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COMPACT AND HILBERT–SCHMIDT WEIGHTED COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES

CHING-ON LO and ANTHONY WAI-KEUNG LOH

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Abstract

Let u and φ be two analytic functions on the unit disk D such that $\varphi(D) \subset D$. A weighted composition operator uC_φ induced by u and φ is defined on A^2_α, the weighted Bergman space of D, by $uC_\varphi f := u \cdot f \circ \varphi$ for every $f \in A^2_\alpha$. We obtain sufficient conditions for the compactness of uC_φ in terms of function-theoretic properties of u and φ. We also characterize when uC_φ on A^2_α is Hilbert–Schmidt. In particular, the characterization is independent of α when φ is an automorphism of D. Furthermore, we investigate the Hilbert–Schmidt difference of two weighted composition operators on A^2_α.

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1. Introduction

Let D be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and T be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α of D consists of all analytic functions f in $L^p(D, dA_\alpha)$, that is,

$$\|f\|^p_{A^p_\alpha} := \int_D |f(z)|^p \, dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) := (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ and $dA(z) := (1/\pi)\, dx\, dy$ is the normalized area measure on D. It is known that A^2_α is a closed subspace of $L^2(D, dA_\alpha)$ and is thus a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle := \int_D f(z)\overline{g(z)}\, dA_\alpha(z) \quad \text{for every } f, g \in A^2_\alpha.$$

In what follows, we denote the norm on A^2_α by $\|\cdot\|$ for brevity. By writing $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we have

$$\|f\|^2 = \sum_{k=0}^{\infty} \frac{k!\ \Gamma(\alpha+2)}{\Gamma(\alpha+2+k)} |a_k|^2,$$

where Γ is the usual gamma function. If we let

$$e_k(z) = \sqrt{\frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)}}\, z^k \quad \text{for } k = 0, 1, \ldots, \tag{1-1}$$

then $\{e_k\}_{k=0}^{\infty}$ is the standard orthonormal basis for A^2_α. Furthermore, if w is an arbitrary point in D, then $\langle f, k_w \rangle = f(w)$ for all $f \in A^2_\alpha$, where $k_w(z) := 1/(1-\overline{w}z)^{\alpha+2}$ is the reproducing kernel representing the point evaluation functional on A^2_α at $z = w$. Moreover, $\|k_w\|^2 = 1/(1-|w|^2)^{\alpha+2}$.

Let u and φ be two analytic functions on D such that $\varphi(D) \subset D$. They induce a *weighted composition operator* uC_φ from A^2_α into the linear space of all analytic functions on D by

$$uC_\varphi(f)(z) := u(z)f(\varphi(z)) \quad \text{for every } f \in A^2_\alpha \text{ and } z \in D.$$

When $u \equiv 1$, the corresponding operator, denoted by C_φ, is known as a *composition operator*. From exercise 3.1.3 in [3, page 127], C_φ is always bounded. However, this is not necessarily true for weighted composition operators. When uC_φ maps A^2_α into itself, we say uC_φ is a weighted composition operator on A^2_α. In this case, $u = uC_\varphi 1 \in A^2_\alpha$. An appeal to the closed graph theorem shows that every operator uC_φ on A^2_α is bounded. Furthermore, if $g \in A^2_\alpha$ and $w \in D$, then

$$\langle (uC_\varphi)^* k_w, g \rangle = \langle k_w, uC_\varphi g \rangle = \overline{u(w)g(\varphi(w))} = \langle \overline{u(w)} k_{\varphi(w)}, g \rangle.$$

Thus,

$$(uC_\varphi)^* k_w = \overline{u(w)} k_{\varphi(w)}.$$

During the past two decades, several authors have studied the properties of (weighted) composition operators on A^p_α with Berezin transforms and Carleson-type measures (see for example [4, 5, 11, 13]). In Section 2, we obtain sufficient conditions for the compactness of uC_φ in terms of function-theoretic properties of u and φ. In Section 3, we characterize Hilbert–Schmidt weighted composition operators and the Hilbert–Schmidt difference of two weighted composition operators on A^2_α.

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2. Compact weighted composition operators

A bounded linear operator T from a Banach space B_1 to a Banach space B_2 is said to be *compact* if it maps bounded subsets of B_1 into relatively compact subsets of B_2. Equivalently, T is compact if and only if it maps every bounded sequence $\{x_n\}_{n=1}^{\infty}$ in B_1 onto a sequence $\{Tx_n\}_{n=1}^{\infty}$ in B_2 which has a convergent subsequence. It was shown in [13, Theorem 4.3] that uC_φ is compact on A_α^p if and only if

$$\lim_{\delta\to 0^+} \sup_{\zeta\in T} \frac{\mu_{\alpha,p}\circ\varphi^{-1}(S(\zeta,\delta))}{\delta^{\alpha+2}} = 0,$$

where $S(\zeta,\delta) := \{z \in D : |z-\zeta| < \delta\}$ and $\mu_{\alpha,p}\circ\varphi^{-1}$ is the measure such that $\|uC_\varphi f\|_{A_\alpha^p}^p = \int_D |f|^p \, d\mu_{\alpha,p}\circ\varphi^{-1}$ for all $f \in A_\alpha^p$. Later, Čučković and Zhao estimated the essential norm of uC_φ and deduced that uC_φ is compact on A_0^2 if and only if

$$\lim_{|a|\to 1^-} \int_D \frac{(1-|a|^2)^2|u(z)|^2}{|1-\bar{a}\varphi(z)|^4}\, dA(z) = 0$$

[4, Corollary 2]. These characterizations, however, are rather implicit and less tractable. In this section, we provide more explicit sufficient conditions that guarantee uC_φ is compact on A_α^2. To this end, we first state a useful result to the study of compact weighted composition operators on A_α^2.

LEMMA 2.1. *Let uC_φ be a weighted composition operator on A_α^2. The following two statements are equivalent:*

(i) *uC_φ is compact on A_α^2;*
(ii) *if $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in A_α^2 and $f_n \to 0$ uniformly on compact subsets of D, then $\|uC_\varphi f_n\| \to 0$.*

While the above lemma is a generalization of [3, Proposition 3.11], it can also be obtained by a Hilbert space argument. From exercise 4.7.1 in [17, page 97], a sequence of functions in A_α^2 is weakly convergent to zero if and only if this sequence is norm bounded and converges to zero uniformly on compact subsets of D. Lemma 2.1 now follows from this fact and [17, Theorem 1.14].

One simple sufficient condition for the compactness of uC_φ, which is analogous to [7, Theorem 2], is given below.

THEOREM 2.2. *Suppose that uC_φ is a weighted composition operator on A_α^2. If $\overline{\varphi(D)} \subset D$, then uC_φ is compact.*

PROOF. Since $\overline{\varphi(D)} \subset D$, there is a constant M such that $0 < M < 1$ and $|\varphi(z)| \le M$ for all $z \in D$. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in A_α^2 such that $f_n \to 0$ uniformly on compact subsets of D. In particular, this sequence converges to zero uniformly on $\overline{S(0,M)}$. Then there exists some $N \in \mathbb{N}$ for which $|f_n(\varphi(z))| < \epsilon$ whenever $n > N$ and $z \in D$. With $u \in A_\alpha^2$, it follows that $\|uC_\varphi f_n\| \le \epsilon\|u\|$ for all $n > N$. By Lemma 2.1, uC_φ is compact. □

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We remark that the condition $\overline{\varphi(D)} \subset D$ in Theorem 2.2 is sufficient, but not necessary for the compactness of uC_φ. This is shown below.

EXAMPLE 2.3. Let $u(z) = z - 1$ and $\varphi(z) = (z + 1)/2$. Note that $1 \in \overline{\varphi(D)}$. Choose any $\varepsilon > 0$. With $u(1) = 0$ and the continuity of u at $z = 1$, there is a sufficiently small $\delta > 0$ such that $|u|^2 < \varepsilon$ on $S(1, \delta)$. We show that uC_φ is compact by using Lemma 2.1.

Let $\{f_n\}_{n=1}^\infty$ be a sequence in A^2_α such that $\|f_n\| \le 1$ for all $n \in \mathbb{N}$ and $f_n \to 0$ uniformly on compact subsets of D. Since φ is continuous on the compact set $\overline{D \setminus S(1, \delta)}$, the set $\varphi(\overline{D \setminus S(1, \delta)})$ is compact in D. Then there exists some $N \in \mathbb{N}$ for which if $n > N$ and $z \in D \setminus S(1, \delta)$, we have

$$|f_n(\varphi(z))|^2 < \varepsilon.$$

These, together with the fact that C_φ is bounded on A^2_α, imply

$$\begin{aligned}
\|uC_\varphi f_n\|^2 &= \int_{S(1,\delta)} |u(z)|^2 |f_n(\varphi(z))|^2 \, dA_\alpha(z) + \int_{D\setminus S(1,\delta)} |u(z)|^2 |f_n(\varphi(z))|^2 \, dA_\alpha(z) \\
&\le \varepsilon \int_{S(1,\delta)} |f_n(\varphi(z))|^2 \, dA_\alpha(z) + \varepsilon \int_{D\setminus S(1,\delta)} |u(z)|^2 \, dA_\alpha(z) \\
&\le \varepsilon \|C_\varphi f_n\|^2 + \varepsilon \int_D |u(z)|^2 \, dA_\alpha(z) \\
&\le (\|C_\varphi\|^2 + 4)\varepsilon
\end{aligned}$$

whenever $n > N$.

In this example, φ has an angular derivative at $z = 1$ because $(1 - \varphi(z))/(1 - z) = 1/2$. Then it follows from [3, Corollary 3.14] that C_φ is *not* compact on A^2_α. However, uC_φ is compact.

There is another question of interest: does the compactness of C_φ guarantee that of uC_φ? The answer to this question is generally *no*, at least when u is unbounded on D. To see this, we first state a necessary condition for uC_φ to be compact.

THEOREM 2.4. *If uC_φ is a compact weighted composition operator on A^2_α, then*

$$\lim_{|z|\to 1^-} |u(z)| \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha/2+1} = 0. \tag{2-1}$$

This theorem is a simple generalization of [4, Proposition 1]: since

$$\|(uC_\varphi)^* K_z\| = (1 - |z|^2)^{\alpha/2+1} |u(z)| \|k_{\varphi(z)}\| = |u(z)| \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha/2+1},$$

where K_z is the normalized reproducing kernel corresponding to the point evaluation functional on A^2_α at z, the condition in Equation (2-1) follows from the compactness of $(uC_\varphi)^*$ and the result that $K_z \to 0$ weakly in A^2_α as $|z| \to 1^-$.

While the validity of the converse of Theorem 2.4 awaits further investigation, the condition in Equation (2-1) actually is equivalent to the compactness of composition

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operators on A^2_α [17, Theorem 11.8]. Under additional assumptions on u and φ, however, the condition in Equation (2-1) does characterize the compactness of uC_φ. This will be shown in Theorem 2.8.

EXAMPLE 2.5. Let $u(z) = 1/(1-z)^{1/2+\alpha/4}$ and $\varphi(z) = 1-(1-z)^{1/2}$. From [10, Example 3.4], φ has no finite angular derivative at any point of T. Thus, C_φ is compact by [3, Theorem 3.22]. However,

$$\lim_{r\to 1^-} u(r)\Big[\frac{1-r^2}{1-(\varphi(r))^2}\Big]^{\alpha/2+1} = \lim_{r\to 1^-}\Big[\frac{1+r}{2-(1-r)^{1/2}}\Big]^{\alpha/2+1} = 1\ (\neq 0).$$

According to Theorem 2.4, uC_φ is not compact on A^2_α.

When C_φ is compact, how can we choose u such that uC_φ is compact? The next result provides one criterion. Its statement and proof are similar to those of [10, Theorem 4.1].

THEOREM 2.6. *Suppose $u \in A^2_\alpha$ and C_φ is compact on A^2_α. If there is a constant c with $0 < c < 1$ such that u is bounded on the set $\{z \in D : |\varphi(z)| > c\}$, then uC_φ is compact on A^2_α.*

We prove a 'converse' of Theorem 2.4 with extra assumptions on u and φ. While Moorhouse showed that the condition in Equation (2-1) characterizes the compactness of uC_φ when u is bounded on D [14, Corollary 1], the validity of our result does *not* require the boundedness of u. The following lemma is needed.

LEMMA 2.7. *If $f \in A^2_\alpha$, then*

$$c\,\|f\|^2 \le |f(0)|^2 + \int_D |f'(z)|^2(1-|z|^2)^2\,dA_\alpha(z) \le d\,\|f\|^2,$$

where $c := \min\{1, [(\alpha+1)(\alpha+2)]/(\alpha+3)\}$ *and* $d := \max\{1, (\alpha+1)(\alpha+2)\}$.

The proof of this lemma is direct and follows from a straightforward computation of the integral $\int_D |f'(z)|^2(1-|z|^2)^2\,dA_\alpha(z)$ in terms of the Taylor coefficients of f. An immediate consequence of Lemma 2.7 is that $f \in A^2_\alpha$ if and only if $f' \in L^2(D, dA_{\alpha+2})$. Indeed, this is a particular case of a more general result in [8, Proposition 1.11]. Moreover, the lemma implies that $\|f\|$ is equivalent to $\|f'\|_{A^2_{\alpha+2}}$ if $f \in A^2_\alpha$ and $f(0) = 0$.

THEOREM 2.8. *Let uC_φ be a weighted composition operator on A^2_α. If*

(i) *φ is univalent on D;*
(ii) $\lim_{|z|\to 1^-} |u'(z)|(1-|z|^2) = 0$*; and*
(iii) $\lim_{|z|\to 1^-} |u(z)|((1-|z|^2)/(1-|\varphi(z)|^2))^{\alpha/2+1} = 0$*;*

then uC_φ is compact on A^2_α.

PROOF. Fix any $\varepsilon > 0$. By conditions (ii) and (iii), there is a constant r with $1/2 < r < 1$ such that

$$|u'(z)|^2(1-|z|^2)^2 < \varepsilon \quad \text{and} \quad |u(z)|^2(1-|z|^2)^{\alpha+2} < \varepsilon(1-|\varphi(z)|^2)^{\alpha+2}$$

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whenever $r < |z| < 1$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in A_α^2 with $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$ and $f_n \to 0$ uniformly on compact subsets of D. By Lemma 2.7,

$$\|uC_\varphi f_n\|^2 \leq \frac{1}{c}\Big[|u(0)f_n(\varphi(0))|^2 + \int_D |(u \cdot f_n \circ \varphi)'(z)|^2(1-|z|^2)^2\, dA_\alpha(z)\Big], \tag{2-2}$$

where c is the constant defined in Lemma 2.7. Then

$$|(u \cdot f_n \circ \varphi)'(z)|^2 \leq 2(|(u(z)f_n'(\varphi(z))\varphi'(z)|^2 + |u'(z)f_n(\varphi(z))|^2),$$

so that

$$\int_D |(u \cdot f_n \circ \varphi)'(z)|^2(1-|z|^2)^2\, dA_\alpha(z) \leq 2(A_n + B_n + C_n + D_n),$$

where

$$A_n := \int_{\overline{S(0,r)}} |u(z)|^2|f_n'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^2\, dA_\alpha(z),$$

$$B_n := \int_{D\setminus\overline{S(0,r)}} |u(z)|^2|f_n'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^2\, dA_\alpha(z),$$

$$C_n := \int_{\overline{S(0,r)}} |u'(z)|^2|f_n(\varphi(z))|^2(1-|z|^2)^2\, dA_\alpha(z),$$

and

$$D_n := \int_{D\setminus\overline{S(0,r)}} |u'(z)|^2|f_n(\varphi(z))|^2(1-|z|^2)^2\, dA_\alpha(z).$$

Both sets $\{\varphi(0)\}$ and $\varphi(\overline{S(0,r)})$ are compact in D. Thus, there exists some $N \in \mathbb{N}$ for which if $n > N$ and $z \in \overline{S(0,r)}$, then

$$|f_n(\varphi(0))|^2,\ |f_n(\varphi(z))|^2,\ |f_n'(\varphi(z))|^2 < \varepsilon. \tag{2-3}$$

From the continuity of $u\varphi'$ and u' on the compact set $\overline{S(0,r)}$, there is a positive constant M such that

$$|u(z)\varphi'(z)|^2,\ |u'(z)|^2 \leq M$$

for all $z \in \overline{S(0,r)}$. Therefore, if $n > N$, we have

$$A_n + C_n \leq 2M\varepsilon \int_{\overline{S(0,r)}} dA_\alpha(z) \leq 2M\varepsilon \int_D dA_\alpha(z) = 2M\varepsilon. \tag{2-4}$$

The boundedness of C_φ on A_α^2 implies that

$$D_n \leq \varepsilon \int_{D\setminus\overline{S(0,r)}} |f_n(\varphi(z))|^2\, dA_\alpha(z) \leq \varepsilon\|C_\varphi f_n\|^2 \leq \|C_\varphi\|^2\varepsilon. \tag{2-5}$$

It remains to estimate B_n. Note that

$$B_n = (\alpha+1)\int_{D\setminus\overline{S(0,r)}} |u(z)|^2|f_n'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{\alpha+2}\,dA(z)$$
$$\leq (\alpha+1)\varepsilon\int_{D\setminus\overline{S(0,r)}} |f_n'(\varphi(z))|^2|\varphi'(z)|^2(1-|\varphi(z)|^2)^{\alpha+2}\,dA(z).$$

Put $w=\varphi(z)$. By the change-of-variable formula in [11, page 891] and the univalence of φ,

$$\begin{aligned} B_n &\leq (\alpha+1)\varepsilon\int_D |f_n'(w)|^2(1-|w|^2)^{\alpha+2}\,dA(w)\\ &\leq \varepsilon\int_D |f_n'(w)|^2(1-|w|^2)^2\,dA_\alpha(w)\\ &\leq \varepsilon d\|f_n\|^2\\ &\leq \varepsilon d, \end{aligned} \tag{2-6}$$

where d is the constant defined in Lemma 2.7. From Equations (2-2)–(2-6), it now follows that

$$\|uC_\varphi f_n\|^2 \leq \frac{\varepsilon}{c}(|u(0)|^2+4M+2\|C_\varphi\|^2+2d)$$

for all $n>N$. Hence, $\|uC_\varphi f_n\|\to 0$ as $n\to\infty$. □

3. Hilbert–Schmidt weighted composition operators

An important class of compact operators is the Hilbert–Schmidt operators. Let H_1 and H_2 be separable Hilbert spaces and $T: H_1\to H_2$ be a bounded linear operator. Then T is said to be *Hilbert–Schmidt* if $\sum_{k=0}^{\infty}\|Te_k\|_{H_2}^2<\infty$ for some orthonormal basis $\{e_k\}_{k=0}^{\infty}$ of H_1. The value of this sum is independent of the choice of an orthonormal basis. It is well known that every Hilbert–Schmidt operator is compact, but the converse is not necessarily true. In what follows, we take $\{e_k\}_{k=0}^{\infty}$ to be the standard orthonormal basis for A_α^2, as given by Equation (1-1) in Section 1. We also recall a few identities for useful reference:

(a) $1/(1-x)^{\alpha+2}=\sum_{k=0}^{\infty}(\Gamma(\alpha+2+k)/k!\,\Gamma(\alpha+2))x^k$ for $|x|<1$;
(b) $1-|(w-z)/(1-\overline{w}z)|^2=(1-|w|^2)(1-|z|^2)/|1-\overline{w}z|^2$ and
$1-\overline{w}((w-z)/(1-\overline{w}z))=(1-|w|^2)/(1-\overline{w}z)$ for every $w,z\in D$.

Using the criterion for uC_φ to belong to the Schatten class, Čučković and Zhao obtained a characterization for Hilbert–Schmidt weighted composition maps on A_0^2 [4, Corollary 3]. We first generalize this result to the weighted Bergman space and provide a direct proof.

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THEOREM 3.1. *Let uC_φ be a weighted composition operator on A^2_α. Then uC_φ is Hilbert–Schmidt if and only if*

$$\int_D \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}\,dA_\alpha(z) < \infty. \tag{3-1}$$

PROOF. Direct computation gives

$$\begin{aligned}\sum_{k=0}^{\infty} \|uC_\varphi e_k\|^2 &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} \int_D |u(z)|^2 |\varphi(z)|^{2k}\,dA_\alpha(z)\\ &= \int_D |u(z)|^2 \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} |\varphi(z)|^{2k}\,dA_\alpha(z)\\ &= \int_D \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}\,dA_\alpha(z).\end{aligned}$$

Interchanging the summation and integral sums in the second equality is legitimate because the terms are all non-negative. The assertion now follows. □

It is shown in Theorem 2.2 that if $\overline{\varphi(D)} \subset D$, then uC_φ is compact. By Theorem 3.1, uC_φ is also Hilbert–Schmidt. The next result shows that when φ is an automorphism of D, the characterization of when a weighted composition operator is Hilbert–Schmidt becomes simpler.

COROLLARY 3.2. *Let φ be an automorphism of D. Then the weighted composition operator uC_φ is Hilbert–Schmidt on A^2_α if and only if*

$$\int_D \frac{|u(z)|^2}{(1-|z|^2)^2}\,dA(z) < \infty. \tag{3-2}$$

PROOF. By the Schwarz–Pick theorem [3, page 48], we have

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1-|\varphi(0)|}{1+|\varphi(0)|}.$$

Thus,

$$\frac{|u(z)|^2}{(1-|z|^2)^{\alpha+2}} \geq \left(\frac{1}{2}\cdot\frac{1-|\varphi(0)|}{1+|\varphi(0)|}\right)^{\alpha+2} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}. \tag{3-3}$$

Write $\varphi(z) = c(a-z)/(1-\bar{a}z)$, where $a \in D$ and $|c| = 1$. Since

$$1-|\varphi(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} \quad \text{and} \quad |1-\bar{a}z| \geq 1-|a|$$

for every $z \in D$, it follows that

$$\frac{|u(z)|^2}{(1-|z|^2)^{\alpha+2}} \leq \left(\frac{1+|a|}{1-|a|}\right)^{\alpha+2} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}. \tag{3-4}$$

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We obtain the desired result by combining Equations (3-3) and (3-4), Theorem 3.1, and the fact that $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha\, dA(z)$. □

The condition in Equation (3-2) is independent of the parameter α and can be expressed as '$u \in L^2(D, d\tau)$', where τ is the Möbius invariant measure on D defined by

$$d\tau(z) = \frac{1}{(1-|z|^2)^2}\, dA(z). \tag{3-5}$$

Here the term 'invariant measure' is justified by the fact that if φ is an automorphism of D, then

$$\int_D |(f\circ\varphi)'(z)|^2(1-|z|^2)^2\, d\tau(z) = \int_D |f'(z)|^2(1-|z|^2)^2\, d\tau(z)$$

for all analytic functions f on D [17, Section 5.3.1].

Corollary 3.2 is also in contrast to the corresponding result for the Hardy space H^2 of D: if φ is an automorphism, then it follows from [12, Theorem 9] that the *only* Hilbert–Schmidt weighted composition operator on H^2 is the zero operator.

EXAMPLE 3.3. Let $u(z) = 1/(1-z)^{1/4}$ and φ be any automorphism of D. Since, for all $z \in D$ we have $1-|z|^2 \le 2|1-z|$, it follows that

$$\begin{aligned}\int_D \frac{|u(z)|^2}{(1-|z|^2)^2}\, dA(z) &= \int_D \frac{1}{|1-z|^{1/2}(1-|z|^2)^2}\, dA(z)\\ &\ge \frac{1}{4}\int_D \frac{1}{|1-z|^{5/2}}\, dA(z).\end{aligned}$$

By [3, Lemma 7.3], $\int_D (1/|1-z|^{5/2})\, dA(z) = \infty$. According to Corollary 3.2, uC_φ is not Hilbert–Schmidt on A^2_α.

The inequality in Equation (3-3) in fact holds for all analytic self-maps φ of D. Thus, Equation (3-2) provides a sufficient condition for uC_φ to be Hilbert–Schmidt on A^2_α. However, this condition is not necessary, as shown by the following example.

EXAMPLE 3.4. Let $u(z) = (1-z)^{(\alpha+1)/4}$ and $\varphi(z) = 1-(1-z)^{1/2}$. Then $u \in A^2_\alpha$. We claim that uC_φ is Hilbert–Schmidt on A^2_α. Since φ takes D into a polygonal region inscribed in T, there exist positive constants c, δ such that $\delta < 1/2$, and $1-|\varphi(z)| \ge c|1-z|^{1/2}$ on $S(1,\delta)$. Write

$$\begin{aligned}&\int_D \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}\, dA_\alpha(z)\\ &\quad= \int_{S(1,\delta)} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}\, dA_\alpha(z) + \int_{D\setminus S(1,\delta)} \frac{|u(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}\, dA_\alpha(z).\end{aligned}$$

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By choosing $1 + \alpha/2 < \beta < \frac{3}{2} + \alpha$, we have

$$(1 - |\varphi(z)|^2)^{\alpha+2} \geq (1 - |\varphi(z)|)^{\alpha+2} \geq c^{\alpha+2}|1 - z|^{1+\alpha/2} \geq c^{\alpha+2}|1 - z|^{\beta}$$

for $z \in S(1, \delta)$. Thus,

$$\begin{aligned}\int_{S(1,\delta)} \frac{|u(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}}\, dA_\alpha(z) &\leq \frac{1}{c^{\alpha+2}} \int_{S(1,\delta)} \frac{1}{|1 - z|^{\beta-(\alpha+1)/2}}\, dA_\alpha(z) \\ &\leq \frac{1}{c^{\alpha+2}} \int_D \frac{1}{|1 - z|^{\beta-(\alpha+1)/2}}\, dA_\alpha(z) \\ &< \infty,\end{aligned}$$

since $|\beta - (\alpha + 1)/2| < 1 + \alpha/2$. On $D \setminus S(1, \delta)$, the continuity of φ ensures that $|\varphi(z)| \leq d$ for a constant d with $0 < d < 1$. Then

$$\begin{aligned}\int_{D\setminus S(1,\delta)} \frac{|u(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}}\, dA_\alpha(z) &\leq \frac{1}{(1 - d^2)^{\alpha+2}} \int_{D\setminus S(1,\delta)} |u(z)|^2\, dA_\alpha(z) \\ &\leq \frac{1}{(1 - d^2)^{\alpha+2}} \|u\|^2 \\ &< \infty.\end{aligned}$$

From Theorem 3.1, uC_φ is Hilbert–Schmidt. However, since $1 - |z|^2 \leq 2|1 - z|$ on D, we have

$$\int_D \frac{|u(z)|^2}{(1 - |z|^2)^2}\, dA(z) \geq \frac{1}{4} \int_D \frac{1}{|1 - z|^{(3-\alpha)/2}}\, dA(z) = \infty,$$

provided that $|(3 - \alpha)/2| \geq 1$, that is, $-1 < \alpha \leq 1$ or $\alpha \geq 5$.

The rest of this section is devoted to characterizing when $uC_\varphi - vC_\psi$ on A^2_α is Hilbert–Schmidt, where v and ψ are two analytic functions on D such that $\psi(D) \subset D$. This problem originates from the study of the topological structure of the space of (weighted) composition operators on A^2_α. There has been extensive investigation about differences of composition operators on the Hardy space H^2 of D (see for example [1, 6, 16]). The compact difference of two composition operators between weighted Bergman spaces was completely characterized in [9, 14, 15].

In [2], Choe *et al.* topologized the space of composition operators on A^2_α and described its components. By putting

$$\phi(z) = \frac{\psi(z) - \varphi(z)}{1 - \overline{\psi(z)}\varphi(z)}$$

for $z \in D$, they also characterized the Hilbert–Schmidt difference of two composition operators C_φ and C_ψ in terms of $|\phi|$, which is known as the pseudo-hyperbolic distance between φ and ψ. We generalize such characterization to the weighted case and construct an example to illustrate the result.

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THEOREM 3.5. *Let uC_φ and vC_ψ be two weighted composition operators on A^2_α. Then the following statements are equivalent.*

(i) *The operator $uC_\varphi - vC_\psi$ is Hilbert–Schmidt on A^2_α.*
(ii) $|\phi|u/(1-|\varphi|^2)^{1+\alpha/2}$, $v/(1-|\psi|^2)^{1+\alpha/2} - u(1-|\psi|^2)^{1+\alpha/2}/(1-\overline{\psi}\varphi)^{\alpha+2} \in L^2(D, dA_\alpha)$.
(iii) $|\phi|v/(1-|\psi|^2)^{1+\alpha/2}$, $u/(1-|\varphi|^2)^{1+\alpha/2} - v(1-|\varphi|^2)^{1+\alpha/2}/(1-\overline{\varphi}\psi)^{\alpha+2} \in L^2(D, dA_\alpha)$.

PROOF. We first compute $\sum_{k=0}^{\infty} \|(uC_\varphi - vC_\psi)e_k\|^2$:

$$\begin{aligned}\sum_{k=0}^{\infty} &\|(uC_\varphi - vC_\psi)e_k\|^2\\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} \|u\varphi^k - v\psi^k\|^2\\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} \int_D |u\varphi^k - v\psi^k|^2\, dA_\alpha\\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} \int_D [|u|^2|\varphi|^{2k} + |v|^2|\psi|^{2k} - 2\,\mathrm{Re}(u\overline{v}(\varphi\overline{\psi})^k)]\, dA_\alpha.\end{aligned}$$

Interchanging the summation and integral signs in the last equality is valid because all the terms $\Gamma(\alpha+2+k)/k!\ \Gamma(\alpha+2)|u\varphi^k - v\psi^k|^2$ are nonnegative. Then

$$\begin{aligned}\sum_{k=0}^{\infty} &\|(uC_\varphi - vC_\psi)e_k\|^2\\ &= \int_D \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} [|u|^2|\varphi|^{2k} + |v|^2|\psi|^{2k} - 2\,\mathrm{Re}(u\overline{v}(\varphi\overline{\psi})^k)]\, dA_\alpha\\ &= \int_D \Big[|u|^2 \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} |\varphi|^{2k} + |v|^2 \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} |\psi|^{2k}\\ &\quad - 2\,\mathrm{Re}\Big(u\overline{v} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2+k)}{k!\ \Gamma(\alpha+2)} (\varphi\overline{\psi})^k\Big)\Big]\, dA_\alpha\\ &= \int_D \Big[\frac{|u|^2}{(1-|\varphi|^2)^{\alpha+2}} + \frac{|v|^2}{(1-|\psi|^2)^{\alpha+2}} - 2\,\mathrm{Re}\Big(\frac{u\overline{v}}{(1-\varphi\overline{\psi})^{\alpha+2}}\Big)\Big]\, dA_\alpha.\end{aligned}$$

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Since $\varphi = (\psi - \phi)/(1 - \overline{\psi}\phi)$, it follows that

$$\begin{aligned}
&\frac{|u|^2}{(1-|\varphi|^2)^{\alpha+2}} + \frac{|v|^2}{(1-|\psi|^2)^{\alpha+2}} - 2\,\mathrm{Re}\Big(\frac{u\overline{v}}{(1-\varphi\overline{\psi})^{\alpha+2}}\Big)\\
&= |u|^2\Big[\frac{|1-\overline{\psi}\phi|^2}{(1-|\psi|^2)(1-|\phi|^2)}\Big]^{\alpha+2} + \frac{|v|^2}{(1-|\psi|^2)^{\alpha+2}} - 2\,\mathrm{Re}\Big[u\overline{v}\Big(\frac{1-\overline{\psi}\phi}{1-|\psi|^2}\Big)^{\alpha+2}\Big]\\
&= \frac{1}{(1-|\psi|^2)^{\alpha+2}}\Big[|u|^2\Big(\frac{|1-\overline{\psi}\phi|^2}{1-|\phi|^2}\Big)^{\alpha+2} + |v|^2 - 2\,\mathrm{Re}(u\overline{v}(1-\overline{\psi}\phi)^{\alpha+2})\Big]\\
&= \frac{1}{(1-|\psi|^2)^{\alpha+2}}\Big[|u|^2\Big(\frac{|1-\overline{\psi}\phi|^2}{1-|\phi|^2}\Big)^{\alpha+2} + |v-u(1-\overline{\psi}\phi)^{\alpha+2}|^2 - |u|^2|1-\overline{\psi}\phi|^{2\alpha+4}\Big]\\
&= \frac{1}{(1-|\psi|^2)^{\alpha+2}}\Big[|u|^2\Big(\frac{|1-\overline{\psi}\phi|^2}{1-|\phi|^2}\Big)^{\alpha+2}(1-(1-|\phi|^2)^{\alpha+2}) + |v-u(1-\overline{\psi}\phi)^{\alpha+2}|^2\Big]\\
&= \frac{[1-(1-|\phi|^2)^{\alpha+2}]|u|^2}{(1-|\varphi|^2)^{\alpha+2}} + \Big|\frac{v}{(1-|\psi|^2)^{1+\alpha/2}} - \frac{u(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}}\Big|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \|(uC_\varphi - vC_\psi)e_k\|^2\\
&\quad= \int_D \Big[\frac{(1-(1-|\phi|^2)^{\alpha+2})|u|^2}{(1-|\varphi|^2)^{\alpha+2}} + \Big|\frac{v}{(1-|\psi|^2)^{1+\alpha/2}} - \frac{u(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}}\Big|^2\Big]\,dA_\alpha.
\end{aligned}$$

The operator $uC_\varphi - vC_\psi$ is Hilbert–Schmidt on A^2_α if and only if

$$\frac{\sqrt{1-(1-|\phi|^2)^{\alpha+2}}\,u}{(1-|\varphi|^2)^{1+\alpha/2}},\ \frac{v}{(1-|\psi|^2)^{1+\alpha/2}} - \frac{u(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}} \in L^2(D, dA_\alpha).$$

Moreover, write

$$\frac{\sqrt{1-(1-|\phi|^2)^{\alpha+2}}\,u}{(1-|\varphi|^2)^{1+\alpha/2}} = \sqrt{\frac{1-(1-|\phi|^2)^{\alpha+2}}{|\phi|^2}}\cdot\frac{|\phi|u}{(1-|\varphi|^2)^{1+\alpha/2}}.$$

Note that the function $f(x) = [1-(1-x^2)^{\alpha+2}]/x^2$ is continuous and positive on $(0, 1]$. This, in conjunction with the fact $\lim_{x\to 0^+} f(x) = \alpha + 2 > 0$, implies that f is bounded above and away from zero on $(0, 1)$. Thus,

$$\frac{\sqrt{1-(1-|\phi|^2)^{\alpha+2}}\,u}{(1-|\varphi|^2)^{1+\alpha/2}} \in L^2(D, dA_\alpha)$$

if and only if $|\phi|u/(1-|\varphi|^2)^{1+\alpha/2} \in L^2(D, dA_\alpha)$. This establishes the equivalence of statements (i) and (ii).

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Furthermore, upon switching the roles of u, v and the roles of φ, ψ in the preceding calculations, we obtain

$$\sum_{k=0}^{\infty} \|(vC_\psi - uC_\varphi)e_k\|^2 = \int_D \left[\frac{(1-(1-|\phi|^2)^{\alpha+2})|v|^2}{(1-|\psi|^2)^{\alpha+2}} + \left| \frac{u}{(1-|\varphi|^2)^{1+\alpha/2}} - \frac{v(1-|\varphi|^2)^{1+\alpha/2}}{(1-\overline{\varphi}\psi)^{\alpha+2}} \right|^2 \right] dA_\alpha.$$

By a similar argument, statements (i) and (iii) are also equivalent. □

Taking $v = 0$ and $\varphi = \psi$ in the above theorem, we obtain the characterization in Equation (3-1) for a single Hilbert–Schmidt weighted composition operator. There are also two nontrivial consequences of Theorem 3.5. The first one characterizes the Hilbert–Schmidt difference of two composition operators on A^2_α. The second one, which generalizes [2, Corollary 3.8], states that the Hilbert–Schmidt property of the difference of weighted composition operators on a smaller space extends to larger spaces.

COROLLARY 3.6 [2, Corollary 3.7]. *The operator $C_\varphi - C_\psi$ is Hilbert–Schmidt on A^2_α if and only if*

$$\frac{|\phi|}{(1-|\varphi|^2)^{1+\alpha/2}}, \frac{|\phi|}{(1-|\psi|^2)^{1+\alpha/2}} \in L^2(D, dA_\alpha).$$

PROOF. The 'only if' part is evident by taking $u = v = 1$ in Theorem 3.5. To prove the 'if' part, assume that both $|\phi|/(1-|\varphi|^2)^{1+\alpha/2}$ and $|\phi|/(1-|\psi|^2)^{1+\alpha/2}$ are in $L^2(D, dA_\alpha)$. Write

$$\frac{1}{(1-|\psi|^2)^{1+\alpha/2}} - \frac{(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}} = \frac{1-(1-\overline{\psi}\phi)^{\alpha+2}}{(1-|\psi|^2)^{1+\alpha/2}}$$
$$= \frac{1-(1-\overline{\psi}\phi)^{\alpha+2}}{\phi} \cdot \frac{\phi}{(1-|\psi|^2)^{1+\alpha/2}}.$$

By the continuity of the function $g(z) = [1-(1-\overline{w}z)^{\alpha+2}]/z$ $(w \in D)$ on $D \setminus \{0\}$ and the fact that $\lim_{z\to 0} g(z)$ exists (and equals $(\alpha+2)\overline{w}$), the expression $[1-(1-\overline{\psi}\phi)^{\alpha+2}]/\phi$ is bounded on the set $\{z \in D : \phi(z) \neq 0\}$. Thus,

$$\frac{1}{(1-|\psi|^2)^{1+\alpha/2}} - \frac{(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}} \in L^2(D, dA_\alpha)$$

as well. In light of Theorem 3.5, $C_\varphi - C_\psi$ is Hilbert–Schmidt on A^2_α. □

COROLLARY 3.7. *Let uC_φ and vC_ψ be two weighted composition operators on A^2_α. If $uC_\varphi - vC_\psi$ is Hilbert–Schmidt on A^2_α, then $uC_\varphi - vC_\psi$ is also Hilbert–Schmidt on A^2_β for every $\beta > \alpha > -1$.*

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PROOF. Since $uC_\varphi - vC_\psi$ is Hilbert–Schmidt on A^2_α, the following functions are all in $L^2(D, d\tau)$, where τ is the measure defined in Equation (3-5):

(i) $|\phi| u((1-|z|^2)/(1-|\varphi|^2))^{1+\alpha/2}$;
(ii) $|\phi| v((1-|z|^2)/(1-|\psi|^2))^{1+\alpha/2}$;
(iii) $[v/(1-|\psi|^2)^{1+\alpha/2} - u(1-|\psi|^2)^{1+\alpha/2}/(1-\overline{\psi}\varphi)^{\alpha+2}](1-|z|^2)^{1+\alpha/2}$;

since

$$\left(\frac{1-|z|}{1-|\varphi|}\right)^{\beta-\alpha} \le \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\beta-\alpha} \tag{3-6}$$

[3, page 48] and

$$\frac{1}{2}\left(\frac{1-|z|}{1-|\varphi|}\right) \le \frac{1-|z|^2}{1-|\varphi|^2} \le 2\left(\frac{1-|z|}{1-|\varphi|}\right) \tag{3-7}$$

(both Equations (3-6) and (3-7) hold if φ is replaced by ψ), we have

$$\begin{aligned}\int_D |\phi|^2 |u|^2 &\left(\frac{1-|z|^2}{1-|\varphi|^2}\right)^{\beta+2} d\tau(z) \\ &\le 2^{\alpha+\beta+4}\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\beta-\alpha} \int_D |\phi|^2|u|^2\left(\frac{1-|z|^2}{1-|\varphi|^2}\right)^{\alpha+2} d\tau(z) \\ &< \infty.\end{aligned}$$

From the proof of Corollary 3.6,

$$\begin{aligned}&\left|\frac{v}{(1-|\psi|^2)^{1+\beta/2}} - \frac{u(1-|\psi|^2)^{1+\beta/2}}{(1-\overline{\psi}\varphi)^{\beta+2}}\right|^2 \\ &\quad = \left|v\left[\frac{1}{(1-|\psi|^2)^{1+\beta/2}} - \frac{(1-|\psi|^2)^{1+\beta/2}}{(1-\overline{\psi}\varphi)^{\beta+2}}\right] + (v-u)\frac{(1-|\psi|^2)^{1+\beta/2}}{(1-\overline{\psi}\varphi)^{\beta+2}}\right|^2 \\ &\quad \le 2\left[\left|\frac{1-(1-\overline{\psi}\phi)^{\beta+2}}{\phi}\right|^2 \frac{|\phi|^2|v|^2}{(1-|\psi|^2)^{\beta+2}} + |v-u|^2\frac{(1-|\psi|^2)^{\beta+2}}{|1-\overline{\psi}\varphi|^{2\beta+4}}\right] \\ &\quad \le 2\left[M_\beta\frac{|\phi|^2|v|^2}{(1-|\psi|^2)^{\beta+2}} + |v-u|^2\left(\frac{1-|\phi|^2}{1-|\varphi|^2}\right)^{\beta+2}\right],\end{aligned}$$

where M_β is a constant depending on β only. Then

$$\begin{aligned}&\left|\frac{v}{(1-|\psi|^2)^{1+\beta/2}} - \frac{u(1-|\psi|^2)^{1+\beta/2}}{(1-\overline{\psi}\varphi)^{\beta+2}}\right|^2(1-|z|^2)^{\beta+2} \\ &\quad \le 2\left[M_\beta|\phi|^2|v|^2\left(\frac{1-|z|^2}{1-|\psi|^2}\right)^{\beta+2} + (1-|\phi|^2)^{\beta+2}|v-u|^2\left(\frac{1-|z|^2}{1-|\varphi|^2}\right)^{\beta+2}\right].\end{aligned}$$

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Note that

$$\begin{aligned}\int_D |\phi|^2|v|^2\Big(\frac{1-|z|^2}{1-|\psi|^2}\Big)^{\beta+2}\,d\tau(z) &\le 2^{\alpha+\beta+4}\Big(\frac{1+|\psi(0)|}{1-|\psi(0)|}\Big)^{\beta-\alpha}\int_D |\phi|^2|v|^2\Big(\frac{1-|z|^2}{1-|\psi|^2}\Big)^{\alpha+2}\,d\tau(z)\\ &< \infty. \end{aligned} \tag{3-8}$$

Moreover, if we put $c = 2^{\alpha+\beta+4}((1+|\varphi(0)|)/(1-|\varphi(0)|))^{\beta-\alpha}$, then appealing to the proof of Corollary 3.6 again gives

$$\begin{aligned}&(1-|\phi|^2)^{\beta+2}|v-u|^2\Big(\frac{1-|z|^2}{1-|\varphi|^2}\Big)^{\beta+2}\\
&\le c(1-|\phi|^2)^{\alpha+2}|v-u|^2\Big(\frac{1-|z|^2}{1-|\varphi|^2}\Big)^{\alpha+2}\\
&= c\bigg|v\Big[\frac{1}{(1-|\psi|^2)^{1+\alpha/2}}-\frac{(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}}\Big]+(v-u)\frac{(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}}\\
&\quad - v\Big[\frac{1}{(1-|\psi|^2)^{1+\alpha/2}}-\frac{(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}}\Big]\bigg|^2(1-|z|^2)^{\alpha+2}\\
&\le 2c\bigg[\bigg|\frac{v}{(1-|\psi|^2)^{1+\alpha/2}}-\frac{u(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}}\bigg|^2\\
&\quad+\bigg|\frac{1-(1-\overline{\psi}\phi)^{\alpha+2}}{\phi}\bigg|^2\frac{|\phi|^2|v|^2}{(1-|\psi|^2)^{\alpha+2}}\bigg](1-|z|^2)^{\alpha+2}\\
&\le 2c\bigg[\bigg|\frac{v}{(1-|\psi|^2)^{1+\alpha/2}}-\frac{u(1-|\psi|^2)^{1+\alpha/2}}{(1-\overline{\psi}\varphi)^{\alpha+2}}\bigg|^2(1-|z|^2)^{\alpha+2}\\
&\quad+M_\alpha|\phi|^2|v|^2\Big(\frac{1-|z|^2}{1-|\psi|^2}\Big)^{\alpha+2}\bigg],\end{aligned}$$

where M_α is a constant depending on α only. Thus,

$$\int_D (1-|\phi|^2)^{\beta+2}|v-u|^2\Big(\frac{1-|z|^2}{1-|\varphi|^2}\Big)^{\beta+2}\,d\tau(z) < \infty.$$

This, in conjunction with Equation (3-8), implies that

$$\Big[\frac{v}{(1-|\psi|^2)^{1+\beta/2}}-\frac{u(1-|\psi|^2)^{1+\beta/2}}{(1-\overline{\psi}\varphi)^{\beta+2}}\Big](1-|z|^2)^{1+\beta/2}\in L^2(D,d\tau).$$

According to Theorem 3.5, $uC_\varphi - vC_\psi$ is also Hilbert–Schmidt on A^2_β. □

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EXAMPLE 3.8. Let $u(z) = v(z) = z$, $\varphi(z) = az + 1 - a$ and $\psi(z) = \varphi(z) + \varepsilon(1 - \varphi(z))^b$, where a, b, ε are positive constants such that $a \le 1/2$, $b > 2$, and ε is to be determined. Since $\mathrm{Re}(z) < 1$ for $z \in D$, we have

$$\begin{aligned} 1 - |\varphi(z)|^2 - |1 - \varphi(z)|^2 &= 2a - 2a^2 - 2a^2|z|^2 + 2a(2a-1)\mathrm{Re}(z) \\ &\ge 2a - 2a^2 - 2a^2|z|^2 + 2a(2a-1) \\ &= 2a^2(1 - |z|^2) \\ &> 0, \end{aligned}$$

that is,

$$1 - |\varphi(z)|^2 > |1 - \varphi(z)|^2 = a^2|1 - z|^2.$$

Note that $0 < |1 - \varphi(z)| = a|1 - z| < 1$ on D. In what follows, we choose $\varepsilon < 1/4$. Then

$$\begin{aligned} 1 - |\psi(z)|^2 &> 1 - |\varphi(z)|^2 - 2\varepsilon|1 - \varphi(z)|^b - \varepsilon^2|1 - \varphi(z)|^{2b} \\ &> (1 - 2\varepsilon - \varepsilon^2)|1 - \varphi(z)|^2 \\ &> \tfrac{7}{16}a^2|1 - z|^2 \\ &> 0, \end{aligned}$$

or $\psi(D) \subset D$. We claim that $uC_\varphi - vC_\psi$ is Hilbert–Schmidt on A^2_α if $3\alpha/4 + \frac{7}{2} < b < 5\alpha/4 + \frac{9}{2}$. Since

$$\begin{aligned} |1 - \overline{\psi(z)}\varphi(z)| &= |1 - |\psi(z)|^2 + \overline{\psi(z)}(\psi(z) - \varphi(z))| \\ &\ge 1 - |\psi(z)|^2 - \varepsilon|1 - \varphi(z)|^b \\ &> \tfrac{7}{16}a^2|1 - z|^2 - \tfrac{1}{4}a^2|1 - z|^2 \\ &= \tfrac{3}{16}a^2|1 - z|^2, \end{aligned}$$

we have

$$|\phi(z)| = \left|\frac{\psi(z) - \varphi(z)}{1 - \overline{\psi(z)}\varphi(z)}\right| < \frac{16\varepsilon|1 - \varphi(z)|^b}{3a^2|1 - z|^2} < \frac{4}{3}a^{b-2}|1 - z|^{b-2}.$$

Thus,

$$\int_D \frac{|\phi(z)|^2|u(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+2}}\, dA_\alpha(z) \le M_1 \int_D \frac{1}{|1 - z|^{8-2b+2\alpha}}\, dA_\alpha(z);$$

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and from the proof of Corollary 3.6,

$$\begin{aligned}\int_D \left| \frac{v(z)}{(1-|\psi(z)|^2)^{1+\alpha/2}} - \frac{u(z)(1-|\psi(z)|^2)^{1+\alpha/2}}{(1-\overline{\psi(z)}\varphi(z))^{\alpha+2}} \right|^2 dA_\alpha(z) \\ = \int_D \left| \frac{1-(1-\overline{\psi(z)}\phi(z))^{\alpha+2}}{\phi(z)} \right|^2 \frac{|\phi(z)|^2|z|^2}{(1-|\psi(z)|^2)^{\alpha+2}}\, dA_\alpha(z) \\ \le M_2 \int_D \frac{1}{|1-z|^{8-2b+2\alpha}}\, dA_\alpha(z),\end{aligned}$$

where M_1 and M_2 are positive constants depending on a, b, and α. The integral $\int_D (1/|1-z|^{8-2b+2\alpha})\, dA_\alpha(z)$ is finite if and only if $|8-2b+2\alpha| < 1+\alpha/2$, that is, $3\alpha/4 + \frac{7}{2} < b < 5\alpha/4 + \frac{9}{2}$. The claim now follows from Theorem 3.5.

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CHING-ON LO, Division of Science, Engineering and Health Studies,
College of Professional and Continuing Education,
The Hong Kong Polytechnic University, Hung Hom, Hong Kong
e-mail: co.lo@cpce-polyu.edu.hk

ANTHONY WAI-KEUNG LOH, Division of Science,
Engineering and Health Studies, College of Professional and Continuing Education,
The Hong Kong Polytechnic University, Hung Hom, Hong Kong
e-mail: anthony.wk.loh@cpce-polyu.edu.hk