.

2.2.1. *The case when m is not an integer.* Vakeroudis and Yor [30, Remark 3.2] conjectured that (2.4) yields infinite divisibility properties for every $m > 0$ (not necessarily an integer). In the next proposition we prove this conjecture.

Proposition 2.4. *For every* $m > 0$ *and* $i = 1, 2$, *the random variable* $X_{i,c}$ *given in (2.14) belongs to the class BO and, hence, is infinitely divisible. Moreover, as* $c \to \infty$, $X_{i,c}$ *converges in distribution to a gamma distribution with scale parameter 1 and shape parameter* $\frac{1}{2}$.

Proof. Observe that (2.15) can be also restated, for $i = 1, 2$, as

$$\mathbb{E}[\exp(-xX_{i,c})] = \frac{1}{\sqrt{1+x}}\mathrm{e}^{-\psi_i(x)} = \exp\big(-\big(\tfrac{1}{2}\log(1+x) + \psi_i(x)\big)\big), \qquad x \geq 0, \tag{2.16}$$

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

where, with elementary computations, and using the infinite product form of the function

$$\cosh(x) = \prod_{k=1}^{\infty}\Big(1 + \frac{x^2}{d_k}\Big), \qquad d_k = \Big(\frac{\pi}{2}(2k-1)\Big)^2,$$

the functions ψ_i, $i = 1, 2$, become

$$\psi_1(x) = \log\Big(1 + \frac{\varphi(x)}{c^2}\Big), \tag{2.17}$$

$$\psi_2(x) = \log\cosh\big(\sqrt{m^2\,\varphi(x)}\big) = \sum_{k=1}^{\infty}\log\Big(1 + \frac{m^2\,\varphi(x)}{d_k}\Big), \tag{2.18}$$

where $\varphi \in \mathrm{TBF}$ is given by (2.9). At this stage, we trivially extract the second assertion in the proposition by letting $c \to \infty$ in (2.17) and (2.18). It remains to show that ψ_i belongs to the class CBF. Using the fact that $x \mapsto \log(1+x) \in \mathrm{TBF}$, (2.9), (2.16)–(2.18), and the stability by composition property in CBF, we conclude that

$$x \mapsto \tfrac{1}{2}\log(1+x) + \psi_i(x) \in \mathrm{CBF},$$

and, hence, $X_{i,c} \sim \mathrm{BO}$. □

2.2.2. *Open problem in the case when m is not an integer.* Recalling the second assertion of Proposition 2.3, which holds for any integer m, we surmise that $X_{i,c} \sim \mathrm{GGC}$ for $i = 1, 2$ and for every positive number m. This result is not proved here because we have not obtained a closed form of the Lévy measures of $X_{i,c}$. To this end, we need to check whether the functions ψ_i, $i = 1, 2$, given by (2.17) and (2.18) belong to the class TBF. Then it is tempting to check whether the function φ given by (2.9) satisfies (2.8) or (2.11) and this does not seem to be a straightforward problem to deal with; see the comments in Remark 2.2. To summarize, if the assertion holds then (2.13) yields

$$\frac{1}{A^Z_{T^\gamma_c}} = 2(X_{1,c})_{[-1/2]} \sim \mathrm{GGC} \quad \text{and} \quad \frac{1}{A^Z_{T^{|\gamma|}_c}} = 2\,(X_{2,c})_{[-1/2]} \sim \mathrm{GGC}.$$

If, furthermore, the distribution of $X_{i,c}$ belongs to the subclass HCM of GGC (i.e. if the density function of $X_{i,c}$ is hyperbolically completely monotone, see [6, p. 55] for the definition) then (see the comments in [6, p. 69]) the random variables $A^Z_{T^\gamma_c}$ and $A^Z_{T^{|\gamma|}_c}$ will also have a HCM density and, hence, they will be infinitely divisible.

2.3. DeBlassie's result: a new proof

In this section we present a new simple proof of DeBlassie's result in [10] and [11] concerning the Bessel clock $H^Z_t := \int_0^t \mathrm{d}s/|Z_s|^2$.

Proposition 2.5. *For every $u > 0$ and for every $\lambda > 0$, we have*

$$\mathbb{P}\Big(\int_0^t \frac{\mathrm{d}s}{|Z_s|^2} \le u\Big) = O(t^{-\lambda}) \quad \text{as } t \to \infty.$$

In order to prove Proposition 2.5, we make use of Williams' pinching method (see, e.g. [21] and [33]). Roughly speaking, when Williams studied windings of BM, instead of calculating

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

directly the asymptotics of the winding process θ, he studied the asymptotic behavior of this process taken at a random time depending on θ (for similar results but with the use of a random time independent of θ, see, e.g. [29]). Next, we simply remark that the difference between the initial winding process and the subordinated process is finite, and renormalizing appropriately, this difference converges to 0. Hence, the asymptotic study of the renormalized subordinated process yields similar results for the renormalized initial one.

Proof of Proposition 2.5. Recall that the first passage time T_t^β, defined by (1.2), inherits the scaling property of BM as follows:

$$\frac{T^\beta_{\sqrt{t}}}{t} \stackrel{\mathrm{D}}{=} T_1^\beta.$$

Williams' pinching method allows us to replace t by $T^\beta_{\sqrt{t}}$ when $t \to \infty$. Indeed, $H_t^Z - H^Z_{T^\beta_{\sqrt{t}}}$ converges to a finite variable as $t \to \infty$ (see also [27] and [29]).

First, we choose $A, B > 0$ such that

$$\mathbb{P}\bigg(A < \frac{T^\beta_{\sqrt{t}}}{t} < B\bigg) = \frac{1}{2}.$$

We also remark that T^β and H^Z are independent; thus,

$$\mathbb{P}(H_t^Z \le u) = 2\mathbb{P}\bigg(A < \frac{T^\beta_{\sqrt{t}}}{t} < B;\ H_t^Z \le u\bigg) = 2\mathbb{P}\big(A\,t < T^\beta_{\sqrt{t}} < B\,t;\ H_t^Z \le u\big). \tag{2.19}$$

Moreover, since H is increasing, we obtain

$$At < T^\beta_{\sqrt{t}} < B\,t \quad\Longleftrightarrow\quad \frac{T^\beta_{\sqrt{t}}}{B} < t < \frac{T^\beta_{\sqrt{t}}}{A};$$

hence, the first part of this inequality yields

$$H_t^Z \le u \quad\Longrightarrow\quad H^Z_{(T^\beta_{\sqrt{t}}/B)} \le u. \tag{2.20}$$

Now, (2.19) and (2.20) yield

$$\mathbb{P}(H_t^Z \le u) \le 2\mathbb{P}\big(H^Z_{T^\beta_{\sqrt{t}}/B} \le u\big). \tag{2.21}$$

With $a(y) := \sqrt{\varphi(y^2)} = \sinh^{-1}(y) = \arg\sinh(y) = \log(y + \sqrt{y^2+1})$, $y \in \mathbb{R}$, it follows that asymptotically

$$a(\sqrt{t}) \approx \tfrac{1}{2}\log t \quad \textit{as } t \to \infty.$$

If $\hat{T}^\beta$ is an independent copy of T^β, we have

$$\mathbb{P}(H_t^Z \le u) \simeq \mathbb{P}\big(H^Z_{\hat{T}^\beta_{\sqrt{t}}} \le u\big) \quad \text{as } t \to \infty. \tag{2.22}$$

Following [27] and [29], the skew-product representation of planar BM (1.3) and Bougerol's identity in law (2.1) yield that if $(\delta_t,\ t \ge 0)$ denotes another independent real BM, then, with obvious notation, we have (see also [29, Proposition 2.3])

$$H^Z_{T^\delta_b} \stackrel{\mathrm{D}}{=} T^\beta_{a(b)} \quad \text{for every } b \ge 0. \tag{2.23}$$

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

Indeed, using the symmetry principle (see [2] for the original note and [14] for a detailed discussion), Bougerol's identity in law (2.1) is equivalently stated as

$$\sinh(\bar{\beta}_u) \stackrel{\mathrm{D}}{=} \bar{\delta}_{A_u^Z(\beta)} \quad \text{for any fixed } u > 0.$$

Hence, identifying the densities of the two parts and recalling that H^Z is given by (1.1), we easily obtain (2.23).

Now, using (2.23) with $b = \sqrt{t}$ and adapting appropriately the notation, (2.22) becomes

$$\mathbb{P}(H_t^Z \le u) = \mathbb{P}\big(T^{\beta}_{a(\sqrt{t})} \le u\big)\mathbb{P}\big(T^{\beta}_{(1/2)\log t} \le u\big) \quad \text{as } t \to \infty.$$

Below, we denote '$\approx$' to mean 'is of order'. We have

$$\mathbb{P}(T_h^{\beta} \le u) \approx \frac{\sqrt{u}}{h} \exp\bigg(-\frac{h^2}{2u}\bigg) \approx \frac{2\sqrt{u}}{\log t} \exp\bigg(-\frac{(\log t)^2}{8u}\bigg)$$

when $h = \frac{1}{2}\log t$. Thus, with $h = \frac{1}{2}\log t$, which corresponds to the asymptotic behavior for $t \to \infty$, we obtain

$$\mathbb{P}(T_h^{\beta} \le u) \le \frac{2\sqrt{u}}{\log t} \exp\bigg(-\frac{(\log t)^2}{8u}\bigg). \tag{2.24}$$

Observe that for large values of d, we have $\exp(-d^2) \le \exp(-\lambda d)$ for all $\lambda > 0$, hence, with $d = \log t$, we have

$$\exp(-(\log t)^2) \le \exp(-\lambda(\log t)) = \frac{1}{t^{\lambda}} \quad \text{for all } \lambda > 0.$$

Using (2.21), (2.24), and the last elementary remark, we obtain

$$\mathbb{P}(H_t^Z \le u) \le \frac{C_{u,B,\lambda}}{t^{\lambda}} \quad \text{for all } \lambda > 0,$$

where $C_{u,B,\lambda}$ denotes a positive constant depending on u, B, and λ. □

3. Planar stable processes

3.1. The winding process

In this section we focus on isotropic planar stable processes. Bertoin and Werner [15] obtained the following results for $\alpha \in (0, 2)$ (see [3] for the proofs). We now denote by $\mathrm{d}z$ the Lebesgue measure on $\mathbb{C}$ and for every complex number $z \ne 0$, $\omega(z)$ is the determination of its argument valued in $(-\pi, \pi]$.

Lemma 3.1. *The time-changed process $(\theta^U_{A^U(u)}, u \ge 0)$ is a real-valued symmetric Lévy process, say ρ. It has no Gaussian component and its Lévy measure has support in $[-\pi, \pi]$. Moreover, the Lévy measure of $\theta_{A^U(\cdot)}$ is the image of the Lévy measure of U by the mapping $z \to \omega(1 + z)$. Consequently, $\mathbb{E}[(\theta_{A^U(u)})^2] = u\, k(\alpha)$, where*

$$k(\alpha) = \frac{\alpha 2^{-1+\alpha/2}\Gamma(1 + \alpha/2)}{\pi\Gamma(1 - \alpha/2)} \int_{\mathbb{C}} |z|^{-2-\alpha} |\omega(1 + z)|^2 \,\mathrm{d}z.$$

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

For the process U, we use the analogue of the skew-product representation for planar BM which is the Lamperti correspondence for stable processes. Hence, there exist two real-valued Lévy processes $(\xi_u,\ u \geq 0)$ and $(\rho_u,\ u \geq 0)$, the first one is nonsymmetric whereas the second one is symmetric, both starting from 0, such that

$$\log |U_t| + \mathrm{i}\theta_t^U = (\xi_u + \mathrm{i}\rho_u)|_{u=H_t^U}.$$

Remark 3.1. The processes $|Z|$ and $Z_{A^U(\cdot)}/|Z_{A^U(\cdot)}|$ are not independent. This is easily seen since $|Z_{A^U(\cdot)}|$ and $Z_{A^U(\cdot)}/|Z_{A^U(\cdot)}|$ jump at the same time, hence, they cannot be independent. Moreover, $A^U(\cdot)$ depends only upon $|Z|$; hence, $|Z|$ and $Z_{A^U(\cdot)}/|Z_{A^U(\cdot)}|$ are not independent. For further discussion on the independence, see, e.g. [18], where it was shown that an isotropic α-self-similar Markov process has a skew-product structure if and only if its radial and its angular part do not jump at the same time.

3.2. The asymptotic behavior of windings

Bertoin and Werner [3] obtained an asymptotic result, which is, in some sense, a version of Spitzer's asymptotic Theorem 2.1 for isotropic stable Lévy processes of index $\alpha \in (0, 2)$.

Theorem 3.1. *As $c \to \infty$, the family of processes $(c^{-1/2}\theta^U_{\exp(ct)},\ t \geq 0)$ converges in distribution, on the space $D([0,\infty),\mathbb{R})$ endowed with the Skorokhod topology, to $(\sqrt{r(\alpha)}\beta_t,\ t \geq 0)$, where $(\beta_s,\ s \geq 0)$ is a real-valued BM and*

$$r(\alpha) = \frac{\alpha 2^{-1-\alpha/2}}{\pi} \int_{\mathbb{C}} |z|^{-2-\alpha} |\omega(1+z)|^2 \,\mathrm{d}z. \tag{3.1}$$

Proof. We refer the reader to two different proofs. First, Bertoin and Werner [3], where the authors used an Ornstein–Uhlenbeck-type process and ergodicity arguments.

Second, Doney and Vakeroudis [12], where the continuity of the composition function $\rho_{H^U(\cdot)}$ (see [32]) was employed. □

4. Applications to the pricing of Asian options

4.1. Asian options and exponential functionals of BM

In this subsection we return to the initial financial mathematics problem, i.e. the characterization of the distribution of

$$A_t^Z = \int_0^t \exp(2\beta_u)\,\mathrm{d}u,$$

in order to compute $\mathbb{E}[((1/t)A_t^Z - K)^+]$. To that end, one may use the previously stated results to access the distribution of A_t via Williams' so-called pinching method (see [21] and [33]) that we also used in Subsection 2.3. We propose here to mimic again this method for our benefit, by invoking the time changes discussed in the previous sections.

Proposition 4.1. *The following convergence in law holds:*

$$\frac{1}{t}\log A^Z_{t^2} \xrightarrow{\mathrm{D}} 2|\beta|_{T_1^\gamma} \stackrel{\mathrm{D}}{=} 2|C|_1, \qquad t \to \infty,$$

where C_1 is a standard Cauchy random variable.

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

Proof. First, observe that

$$\log\left(\frac{A^Z_{T^\gamma_t}}{A^Z_{t^2}}\right) = \log\left(\frac{\int_0^{T^\gamma_t} \exp(2\beta_u)\,\mathrm{d}u}{\int_0^{t^2} \exp(2\beta_u)\,\mathrm{d}u}\right),$$

which is a random variable that exists (and which seems to be of no other interest here). Renormalizing by t, we obtain

$$\frac{1}{t}(\log A^Z_{T^\gamma_t} - \log A^Z_{t^2}) = \frac{1}{t}\log\left(\frac{A^Z_{T^\gamma_t}}{A^Z_{t^2}}\right) \overset{\mathrm{D}}{\to} 0, \qquad t\to\infty.$$

Hence, studying asymptotically

$$t^{-1}\log A^Z_{T^\gamma_t} \quad \text{as } t\to\infty$$

would yield similar results for $t^{-1}\log A^Z_{t^2}$. Following [31], applying the scaling property of BM and making a change of variables, we have

$$A^Z_{T^\gamma_t} = \int_0^{T^\gamma_t} \mathrm{e}^{2\beta_v}\,\mathrm{d}v \overset{\mathrm{D}}{=} t^2\int_0^{T^\gamma_1} \mathrm{e}^{2t\beta_u}\,\mathrm{d}u$$

(recall that $T^\gamma_t \overset{\mathrm{D}}{=} t^2 T^\gamma_1$), so that, for all $t>0$, we have

$$\frac{1}{t}\log A^Z_{T^\gamma_t} \overset{\mathrm{D}}{=} \frac{1}{t}\log\left(t^2\int_0^{T^\gamma_1} \mathrm{e}^{2t\beta_u}\,\mathrm{d}u\right) = \frac{2\log t}{t} + \log\left(\int_0^{T^\gamma_1} \mathrm{e}^{2t\beta_u}\,\mathrm{d}u\right)^{1/t}.$$

Using the fact that the p-norm converges to the ∞-norm when $p\to\infty$, the latter converges for $t\to\infty$ towards $2\sup_{0\le u\le T^\gamma_1}\beta_u$. By the reflexion principle (see, e.g. [24]), we have

$$\sup_{0\le u\le T^\gamma_1}\beta_u \overset{\mathrm{D}}{=} |\beta|_{T^\gamma_1} \overset{\mathrm{D}}{=} |C_1|$$

and we deduce that

$$\frac{1}{t}\log A^Z_{T^\gamma_t} \overset{\mathrm{D}}{\to} 2|C_1|, \qquad t\to\infty.$$

The result for A^Z_t follows immediately. □

The distribution of A^Z_t may also be characterized by a result due to Dufresne [13] that we state now. For the sake of completeness, we also sketch the proof.

Proposition 4.2. *For every $t>0$, $x\ge 0$, and with φ given in (2.9), we have*

$$\mathbb{E}\left[\frac{1}{\sqrt{2\pi A^Z_t}}\exp\left(-\frac{x}{2A^Z_t}\right)\right] = \frac{1}{\sqrt{2\pi t}}\frac{1}{\sqrt{1+x}}\exp\left(-\frac{\varphi(x)}{2t}\right).$$

Proof. We appeal again to Bougerol's identity in law. For every $t>0$ fixed,

$$\sinh(\beta_t) \overset{\mathrm{D}}{=} \hat\beta_{A^Z_t(\beta)},$$

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

and we identify the densities of the two parts, i.e.

$$\frac{1}{\sqrt{2\pi t}}\frac{1}{\sqrt{1+y^2}}\exp\biggl(-\frac{\varphi(y^2)}{2t}\biggr)$$

for the left-hand side and

$$\mathbb{E}\biggl[\frac{1}{\sqrt{2\pi A_t^Z}}\exp\biggl(-\frac{y^2}{2A_t^Z}\biggr)\biggr]$$

for the right-hand side. The proof is complete by the change of variable $x = y^2$. □

Corollary 4.1. *For every $t > 0$, it follows that $1/A_t^Z$ ∼GGC.*

Proof. With the notation of (2.12), observe that

$$\mathbb{E}\biggl[\exp\biggl(-\frac{x}{2}\biggl(\frac{1}{A_t^Z}\biggr)_{[1/2]}\biggr)\biggr] = \mathrm{e}^{-\chi(x)},$$

where $x \mapsto \chi(x) = \frac{1}{2}\log(1+x) + \varphi(x)/2t \in$ TBF, and conclude as in the proof of Proposition 2.4. □

Remark 4.1. These results may easily be generalized for the functional

$$A_t^{(\nu)} = \int_0^t \exp(\beta_s + \nu s)\,\mathrm{d}s.$$

Indeed, we have access to its distribution by the following relation (see, e.g. [1] or [28]): with ν, μ two real numbers, for every $t > 0$ fixed (β, B, and δ are three independent BMs),

$$\sinh(Y_t^{(\nu,\mu)}) \stackrel{\mathrm{D}}{=} \int_0^t \exp(\beta_s + \nu s)\,\mathrm{d}(B_s + \mu s) = \delta_{\int_0^t \exp(2(\beta_s+\nu s))\,\mathrm{d}s},$$

where $(Y_t^{(\nu,\mu)}, t \geq 0)$ is a diffusion with infinitesimal generator

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \biggl(\nu\tanh(y) + \frac{\mu}{\cosh(y)}\biggr)\frac{\mathrm{d}}{\mathrm{d}y},$$

starting from $y = \arg\sinh(x)$. Here, without loss of generality, we may consider $\mu = 0$ and mimic the approach with $\nu = 0$, which was presented above.

4.2. Asian options and exponential functionals of Lévy processes

In this subsection we discuss the case of Asian options in conjunction with Lévy processes, i.e. the case where the exponential functional of interest is

$$A_t^U := \int_0^t \exp(\alpha\xi_s)\,\mathrm{d}s.$$

Recall from Subsection 3.1 that U is an isotropic planar stable process, and ξ and ρ are two real-valued Lévy processes, the first one is nonsymmetric and the second one is symmetric. Following [4, Section 6.3] and [37], this case may be considered as a natural generalization of the case of Asian options where the exponential functional is associated to a BM. More precisely, from the above references, the computation of the price of Asian options corresponds

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

to the study of the law of the exponential functional associated to a Lévy process ξ at a fixed time t. In particular, the problem may be reduced by substituting t by an exponential (random) time. Hence, we have

$$T_c^{\theta^U} = \inf\{t : \theta_t^U = c\} = (H^U)_u^{-1}|_{u=T_c^\rho} = \int_0^{T_c^\rho} \mathrm{d}s \exp(\alpha\xi_s) =: A_{T_c^\rho}^U \tag{4.1}$$

and, similarly,

$$T_c^{|\theta^U|} = A_{T_c^{|\rho|}}^U.$$

We state the following proposition only for $A_{T_c^\rho}^U$, a similar result may also be obtained for $A_{T_c^{|\rho|}}^U$.

Proposition 4.3. *The following convergence in law holds:*

$$\frac{1}{t}\log A_{t^{\alpha/2}}^U \xrightarrow{\mathrm{D}} T_{\sqrt{1/r(\alpha)}}^\beta, \qquad t \to \infty, \tag{4.2}$$

where $r(\alpha)$ is given by (3.1), β denotes a real BM, and $(T_u^\beta)_{u>0}$, given by (1.2), is a $\frac{1}{2}$-stable subordinator.

Proof. Mimicking the approach of the previous subsection, we can extend this result to A_t^U. Indeed, we easily show that $A_{T_{\sqrt{t}}^\rho}^U - A_{t^{\alpha/2}}^U$ is a variable that exists; hence,

$$\frac{1}{t}(\log A_{T_{\sqrt{t}}^\rho}^U - \log A_{t^{\alpha/2}}^U) = \frac{1}{t}\left(\log \frac{A_{T_{\sqrt{t}}^\rho}^U}{A_{t^{\alpha/2}}^U}\right) \xrightarrow{\mathrm{D}} 0, \qquad t \to \infty.$$

Now, following [12], we use (4.1) and Theorem 3.1 in order to obtain

$$\begin{aligned}
\frac{1}{t}\log A_{T_{\sqrt{t}}^\rho}^U &= \frac{1}{t}\log(T_{\sqrt{t}}^{\theta^U}) \\
&= \frac{1}{t}\log(\inf\{u : \theta_u^U > \sqrt{t}\}) \\
&= \frac{1}{t}\log\left(\inf\left\{\mathrm{e}^{ts} : \frac{1}{\sqrt{t}}\theta_{\exp(ts)}^U > 1\right\}\right) \quad (u = \exp(ts)) \\
&= \inf\left\{s : \frac{1}{\sqrt{t}}\theta_{\exp(ts)}^U > 1\right\} \\
&\xrightarrow{\mathrm{D}} \inf\{s : \beta_{r(\alpha)s} > 1\} \quad (t \to \infty) \\
&= \inf\{s : \sqrt{r(\alpha)}\beta_s > 1\} \\
&=: T_{\sqrt{1/r(\alpha)}}^\beta,
\end{aligned}$$

which completes the proof. □

Corollary 4.2. *Let $N \sim \mathcal{N}(0,1)$, The following convergence in law holds:*

$$(A_{t^{\alpha/2}}^U)^{1/t} \xrightarrow{\mathrm{D}} \exp\left(\frac{1}{r(\alpha)}\frac{1}{N^2}\right) \sim \mathrm{GGC}, \qquad t \to \infty.$$

Proof. The proof follows using (4.2), the scaling property of stable processes, the fact that $1/N^2 \stackrel{\mathrm{D}}{=} T_1^\beta$ ~GGC, and [7, Theorem 3]. □

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

Remark 4.2. The result of Corollary 4.2 could be useful in order to obtain asymptotic closed formulae about 'jump type' Asian option prices by following the spirit of Geman and Yor [38]. This problem will be further discussed in a forthcoming paper.

Acknowledgements

The first author would like to extend his sincere appreciation to the Deanship of Scientific Research at King Saud University for funding (grant number RG-1437-020). The research of the second author was partly financed by the project Postdoctoral Researchers 2016–2017 of the University of Cyprus. He is indebted to Professor Konstantinos Fokianos for supervising his postdoctoral stay at the University of Cyprus where he prepared several parts of this work. Moreover, the authors would like to thank the anonymous referees for useful comments that improved the paper and, in particular, Subsection 2.2. Finally, they would like to express their gratitude to the late Professor Marc Yor; the stimulating discussions they had with him will always remain a source of inspiration.

References

[1] Alili, L., Dufresne, D. and Yor, M. (1997). Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement Brownien avec drift. In *Exponential Functionals and Principal Values Related to Brownian Motion*, Biblioteca de la Revista Matematica, Madrid, pp. 3–14.

[2] André, D. (1887). Solution directe du problème résolu par M. Bertrand. *C. R. Acad. Sci. Paris* **105,** 436–437.

[3] Bertoin, J. and Werner, W. (1996). Stable windings. *Ann. Prob.* **24,** 1269–1279.

[4] Bertoin, J. and Yor, M. (2005). Exponential functionals of Lévy processes. *Prob. Surveys* **2,** 191–212.

[5] Biane, P. and Yor, M. (1987). Valeurs principales associées aux temps locaux Browniens. *Bull. Sci. Math.* **111,** 23–101.

[6] Bondesson, L. (1992). *Generalized Gamma Convolutions and Related Classes of Distributions and Densities* (Lecture Notes Statist. **76**), Springer, New York.

[7] Bondesson, L. (2015). A class of probability distributions that is closed with respect to addition as well as multiplication of independent random variables. *J. Theoret. Prob.* **28,** 1063–1081.

[8] Bosch, P. and Simon, T. (2015). On the infinite divisibility of inverse beta distributions. *Bernoulli* **21,** 2552–2568.

[9] Bougerol, P. (1983). Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré B* **19,** 369–391.

[10] DeBlassie, R. D. (1987). Exit times from cones in $\mathbb{R}^n$ of Brownian motion. *Prob. Theory Relat. Fields* **74,** 1–29.

[11] DeBlassie, R. D. (1988). Remark on exit times from cones in $\mathbb{R}^n$ of Brownian motion. *Prob. Theory Relat. Fields* **79,** 95–97.

[12] Doney, R. A. and Vakeroudis, S. (2013). Windings of planar stable processes. In *Séminaire de Probabilités XLV* (Lecture Notes Math. **2078**), Springer, Cham, pp. 277–300.

[13] Dufresne, D. (2000). Laguerre series for Asian and other options. *Math. Finance* **10,** 407–428.

[14] Gallardo, L. (2008). *Mouvement Brownien et Calcul d'Itô.* Hermann, Paris.

[15] Graversen, S. E. and Vuolle-Apiala, J. (1986). α-self-similar Markov processes. *Prob. Theory Relat. Fields* **71,** 149–158.

[16] Itô, K. and McKean, H. P., Jr. (1965). *Diffusion Processes and Their Sample Paths.* Springer, Berlin.

[17] Kyprianou, A. E. and Vakeroudis, S. M. (2018). Stable windings at the origin. To appear in *Stoch. Process. Appl.*

[18] Liao, M. and Wang, L. (2011). Isotropic self-similar Markov processes. *Stoch. Process. Appl.* **121,** 2064–2071.

[19] Matsumoto, H. and Yor, M. (2005). Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Prob. Surveys* **2,** 312–347.

[20] Matsumoto, H. and Yor, M. (2005). Exponential functionals of Brownian motion. II. Some related diffusion processes. *Prob. Surveys* **2,** 348–384.

[21] Messulam, P. and Yor, M. (1982). On D. Williams' "pinching method" and some applications. *J. London Math. Soc. 2* **26,** 348–364.

[22] Pitman, J. and Yor, M. (1986). Asymptotic laws of planar Brownian motion. *Ann. Prob.* **14,** 733–779.

[23] Port, S. C. and Stone, C. J. (1978). *Brownian Motion and Classical Potential Theory.* Academic Press, New York.

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press

[24] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion,* 3rd edn. Springer, Berlin.
[25] Schilling, R. L., Song, R. and Vondraček, Z. (2012). *Bernstein Functions: Theory and Applications,* 2nd edn. De Gruyter, Berlin.
[26] Spitzer, F. (1958). Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87,** 187–197.
[27] Vakeroudis, S. (2011). Nombres de tours de certains processus stochastiques plans et applications à la rotation d'un polymère. Doctoral Thesis, Université Pierre et Marie Curie.
[28] Vakeroudis, S. (2012). Bougerol's identity in law and extensions. *Prob. Surveys* **9,** 411–437.
[29] Vakeroudis, S. (2012). On hitting times of the winding processes of planar Brownian motion and of Ornstein–Uhlenbeck processes, via Bougerol's identity. *Theory Prob. Appl.* **56,** 485–507.
[30] Vakeroudis, S. and Yor, M. (2012). Some infinite divisibility properties of the reciprocal of planar Brownian motion exit time from a cone. *Electron. Commun. Prob.* **17,** 23.
[31] Vakeroudis, S. and Yor, M. (2013). Integrability properties and limit theorems for the exit time from a cone of planar Brownian motion. *Bernoulli* **19,** 2000–2009.
[32] Whitt, W. (1980). Some useful functions for functional limit theorems. *Math. Operat. Res.* **5,** 67–85.
[33] Williams, D. (1974). A simple geometric proof of Spitzer's winding number formula for 2-dimensional Brownian motion. Unpublished, manuscript.
[34] Yor, M. (1980). Loi de l'indice du lacet Brownien, et distribution de Hartman-Watson. *Z. Wahrscheinlichkeitsth.* **53,** 71–95.
[35] Yor, M. (1992). Sur les lois des fonctionnelles exponentielles du mouvement Brownien, considérées en certains instants aléatoires. *C. R. Acad. Sci. Paris I* **314,** 951–956.
[36] Yor, M. (1993). From planar Brownian windings to Asian options. *Insurance Math. Econom.* **13,** 23–34.
[37] Yor, M. (2001). *Exponential Functionals of Brownian Motion and Related Processes.* Springer, Berlin.
[38] Yor, M. and Geman, H. (2001). Bessel processes, Asian options, and perpetuities. In *Exponential Functionals of Brownian Motion and Related Processes,* Springer, Berlin, pp. 63–92.

https://doi.org/10.1017/apr.2018.33 Published online by Cambridge University Press