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Path integrals for mean-field equations in nonlinear dynamos

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Mean-field dynamo equations are addressed with the aid of the path integral method. The evolution of magnetic field is treated as a three-dimensional Wiener random process, and the mean magnetic-field equations are obtained with the Wiener integrals taken over all the trajectories of the fluid particles. The form of the equations is just the same as the conventional mean-field equations, but here the equations are derived with the velocity field realisation affected by the force exerted by the magnetic field. In this sense, we derive nonlinear dynamo equations.

Key words: astrophysical plasmas, space plasma physics

1. Introduction

Mean-field equations were introduced in dynamo theory at quite an early stage of its development (see e.g. Krause & Rädler (1980)). At that time it was the main tool to produce dynamo models for magnetic-field evolution in various celestial bodies including the Sun. Contemporary science has for this aim other tools primarily including direct numerical simulations; however the mean-field equations remain an important approach. In particular, the mean-field description is useful to understand the physical mechanism of dynamo action based on mirror asymmetry of turbulence or convection, in the form of the famous α effect, which explains the solar dynamo mechanism suggested in Parker (1955).

Several methods to perform statistical averaging of the induction equation in order to obtain mean-field dynamo equations have been suggested. The main issue here is how to split correlations into terms which contain both the velocity field $\boldsymbol{v}$ as well as the magnetic field $\boldsymbol{H}$ which are obviously statistically dependent. One such method is the path integral method, initially suggested in Molchanov, Ruzmaikin & Sokoloff (1983).

The idea of the method can be presented as follows. Performing statistical averaging looks attractive to deal with the solution of the induction equation rather with the induction equation itself, because the solution by definition contains only the velocity field and initial magnetic field. The problem is how to obtain such a solution. It is

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achievable if we neglect ohmic losses and use a Lagrangian reference frame, so the induction equation reduces to a system of linear ordinary differential equations. Then the desired solution can be obtained following magnetic-field evolution over the Lagrangian trajectories.

Including ohmic losses, we have to consider a bundle of random paths surrounding the Lagrangian trajectory. These paths arise from the joint action of advection and random walks which mimic magnetic-field diffusion. Magnetic-field evolution is considered here as a superposition of contributions associated with a particular random trajectory. In this sense the path integral approach can be considered as an extension of the Green's function approach (Kraichnan 1965).

The path integral technique was initially invented by Feynman for quantum mechanics and then generalised by Kac to various transport problems, while initial ideas came even from Wiener (see Zeldovich, Ruzmaikin & Sokoloff (1990) for a historical review). Because the technique exploits substantially the superposition concept, the equations when averaged have to be linear. This is natural in quantum mechanics which is a basically linear science, however it is severely restrictive for dynamo theory because magnetic force begins to play a role at quite an early stage in the evolution of a dynamo driven magnetic configuration.

Of course, people have exploited path integral techniques for nonlinear dynamo studies (e.g. Kleeorin & Rogachevskii (1994)) however the results are considered as not mathematically justified because, strictly speaking, the Kac–Feynman formula is valid for linear equations only. The point however is that the induction equation taken alone is a linear equation and nonlinearity enters the problem only because the magnetic field affects motions and this effect is addressed by the hydrodynamical equations. Such equations are known in mathematical studies as quasilinear. The path integral method can be applied to quasilinear equations by using a specific 'trick' (e.g. Peng (1991)) which is exploited in some areas of science (initially in financial mathematics (Shiryaev 1999)); however the technique still deserves introduction into dynamo theory. This is the aim of this paper.

We restrict our presentation to the case similar to the kinematical problem considered by Molchanov *et al.* (1983) and try to present the simplest version of the generalisation under discussion. Correspondingly, our algebra below is as close to that of Molchanov *et al.* (1983) as possible. However, the message of the paper is quite different. The form of kinematic mean-field dynamo equations is very robust. It is quite easy to obtain a link between expressions for parametrisations of turbulent diffusivity and α-effect obtained in various approaches, say in second-order smoothing and path integral approaches. We discuss in § 5 how the situation for nonlinear dynamos becomes more complicated and the form of dynamo equations obtained in the approach discussed is quite different from that considered in contemporary dynamical quenching models.

2. The Kac–Feynman formula

For the sake of consistency we consider briefly how magnetic field evolution in a given velocity field can be addressed by the path integral method. We start from the standard induction equation for the magnetic field $\boldsymbol{H}$ in the flow $\boldsymbol{v}$ with magnetic diffusivity ν_m

$$\frac{\partial \boldsymbol{H}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{H} = (\boldsymbol{H}\cdot\nabla)\boldsymbol{v} - \nu_m \mathrm{curl}\,\mathrm{curl}\,\boldsymbol{H}\,. \tag{2.1}$$

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For the inviscid case in a Lagrangian frame of reference this equation reads

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{H}\hat{\boldsymbol{A}}, \tag{2.2}$$

where the vector $\boldsymbol{H}$ is considered to be a row (not column) and the matrix $\hat{\boldsymbol{A}}$ consists of derivatives $\partial v_i/\partial x_j$. The variable $\boldsymbol{x}$ in (2.2), position of the 'fluid element' at instant t, is governed by the equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}. \tag{2.3}$$

Equation (2.2) can be solved in an explicit form in terms of what are known as T-exponents or, more mathematically, Volterra multiplicative integrals (e.g. Gantmacher (1959))

$$\boldsymbol{H}(\boldsymbol{x}(t), t) = \boldsymbol{H}\Pi^t_{s=0}[\hat{\boldsymbol{E}} + \hat{\boldsymbol{A}}(s)\mathrm{d}s], \tag{2.4}$$

where $\hat{\boldsymbol{E}}$ is a unit matrix and s is a past time ($s \leqslant t$).

Equation (2.4) can be extended to include the dissipative term if we add diffusive random walks $\sqrt{2}\nu_m \boldsymbol{w}_t$ to the advection velocity $\boldsymbol{v}$ in (2.3). Here $\boldsymbol{w}_t$ means the standard three-dimensional Wiener random process, i.e. a process with independent increments, zero mean and correlation matrix $\delta_{ij}t$. We have to consider a bundle of diffusion paths, summarise corresponding contributions and perform averaging taken over all paths (still not over all realisations of the velocity field). This is the step that requires linearity of (2.1). From the mathematical viewpoint the last operation can be described as a Wiener integral taken over all trajectories of fluid particles.

We have to take into account that the random walk $\boldsymbol{w}_t$ has no finite velocity (no finite limit $(\boldsymbol{w}_{t+\Delta t} - \boldsymbol{w}_t)/\Delta t$ has no finite limit for $\Delta t \to 0$) because $\boldsymbol{w}_{t+\Delta t} - \boldsymbol{w}_t \propto \sqrt{t}$ (Einstein relation), and we have to rewrite equations (2.3) and (2.4) in integral form. This yields in

$$\boldsymbol{H}(\boldsymbol{x}, t) = M_{\boldsymbol{x}}\boldsymbol{H}(\boldsymbol{\xi}_{\boldsymbol{x},s})\Pi^t_{s=0}[\hat{\boldsymbol{E}} + \hat{\boldsymbol{A}}(\boldsymbol{\xi}_{\boldsymbol{x},s'})\,\mathrm{d}s'], \tag{2.5}$$

where

$$\boldsymbol{\xi}_{\boldsymbol{x},s} = \boldsymbol{x} + \sqrt{2}\nu_m^{1/2}\boldsymbol{w}_t - \int_t^s \boldsymbol{v}\,\mathrm{d}s' \tag{2.6}$$

(the so-called Ito equation; see Ito (1946)). Equation (2.5) belongs to the type of expressions referred as the Kac–Feynman formulae. $M_{\boldsymbol{x}}$ mean averaging taken over trajectories obtained from (2.6).

The fact that (2.5) gives the solution of (2.1) can be verified directly by taking derivatives with respect to t of this function.

Equations (2.5) and (2.6) do require linearity; however, we use them in particular circumstances only and this gives freedom to use the technique to be presented here.

3. Short-correlated model for the nonlinear dynamo

To be specific, we present the 'trick' for the short-correlated model, i.e. we consider the correlation time of turbulence or convection to be short enough to ignore details of the random path ξ evolution during this time. Formally it means the following. We consider a family of random velocity fields $\boldsymbol{v}^{\Delta}$ which are statistically independent and identically distributed at time intervals $[n\Delta, (n+1)\Delta)$. (Note that the left-hand boundary is included in the interval while the right-hand is excluded; this solves the problem concerning the memory of what happens just at the instant $t = n\Delta$.) In order

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to avoid lengthy algebra we assume that the random velocity field is statistically homogeneous and isotropic in space and the mean velocity vanishes (see Tomin & Sokoloff (2010) concerning implementation of a non-zero and inhomogeneous mean velocity in the procedure, and Yokoi (2013) in connection with the cross-helicity problem; it is however difficult to use the method for steady flows).

We are going to consider the case $\varDelta \to 0$. To avoid vanishing of induction effects in this limiting case we have to assume that

$$\boldsymbol{v}^{\varDelta} \propto \varDelta^{-1/2}. \tag{3.1}$$

The scaling (3.1) looks similar to the Einstein relation $w_{t+\Delta t} - w_t \propto \sqrt{t}$. This means that the hydrodynamical flow in the framework of short-correlated model is assumed to be similar to Brownian motion.

Then we apply (2.5), considering the instant $t = n\varDelta$ as the initial state and calculate integrals participating in (2.5) and (2.6) using Taylor expansions taken with respect to the parameter $\varDelta$. This means that we need to apply (2.5) for the short time interval from $n\varDelta$ until $(n+1)\varDelta$ only, rather than to the whole time from 0 until t.

Let us take the random field $\boldsymbol{v}^{\varDelta}$ to depend on magnetic field on the times before the instant $n\varDelta$ only, and not to depend on magnetic field in the interval $n\varDelta \leqslant t \leqslant (n+1)\varDelta$. Then we can still can apply (2.5) to the interval of interest in spite of the fact that the problem remains nonlinear.

This is the technique reported here. The consequences of this are hard to foresee. We discuss them at the end of the paper.

As a comment on the short-correlated model we note the following. In magnetohydrodynamics (MHD), in general, we can introduce four Green's functions: G_{uu}, G_{ub}, G_{bu}, G_{bb}, where G_{fg} means the response of the field f to an infinitesimal change of the g field. The short correlated model in the present formulation may correspond to assuming that the correlation times associated with G_{bu} and G_{bb} are much shorter than the counterpart of G_{ub}.

4. Obtaining the mean-field equations

We are now going to finalise the derivation of mean field in the framework of our model. We perform a Taylor expansion of equations (2.5) and (2.6) for $\varDelta \to 0$ taking into account that

$$\mathrm{d}F(w_t) = F' w_{dt} + \tfrac{1}{2} F'' \,\mathrm{d}t, \tag{4.1}$$

where F is a smooth function (known as the Ito formula). Equation (4.1) allows restoration of the second derivative term in (2.1) from (2.5).

Equation (2.6) yields

$$\xi_{i,\varDelta} - x_i = -v_i(n\varDelta, x)\varDelta + \sqrt{2}\nu_m^{1/2} w_{i,\varDelta} - \frac{\partial v_i}{\partial x_j}\int_0^{\varDelta} w_{t,j}\,\mathrm{d}t + \frac{1}{2} v_j \frac{\partial v_i}{\partial x_j}\ldots, \tag{4.2}$$

where the ellipsis represents the terms smaller than $\varDelta$ (remember that the second term on the right-hand side is of order $\varDelta$ because of (3.1), while the multiplicative integral in (2.5) can be presented as

$$\begin{aligned} \varPi^t_{s=0}[\hat{\boldsymbol{E}} + \hat{\boldsymbol{A}}(\boldsymbol{\xi_x}, s')d s'] = \delta_{ij} - \int_0^{\varDelta} \frac{\partial v_i(t-s, \boldsymbol{\xi}_s)}{\partial x_j} \\ + \int_o^{\varDelta} \frac{\partial v_i(t-\sigma, \boldsymbol{\xi}_\sigma)}{\partial x_l} \int_0^{\varDelta} \frac{\partial v_l(t-s, \boldsymbol{\xi}_s)}{\partial x_j}\,\mathrm{d}s\,\mathrm{d}\sigma + \cdots . \end{aligned} \tag{4.3}$$

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Taking into account that

$$\frac{\partial v_i(t-s, \boldsymbol{\xi}_s)}{\partial x_j} = \frac{\partial v_i(t-s, x)}{\partial x_j} + \frac{\partial^2 v_i}{\partial x_j \partial x_k}(\boldsymbol{\xi}_{s,k} - x_k) + \cdots , \tag{4.4}$$

collecting terms in (2.5) and performing averaging following Molchanov *et al.* (1983) we obtain the mean-field equation for $\boldsymbol{B} = \langle \boldsymbol{H} \rangle$ as follows

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl} \alpha \boldsymbol{B} - \beta \operatorname{curl} \operatorname{curl} \boldsymbol{B} , \tag{4.5}$$

where

$$\alpha = - \lim_{\Delta \to 0} \Delta \frac{\langle \boldsymbol{v} \operatorname{curl} \boldsymbol{v} \rangle}{3} \tag{4.6}$$

and

$$\beta = \lim_{\Delta \to 0} \Delta \frac{\langle \boldsymbol{v}^2 \rangle}{3}. \tag{4.7}$$

The physical meaning of the limiting procedure $\Delta \to 0$ is that we are not interested in what happens on time scales shorter than Δ, which plays the role of the memory time τ. Standard expressions of mixing length theory for α and β follow from (4.5) and (4.7) using $l = v\Delta$ as a spatial scale of the flow.

5. Conclusions and discussion

Using the path integral method in the framework of the model described above we obtained the mean-field dynamo equation (4.5) for the nonlinear dynamo. The form of this equation is just the same as the conventional mean-field equation obtained by Steenbeck, Krause and Rädler (see Krause & Rädler (1980)). The important difference from the classical mean-field equation is that the averaging in (4.6) and (4.7) is taken over the ensemble of velocity field realisations affected by the magnetic force while the classical mean-field equation assumes that this ensemble as well as the quantities (4.6), (4.7) are given in advance. This departure from the kinematic approach is the most important point in the present formulation of the nonlinear dynamo.

In this sense (4.5) is not a closed linear equation, but however should be combined with equations to calculate the quantities of (4.6), (4.7). Of course, these equations have to be obtained from some arguments external to those considered here. As a possible source for a procedure for calculation of quantities from (4.6) and (4.7) we might suggest the technique of shell models (Plunian, Stepanov & Frick 2013). At the moment this looks to be the most promising technique for quantification of locally homogeneous and isotropic turbulence at very high hydrodynamic and magnetic Reynolds numbers. A generalisation to the anisotropic case looks feasible as well.

Our approach can be compared with that of Brandenburg *et al.* (2008), where the velocity fluctuations are affected by the magnetic field. This is called quasi-kinematic by Rheinhardt & Brandenburg (2010) and is obviously not the fully nonlinear formulation of the problem. A formal manifestation of this fact is that we deal with quasilinear rather fully nonlinear equations. Courvoisier, Hughes & Proctor (2010) insist that a mean-field description of the nonlinear regime based solely on a quenched α-coefficient is incorrect. We agree that nonlinear modification of other transport coefficients has also to be taken into account. However we do not focus our presentation on this point.

The evaluation of the mean-field equation demonstrated above becomes possible because we assume that the action of the nonlinear magnetic force occurs before the renewal instant $t = n\Delta$, while the corresponding induction effect acts from the instant $t = n\Delta$ until $t = (n+1)\Delta$. As for the other assumptions which allow the derivation of the closed mean-field equations, this assumption is also a simplification and its applicability is limited. If the velocity variation through the magnetic force action occurs simultaneously with that of the magnetic-field induction, the mean-field evolution has to be treated in fully nonlinear manner.

An important feature of (4.5) is that it yields a mean-field equation which does not explicitly include the idea that current (or magnetic) helicity is a quantity important for nonlinear mean-field dynamo suppression. Indeed, current (or magnetic) helicity does not participate in this equation directly but can participate via (4.6) (or even (4.7), cf. Brandenburg, Schober & Rogachevskii (2017) where contribution of kinetic helicity in turbulent diffusivity is considered) only. Maybe this is a shortcoming of the model of the flow studied here. If this is true, it means that the key issue for the role of current helicity term in the nonlinear dynamo model is the absence of any time lag between the induction effect and the action of the magnetic force. If the response of the velocity fluctuation to the fluctuating Lorentz force $\boldsymbol{j}' \times \boldsymbol{B}$ $[\boldsymbol{B} = \langle \boldsymbol{H} \rangle, \boldsymbol{j}' = \mathrm{curl}(\boldsymbol{H} - \langle \boldsymbol{H} \rangle)]$ is much slower than the counterpart of the magnetic fluctuation to the velocity variation, the former or current helicity effect is negligible as compared with the latter or kinetic helicity effect. As for the physical origins of the kinetic and current helicity effects, see § 3.3.2 of Yokoi (2013). Such a lag is postulated in our model. If there is no time lag between the magnetic induction and the velocity response to the magnetic force action, the current helicity contributes to the turbulent electromotive force through the fluctuating Lorentz force in a sense opposed to the contribution of the kinetic helicity effect. Maybe this means that only a fully nonlinear approach to the dynamo saturation problem is adequate. Of course, the relative importance of the current helicity to the kinetic helicity effects depends on the domain of the magnetohydrodynamic flows. If the magnetic Reynolds number $v\ell/\eta$ (v: characteristic velocity, ℓ: characteristic length, η: magnetic diffusivity) is much smaller than the kinetic Reynolds number $v\ell/\nu$ (ν: kinematic viscosity), the magnitude of induced magnetic-field fluctuation itself and consequently that of the turbulent current helicity density are relatively so small that the current helicity effect is negligible. Another option is that memory effects are important for the current helicity terms in the nonlinear dynamo equations. On the other hand however, in the usual homogeneous turbulence theory, it is often considered that the helicity introduces a time scale other than the eddy turnover time. For instance, the bottleneck effect (energy pile up at small scales) in helical turbulence is often attributed to the time scale change due to turbulence at small scales.

In any case, the final decision as to which parametrisation for nonlinear mean-field dynamo action is more realistic has to come from experience with modelling of particular natural (or, in perspective, laboratory) dynamos, together with numerical experiments.

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