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ZETA FUNCTIONS AND THE LOG BEHAVIOUR OF COMBINATORIAL SEQUENCES

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Abstract In this paper, we use the Riemann zeta function $\zeta(x)$ and the Bessel zeta function $\zeta_\mu(x)$ to study the log behaviour of combinatorial sequences. We prove that $\zeta(x)$ is log-convex for $x > 1$. As a consequence, we deduce that the sequence $\{|B_{2n}|/(2n)!\}_{n\geqslant 1}$ is log-convex, where B_n is the nth Bernoulli number. We introduce the function $\theta(x) = (2\zeta(x)\Gamma(x+1))^{1/x}$, where $\Gamma(x)$ is the gamma function, and we show that $\log\theta(x)$ is strictly increasing for $x \geqslant 6$. This confirms a conjecture of Sun stating that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geqslant 1}$ is strictly increasing. Amdeberhan *et al*. defined the numbers $a_n(\mu) = 2^{2n+1}(n+1)!(\mu+1)_n\zeta_\mu(2n)$ and conjectured that the sequence $\{a_n(\mu)\}_{n\geqslant 1}$ is log-convex for $\mu = 0$ and $\mu = 1$. By proving that $\zeta_\mu(x)$ is log-convex for $x > 1$ and $\mu > -1$, we show that the sequence $\{a_n(\mu)\}_{n\geqslant 1}$ is log-convex for any $\mu > -1$. We introduce another function $\theta_\mu(x)$ involving $\zeta_\mu(x)$ and the gamma function $\Gamma(x)$ and we show that $\log\theta_\mu(x)$ is strictly increasing for $x > 8\mathrm{e}(\mu+2)^2$. This implies that

$$\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)} \quad \text{for } n > 4\mathrm{e}(\mu+2)^2.$$

Based on Dobinski's formula, we prove that

$$\sqrt[n]{B_n} < \sqrt[n+1]{B_{n+1}} \quad \text{for } n \geqslant 1,$$

where B_n is the nth Bell number. This confirms another conjecture of Sun. We also establish a connection between the increasing property of $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ and Hölder's inequality in probability theory.

Keywords: log-convexity; Riemann zeta function; Bernoulli number; Bell number; Bessel zeta function; Narayana number

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1. Introduction

The objective of this paper is to present an analytic approach to the log behaviour of combinatorial sequences.

Let B_n denote the nth Bernoulli number (see [**11, 14**]). Recall that $B_{2n+1} = 0$ for $n \geqslant 1$ and that the B_{2n} alternate in sign for $n \geqslant 1$. We consider the log behaviour of the sequence $\{|B_{2n}|\}_{n\geqslant 1}$. A sequence $\{a_n\}_{n\geqslant 1}$ of real numbers is said to be log-convex if, for $n \geqslant 2$,

$$a_n^2 \leqslant a_{n-1}a_{n+1}.$$

It is well known that

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|, \tag{1.1}$$

where

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

is the Riemann zeta function. By proving that $\zeta(x)$ is log-convex for $x > 1$, we establish the log-convexity of the sequence $\{|B_{2n}|/(2n)!\}_{n\geqslant 1}$. Consequently, the sequence $\{|B_{2n}|\}_{n\geqslant 1}$ is log-convex. Moreover, we introduce the function

$$\theta(x) = (2\zeta(x)\Gamma(x+1))^{1/x}, \tag{1.2}$$

where $\Gamma(x)$ is the gamma function. We show that $\log\theta(x)$ is strictly increasing for $x \geqslant 6$. From (1.1) it can be seen that

$$\sqrt[n]{|B_{2n}|} = \frac{1}{4\pi^2}\theta^2(2n).$$

So we reach the assertion that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geqslant 1}$ is strictly increasing. This confirms a conjecture of Sun [**15**], which has been independently proved by Luca and Stănică [**9**]. We conjecture that $(\log\theta(x))'' < 0$ for $x \geqslant 6$.

Our approach also applies to the sequence of generalized Lasalle numbers. Let C_n denote the nth Catalan number, that is,

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

and let $N_r(z)$ denote the rth Narayana polynomial as given by

$$N_r(z) = \sum_{k=1}^{r} \frac{1}{r}\binom{r}{k-1}\binom{r}{k} z^k.$$

Lasalle [**8**] derived the recurrence relation

$$(z+1)N_r(z) - N_{r+1}(z) = \sum_{n\geqslant 1}(-z)^n\binom{r-1}{2n-1}A_n N_{r-2n+1}(z),$$

where the numbers A_n satisfy the recurrence relation

$$(-1)^{n-1}A_n = C_n + \sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-1}A_j C_{n-j}. \tag{1.3}$$

Let

$$a_n = \frac{2A_n}{C_n}.$$

Lasalle [**8**] showed that $\{a_n\}_{n\geqslant 1}$ is an increasing sequence of positive integers. Amdeberhan *et al*. [**2**] established a connection between a_n and the Bessel zeta functions $\zeta_\mu(x)$.

Recall that for a real number μ, the Bessel function $J_\mu(z)$ of the first kind of order μ is defined by

$$J_\mu(z) = \left(\frac{z}{2}\right)^\mu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\mu+k+1)k!}\left(\frac{z}{2}\right)^{2k}.$$

For $\mu \geqslant -1$, $J_\mu(z)$ has infinitely many positive real zeros $j_{\mu,n}$, where we assume that

$$0 < j_{\mu,1} < j_{\mu,2} < j_{\mu,3} < \cdots$$

(see [**3**, § 4.14]). The Bessel zeta functions $\zeta_\mu(x)$ are defined by

$$\zeta_\mu(x) = \sum_{n=1}^{\infty} \frac{1}{j_{\mu,n}^x}. \tag{1.4}$$

Amdeberhan *et al*. [**2**] found the relation

$$a_n = 2^{2n+1}(n+1)!(n-1)!\zeta_1(2n). \tag{1.5}$$

They also gave the following generalization of a_n for $\mu \geqslant -1$:

$$a_n(\mu) = 2^{2n+1}(n-1)!(\mu+1)_n\zeta_\mu(2n), \tag{1.6}$$

where $(\mu+1)_n = (\mu+1)(\mu+2)\cdots(\mu+n)$.

It is easily seen that $a_n = a_n(1)$. Setting $\mu = 0$ in (1.6), Amdeberhan *et al*. defined the sequence $\{b_n\}_{n\geqslant 1}$ as given by

$$b_n = \tfrac{1}{2}a_n(0) = 2^{2n}n!(n-1)!\zeta_0(2n). \tag{1.7}$$

Note that this sequence has been studied by Carlitz [**6**]. It is listed as Sequence A002190 in [**10**].

Amdeberhan *et al*. conjectured that the sequences $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ are log-convex. We show that $\zeta_\mu(x)$ is log-convex for $x > 1$. This implies that the sequence $\{a_n(\mu)\}_{n\geqslant 1}$ is log-convex for any $\mu > -1$. This confirms the above conjectures, which have been independently proved by Wang and Zhu [**16**].

Moreover, we define the function

$$\theta_\mu(x) = \left(\frac{2}{\mu!}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\mu+1\right)\zeta_\mu(x)\right)^{1/x}. \tag{1.8}$$

It can be easily checked that

$$4\theta_\mu^2(2n) = \sqrt[n]{a_n(\mu)}. \tag{1.9}$$

We show that $\log\theta_\mu(x)$ is strictly increasing for $x > 8\mathrm{e}(\mu+2)^2$. This leads to the increasing property that

$$\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)} \tag{1.10}$$

for $n > 4\mathrm{e}(\mu+2)^2$. We note that for $\mu = 0$ and $\mu = 1$, (1.10) has been independently proved by Wang and Zhu [**16**].

Owing to the formula of Dobinski, we may use our analytic approach to study the log behaviour of Bell numbers. Let B_n be the nth Bell number, that is, the number of partitions of $\{1,2,\dots,n\}$ (see [**5**] and [**12**]). Notice that we have adopted the same notation B_n for both Bell numbers and Bernoulli numbers. Recall that Dobinski's formula for the Bell numbers states that

$$B_n = \frac{1}{\mathrm{e}}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

For $x>0$, we define

$$B(x) = \frac{1}{\mathrm{e}}\sum_{k=0}^{\infty}\frac{k^x}{k!} \tag{1.11}$$

so that we have $B_n = B(n)$ whenever n is a non-negative integer.

We show that $\log B(x)^{1/x}$ is increasing for $x \geqslant 1$. This implies that the sequence $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ is increasing, as conjectured by Sun [**15**]. We conjecture that $(\log B(x)^{1/x})'' < 0$ for $x \geqslant 1$. In the final section, we give a probabilistic proof of the increasing property of the sequence $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ by using Hölder's inequality.

2. The log-convexity of Bernoulli numbers

To prove the log-convexity of Bernoulli numbers, we consider the log behaviour of the Riemann zeta function $\zeta(x)$ for $x>1$. Recall that a positive function f is called log-convex on a real interval $I=[a,b]$ if, for all $x,y\in[a,b]$ and $\lambda\in[0,1]$,

$$f(\lambda x + (1-\lambda)y) \leqslant f(x)^{\lambda} f(y)^{1-\lambda} \tag{2.1}$$

(see, for example, [**4**]). It is known that a positive function f is log-convex if and only if $(\log f(x))'' \geqslant 0$. So, if

$$(\log \zeta(x))'' > 0 \tag{2.2}$$

for $x>1$, then we can deduce that $\zeta(x)$ is log-convex for $x>1$.

Lemma 2.1. *The Riemann zeta function $\zeta(x)$ is log-convex for $x>1$.*

Proof. Clearly, condition (2.2) is equivalent to

$$\zeta(x)\zeta''(x) - (\zeta'(x))^2 > 0. \tag{2.3}$$

Since $\zeta(x)$ converges for $x>1$, we find that, for $x>1$,

$$\begin{aligned}\zeta(x)\zeta''(x) - (\zeta'(x))^2 &= \sum_{m=1}^{\infty}\frac{1}{m^x}\sum_{n=1}^{\infty}\frac{(\log n)^2}{n^x} - \sum_{m=1}^{\infty}\frac{\log m}{m^x}\sum_{n=1}^{\infty}\frac{\log n}{n^x}\\ &= \sum_{n>m\geqslant 1}\frac{(\log n)^2 + (\log m)^2 - 2\log m\log n}{(mn)^x}\\ &= \sum_{n>m\geqslant 1}\frac{(\log n - \log m)^2}{(mn)^x},\end{aligned}$$

which is positive. This completes the proof. □

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The log-convexity of $\zeta(x)$ enables us to deduce the following property of Bernoulli numbers.

Theorem 2.2. *The sequence* $\{|B_{2n}|/(2n)!\}_{n\geqslant 1}$ *is log-convex.*

Proof. Since $\zeta(x)$ is log-convex, setting $x = 2n-2$, $y = 2n+2$ and $\lambda = 1/2$ in the defining relation (2.1), we find that

$$\zeta(2n-2)\zeta(2n+2) \geqslant \zeta(2n)^2. \tag{2.4}$$

Invoking (1.1), we obtain that

$$\left(\frac{|B_{2n}|}{(2n)!}\right)^2 \leqslant \frac{|B_{2n-2}|}{(2n-2)!}\frac{|B_{2n+2}|}{(2n+2)!}.$$

This completes the proof. □

Since $((2n)!)^2 < (2n-2)!(2n+2)!$ for $n \geqslant 1$, the above theorem implies the following property.

Corollary 2.3. *The sequence* $\{|B_{2n}|\}_{n\geqslant 1}$ *is log-convex.*

3. The log behaviour of $\theta(x)$

In this section we consider the log behaviour of the function

$$\theta(x) = (2\zeta(x)\Gamma(x+1))^{1/x}.$$

We begin with the following monotone property of $\log\theta(x)$.

Theorem 3.1. $\log\theta(x)$ *is strictly increasing for* $x \geqslant 6$.

Proof. To prove that $\log\theta(x)$ is increasing for $x \geqslant 6$, we aim to show that

$$(\log\theta(x))' > 0 \tag{3.1}$$

for $x \geqslant 6$. Let

$$g(x) = 2\zeta(x)\Gamma(x+1).$$

We then have

$$\theta(x) = g(x)^{1/x}$$

and

$$(\log\theta(x))' = \frac{1}{x}\left(\frac{g'(x)}{g(x)} - \frac{\log g(x)}{x}\right).$$

Thus, (3.1) can be rewritten as

$$\frac{g'(x)}{g(x)} > \frac{\log g(x)}{x}$$

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for $x \geqslant 6$. Since $\zeta(x)$ and $\Gamma(x)$ are continuous and differentiable on $(1, \infty)$, so is $g(x)$ on $(1, \infty)$. Applying the mean value theorem to $\log g(x)/x$, it can be shown that there exists t in $(2, x)$ such that

$$\frac{g(t)'}{g(t)} > \frac{\log g(x)}{x}. \tag{3.2}$$

Since $\zeta(2) = \pi^2/6$ and $\Gamma(3) = 2$, we find that

$$\log g(2) = \log(2\zeta(2)\Gamma(3)) = \log \frac{2\pi^2}{3} < 2. \tag{3.3}$$

On the other hand, for $x \geqslant 6$, it is easily seen that $\zeta(x) > 1$ and $\Gamma(x+1) > \mathrm{e}^x$. It follows that

$$\log g(x) = \log 2 + \log \zeta(x) + \log \Gamma(x+1) > x. \tag{3.4}$$

In view of (3.3) and (3.4), we deduce that for $x \geqslant 6$,

$$\frac{\log g(x)}{x} = \frac{(1-2/x)\log g(x)}{(1-2/x)x} < \frac{\log g(x) - 2}{x-2} < \frac{\log g(x) - \log g(2)}{x-2}. \tag{3.5}$$

Applying the mean value theorem to $\log g(x)$, we see that there exists $t \in (2, x)$ such that

$$(\log g(t))' = \frac{\log g(x) - \log g(2)}{x-2}, \tag{3.6}$$

that is,

$$\frac{g'(t)}{g(t)} = \frac{\log g(x) - \log g(2)}{x-2}. \tag{3.7}$$

Combining (3.5) and (3.7), we get (3.2).

We now proceed to show that

$$\frac{g(x)'}{g(x)} > \frac{g(t)'}{g(t)}. \tag{3.8}$$

Clearly, (3.8) is equivalent to

$$\left(\frac{g'(y)}{g(y)}\right)' > 0. \tag{3.9}$$

By the definition of $g(x)$, we have

$$\left(\frac{g'(y)}{g(y)}\right)' = (\log g(y))'' = (\log \Gamma(y+1))'' + (\log \zeta(y))''.$$

It is known that $(\log \Gamma(y+1))'' > 0$ for $y > 1$ (see [**3**, Theorem 1.2.5]). On the other hand, in the proof of Lemma 2.1 we have shown that $(\log \zeta(y))'' > 0$. This proves (3.9). In other words, $g'(y)/g(y)$ is strictly increasing for $y > 1$. Thus, for $2 < t < x$, (3.8) holds.

Combining (3.2) and (3.8), we deduce that for $x \geqslant 6$,

$$\frac{g'(x)}{g(x)} - \frac{\log g(x)}{x} > \frac{g'(x)}{g(x)} - \frac{g'(t)}{g(t)} > 0.$$

Hence, $(\log \theta(x))' > 0$ for $x \geqslant 6$. This completes the proof. □

From the log behaviour of $\theta(x)$, we are led to an affirmative answer to a conjecture of Sun [**15**].

Corollary 3.2. *The sequence* $\{\sqrt[n]{|B_{2n}|}\}_{n\geqslant 1}$ *is strictly increasing.*

Proof. From (1.1), we see that for $n \geqslant 1$,

$$\sqrt[n]{|B_{2n}|} = \frac{1}{4\pi^2}\sqrt[n]{2\zeta(2n)(2n)!} = \frac{1}{4\pi^2}\theta^2(2n). \tag{3.10}$$

Since $\log\theta(x)$ is strictly increasing for $x \geqslant 6$, we find that $\theta(x)$ is also strictly increasing for $x \geqslant 6$. It follows from (3.10) that $\sqrt[n]{|B_{2n}|}$ is strictly increasing for $n \geqslant 3$. On the other hand, it is easily checked that

$$|B_2| < \sqrt{|B_4|} < \sqrt[3]{|B_6|}.$$

This completes the proof. □

The conjecture of Sun was independently proved by Luca and Stănică [**9**]. In fact, they proved that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geqslant 1}$ is log-concave, which was also conjectured by Sun [**15**].

We pose the following conjecture concerning the function $\theta(x)$. If it is true, then it implies that the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geqslant 1}$ is log-concave.

Conjecture 3.3. *The function $\theta(x)$ is log-concave for $x \geqslant 6$, that is, for $x \geqslant 6$,* $(\log f(x))'' < 0$.

4. The log behaviour of the sequence $\{a_n(\mu)\}_{n\geqslant 1}$

In this section, we study the log behaviour of the sequence $\{a_n(\mu)\}_{n\geqslant 1}$. We begin with the log behaviour of the Bessel zeta functions $\zeta_\mu(x)$.

Lemma 4.1. *For $\mu > -1$, the Bessel zeta function $\zeta_\mu(x)$ is log-convex for $x > 1$.*

Proof. We proceed to show that for $x > 1$,

$$(\log\zeta_\mu(x))'' > 0,$$

or, equivalently,

$$\zeta_\mu(x)\zeta_\mu''(x) - (\zeta_\mu'(x))^2 > 0. \tag{4.1}$$

By the convergence of $\zeta_\mu(x)$, it is easily seen that

$$\zeta_\mu'(x) = -\sum_{n=1}^{\infty}\frac{\log j_{\mu,n}}{j_{\mu,n}^x}$$

and

$$\zeta_\mu''(x) = \sum_{n=1}^{\infty}\frac{(\log j_{\mu,n})^2}{j_{\mu,n}^x}.$$

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Hence,

$$\begin{aligned}\zeta_\mu(x)\zeta_\mu''(x) - (\zeta_\mu'(x))^2 &= \sum_{m=1}^{\infty} \frac{1}{j_{\mu,m}^x} \sum_{n=1}^{\infty} \frac{(\log j_{\mu,n})^2}{j_{\mu,n}^x} - \sum_{m=1}^{\infty} \frac{\log j_{\mu,m}}{j_{\mu,m}^x} \sum_{n=1}^{\infty} \frac{\log j_{\mu,n}}{j_{\mu,n}^x} \\ &= \sum_{n>m\geqslant 1} \frac{(\log j_{\mu,m})^2 + (\log j_{\mu,n})^2 - 2(\log j_{\mu,m})(\log j_{\mu,n})}{j_{\mu,m}^x j_{\mu,n}^x} \\ &= \sum_{n>m\geqslant 1} \frac{(\log j_{\mu,m} - \log j_{\mu,n})^2}{j_{\mu,m}^x j_{\mu,n}^x},\end{aligned}$$

which is positive. This completes the proof. □

Setting $f(x) = \zeta_\mu(x)$, $x = 2n-2$, $y = 2n+2$ and $\lambda = 1/2$ in the defining relation (2.1) of a log-convex function, we obtain that for $\mu > -1$,

$$\zeta_\mu(2n-2)\zeta_\mu(2n+2) > \zeta_\mu(2n)^2. \tag{4.2}$$

This yields that the sequence $\{\zeta_\mu(2n)\}_{n\geqslant 1}$ is log-convex for $\mu > -1$. On the other hand, it is easily checked that the sequence $\{2^{2n+1}(n+1)!(\mu+1)_n\}_{n\geqslant 1}$ is log-convex for $\mu > -1$. Notice that for two positive log-convex sequences $\{u_n\}_{n\geqslant 1}$ and $\{v_n\}_{n\geqslant 1}$, the sequence $\{u_n v_n\}_{n\geqslant 1}$ is also log-convex. So we arrive at the following property.

Theorem 4.2. *The sequence $\{a_n(\mu)\}_{n\geqslant 1}$ is log-convex for $\mu > -1$.*

For $\mu = 0$ and $\mu = 1$, Theorem 4.2 gives affirmative answers to the two conjectures of Amdeberhan *et al.* [**2**] on the log-convexity of the sequences $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$, where $a_n = a_n(1)$ and $b_n = \frac{1}{2}a_n(0)$.

Next we consider the monotone property of the sequence $\{\sqrt[n]{a_n(\mu)}\}_{n\geqslant 1}$ for $\mu > 0$.

Theorem 4.3. *For $\mu > 0$, the sequence $\{\sqrt[n]{a_n(\mu)}\}_{n\geqslant 1}$ is increasing for $n > 4\mathrm{e}(\mu+2)^2$.*

To prove this theorem, we introduce the function

$$\theta_\mu(x) = \left(\frac{2}{\mu!}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\mu+1\right)\zeta_\mu(x)\right)^{1/x},$$

which has the following monotone property.

Theorem 4.4. *For $\mu \geqslant 0$, $\log\theta_\mu(x)$ is strictly increasing for $x > 8\mathrm{e}(\mu+2)^2$.*

Proof. Assume that $\mu \geqslant 0$. To prove the monotone property in the theorem, we aim to show that for $x > 8\mathrm{e}(\mu+2)^2$,

$$(\log\theta_\mu(x))' > 0. \tag{4.3}$$

Let

$$h(x) = \frac{2}{\mu!}\Gamma(x/2)\Gamma(x/2+\mu+1)\zeta_\mu(x). \tag{4.4}$$

Recalling the definition of $\theta_\mu(x)$ as given by (1.8), we have

$$\theta_\mu(x) = h(x)^{1/x}$$

and

$$\log\theta_\mu(x) = \frac{1}{x}\log h(x).$$

It follows that

$$(\log\theta_\mu(x))' = \frac{1}{x}\left(\frac{h'(x)}{h(x)} - \frac{\log h(x)}{x}\right). \tag{4.5}$$

Since $\zeta_\mu(x)$ and $\Gamma(x)$ are continuous and differentiable on $(1,\infty)$, so is $h(x)$. We shall apply the mean value theorem to $\log h(x)$ on $[2,x]$, where $x > 8\mathrm{e}(\mu+2)^2$ and $\mu > -1$. To this end, we need to show that $h(2) < 1$ and $h(x) > 1$ for $\mu > -1$ and $x > 8\mathrm{e}(\mu+2)^2$.

Recalling the definition of $h(x)$ as given by (4.4), we get

$$h(2) = \frac{2}{\mu!}\Gamma(1)\Gamma(\mu+2)\zeta_\mu(2),$$

where

$$\zeta_\mu(2) = \frac{1}{4(\mu+1)},$$

$\Gamma(1) = 1$ and $\Gamma(\mu+2) = (\mu+1)!$. Hence,

$$h(2) = \frac{2}{\mu!}(\mu+1)!\frac{1}{4(\mu+1)}, \tag{4.6}$$

so $h(2) < 1$.

It remains to show that $h(x) > 1$ for $\mu > -1$ and $x > 8\mathrm{e}(\mu+2)^2$. Recall that

$$j_{\mu,1} < (\mu+1)^{1/2}((\mu+2)^{1/2}+1) \tag{4.7}$$

for $\mu > -1$ (see [**7**]). It follows that for $\mu > -1$,

$$j_{\mu,1} < 2(\mu+2). \tag{4.8}$$

Therefore, we obtain that for $\mu > -1$,

$$\zeta_\mu(x) = \sum_{n=1}^{\infty}\frac{1}{j_{\mu,n}^x} > \frac{1}{j_{\mu,1}^x} > \frac{1}{2^x(\mu+2)^x}. \tag{4.9}$$

On the other hand, it is known that for $x \geqslant 0$,

$$\Gamma(x) > \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x \tag{4.10}$$

(see [**1**]). Combining (4.9) and (4.10), we deduce that for $x > 2$ and $\mu > -1$,

$$2\Gamma\left(\frac{x}{2}\right)\zeta_\mu(x) > 2\sqrt{\pi x}\left(\frac{x}{8\mathrm{e}(\mu+2)^2}\right)^{x/2}.$$

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Consequently, for $\mu > -1$ and $x > 8\mathrm{e}(\mu+2)^2$, we obtain that

$$2\Gamma(\tfrac{1}{2}x)\zeta_\mu(x) > 2\sqrt{\pi x} > 1. \tag{4.11}$$

Clearly, for $x > 0$ we have

$$\frac{\Gamma(x/2+\mu+1)}{\mu!} > 1. \tag{4.12}$$

In view of (4.11) and (4.12), we find that for $\mu > -1$ and $x > 8\mathrm{e}(\mu+2)^2$,

$$h(x) = \frac{2}{\mu!}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\mu+1\right)\zeta_\mu(x) > 1, \tag{4.13}$$

as claimed.

Next we proceed to prove that there exists t in $(2, x)$ such that

$$\frac{h'(t)}{h(t)} > \frac{\log h(x)}{x}. \tag{4.14}$$

By the mean value theorem applied to $\log h(x)$ on $[2, x]$, there exists $t \in (2, x)$ such that

$$\frac{h'(t)}{h(t)} = (\log h(t))' = \frac{\log h(x) - \log h(2)}{x-2}. \tag{4.15}$$

On the other hand, we have shown that $h(2) < 1$ and $h(x) > 1$ for $\mu > -1$ and $x > 8\mathrm{e}(\mu+2)^2$. Consequently, we have $\log h(2) < 0$ and $\log h(x) > 0$. Note that for $\mu > -1$ and $x > 8\mathrm{e}(\mu+2)^2$, we have $x > 2$. Hence,

$$\frac{\log h(x)}{x} < \frac{\log h(x) - \log h(2)}{x-2}. \tag{4.16}$$

Combining (4.15) and (4.16), we obtain (4.14).

Moreover, it can be shown that

$$\frac{h'(x)}{h(x)} > \frac{h'(t)}{h(t)}. \tag{4.17}$$

We claim that for $y > 2$,

$$\left(\frac{h'(y)}{h(y)}\right)' > 0. \tag{4.18}$$

By the definition of $h(x)$ as given by (4.4), we have

$$\begin{aligned}\left(\frac{h'(y)}{h(y)}\right)' &= (\log h(y))'' \\ &= (\log\Gamma(y/2))'' + (\log\Gamma(y/2+\mu+1))'' + (\log\zeta_\mu(x))''.\end{aligned}$$

It is known that $(\log\Gamma(y))'' > 0$ for $y > 1$ (see [**3**, Theorem 1.2.5]). Thus, $(\log\Gamma(y/2))'' > 0$ and $(\log\Gamma(y/2+\mu+1))'' > 0$ for $y > 2$. Moreover, in the proof

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of Lemma 4.1, we have shown that $(\log \zeta_\mu(y))'' > 0$. This proves (4.18). In other words, $h'(y)/h(y)$ is strictly increasing for $y > 2$. Thus, for $2 < t < x$, (4.17) holds.

Combining (4.14) and (4.17), for $\mu > -1$ and $x > 8\mathrm{e}(\mu+2)^2$ we find that

$$\frac{h'(x)}{h(x)} - \frac{\log h(x)}{x} > \frac{h'(x)}{h(x)} - \frac{h'(t)}{h(t)} > 0.$$

Hence, (4.3) follows from (4.5). This completes the proof. □

In view of (1.6), it can be checked that

$$\sqrt[n]{a_n(\mu)} = 4\theta_\mu(2n)^2. \tag{4.19}$$

Thus, Theorem 4.4 implies that for any $\mu \geqslant 0$ and $n > 4\mathrm{e}(\mu+2)^2$, we have $\sqrt[n]{a_n(\mu)} < \sqrt[n+1]{a_{n+1}(\mu)}$.

For $\mu = 1$, it can be verified that

$$\sqrt[n]{a_n(1)} < \sqrt[n+1]{a_{n+1}(1)} \quad \text{for } 2 \leqslant n \leqslant 108.$$

In the meantime, for $\mu = 1$, Theorem 4.4 states that

$$\sqrt[n]{a_n(1)} < \sqrt[n+1]{a_{n+1}(1)} \quad \text{for } n > 101.$$

Thus, we reach the following assertion.

Theorem 4.5. *The sequence* $\{\sqrt[n]{a_n}\}_{n\geqslant 2}$ *is strictly increasing.*

For $\mu = 0$, it can be verified that

$$\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)} \quad \text{for } 2 \leqslant n \leqslant 48.$$

Meanwhile, for $\mu = 0$, Theorem 4.4 states that

$$\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)} \quad \text{for } n > 45.$$

So we have

$$\sqrt[n]{a_n(0)} < \sqrt[n+1]{a_{n+1}(0)} \quad \text{for } n \geqslant 2.$$

Since $b_n = \frac{1}{2}a_n(0)$, we have for $n \geqslant 2$,

$$\sqrt[n]{b_n} = \frac{\sqrt[n]{a_n(0)}}{\sqrt[n]{2}} < \frac{\sqrt[n+1]{a_{n+1}(0)}}{\sqrt[n+1]{2}} = \sqrt[n+1]{b_{n+1}}.$$

Thus, we have the following monotone property.

Theorem 4.6. *The sequence* $\{\sqrt[n]{b_n}\}_{n\geqslant 2}$ *is strictly increasing.*

Note that Wang and Zhu [**16**] independently proved the log-convexity of $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ and the increasing properties of $\{\sqrt[n]{a_n}\}_{n\geqslant 1}$ and $\{\sqrt[n]{b_n}\}_{n\geqslant 1}$.

5. The log behaviour of Bell numbers

In this section, we consider the log behaviour of Bell numbers, which are also denoted by B_n. Recall that the function $B(x)$ is defined by

$$B(x) = \frac{1}{\mathrm{e}} \sum_{k=0}^{\infty} \frac{k^x}{k!}.$$

Lemma 5.1. *The function $B(x)$ is log-convex for $x > 1$.*

Proof. We proceed to show that

$$(\log B(x))'' > 0,$$

that is,

$$B(x)B''(x) - (B'(x))^2 > 0. \tag{5.1}$$

For $x \geqslant 1$, we have

$$B'(x) = \frac{1}{\mathrm{e}} \sum_{n=0}^{\infty} \frac{n^x \log n}{n!}$$

and

$$B''(x) = \frac{1}{\mathrm{e}} \sum_{n=0}^{\infty} \frac{n^x (\log n)^2}{n!}.$$

Thus, for $x > 1$, we have

$$\begin{aligned} B(x)B''(x) - (B'(x))^2 &= \frac{1}{\mathrm{e}^2} \sum_{m=0}^{\infty} \frac{m^x}{m!} \sum_{n=0}^{\infty} \frac{n^x (\log n)^2}{n!} - \frac{1}{\mathrm{e}^2} \sum_{m=0}^{\infty} \frac{m^x \log m}{m!} \sum_{n=0}^{\infty} \frac{n^x \log n}{n!} \\ &= \frac{1}{\mathrm{e}^2} \sum_{n>m\geqslant 0} \frac{m^x n^x}{m!n!} ((\log m)^2 + (\log n)^2 - 2 \log m \log n) \\ &= \frac{1}{\mathrm{e}^2} \sum_{n>m\geqslant 0} \frac{m^x n^x}{m!n!} (\log n - \log m)^2, \end{aligned}$$

which is positive. This completes the proof. □

We now turn to the log behaviour of the function $B(x)^{1/x}$.

Theorem 5.2. $\log B(x)^{1/x}$ *is strictly increasing for $x \geqslant 1$.*

Proof. To prove that $\log B(x)^{1/x}$ is strictly increasing, we wish to show that

$$(\log B(x)^{1/x})' > 0. \tag{5.2}$$

Since

$$(\log B(x)^{1/x})' = \frac{1}{x} \left(\frac{B'(x)}{B(x)} - \frac{\log B(x)}{x} \right),$$

(5.2) can be rewritten as

$$\frac{B'(x)}{B(x)} > \frac{\log B(x)}{x}. \tag{5.3}$$

We claim that there exists t in $(1, x)$ such that

$$\frac{B'(t)}{B(t)} > \frac{\log B(x)}{x}. \tag{5.4}$$

Since $B(1) = 1$ and $B(x) > 1$ for $x > 1$, by the mean value theorem with respect to $\log B(x)$ on $[1, x]$, there exists $t \in (1, x)$ such that

$$\frac{B'(t)}{B(t)} = \frac{\log B(x) - \log B(1)}{x - 1} = \frac{\log B(x)}{x - 1}. \tag{5.5}$$

Since $x > 1$, we have

$$\frac{\log B(x)}{x - 1} > \frac{\log B(x)}{x}. \tag{5.6}$$

Combining (5.5) and (5.6), we obtain (5.4).

Next we show that for $x > t > 1$,

$$\frac{B'(x)}{B(x)} > \frac{B'(t)}{B(t)}. \tag{5.7}$$

In fact, by Lemma 5.1, we see that for $y \geqslant 1$,

$$\left(\frac{B'(y)}{B(y)}\right)' = (\log B(y))'' > 0.$$

This implies that $B'(y)/B(y)$ is strictly increasing for $y > 1$. This proves (5.7).

Combining (5.4) and (5.7), we obtain (5.3). This completes the proof. □

Since $B(n) = B_n$ whenever n is a positive integer, Theorem 5.2 implies the following monotone property conjectured by Sun [**15**].

Corollary 5.3. *The sequence* $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ *is strictly increasing.*

The above property was independently obtained by Wang and Zhu [**16**] via a different approach. We pose the following conjecture that implies the conjecture of Sun [**15**] stating that the sequence $\{\sqrt[n]{B_n}\}_{n\geqslant 1}$ is log-concave.

Conjecture 5.4. *The function* $B(x)^{1/x}$ *is log-concave for* $x \geqslant 1$*, that is, for* $x > 1$*,* $(\log B(x)^{1/x})'' < 0$.

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6. A connection to Hölder's inequality

In this section, we give a derivation of the monotone property of the function $B(x)^{1/x}$ in Theorem 5.2 by applying Hölder's inequality in probability theory. In fact, it can be shown that the condition $1 < x < y$ in Theorem 5.2 can be relaxed to $0 < x < y$.

Let Z be a discrete random variable with Poisson distribution as given by

$$P(Z = k) = \frac{1}{\mathrm{e}}\frac{1}{k!}.$$

From Dobinski's formula, it is easily checked that $B(x) = E[Z^x]$. Hölder's inequality states that for real-valued random variables U and V, and positive numbers p and q satisfying $1/p + 1/q = 1$, we have

$$E[|UV|] \leqslant E[|U|^p]^{1/p} E[|V|^q]^{1/q}$$

and the equality holds if and only if there exist constants $\alpha, \beta > 0$ such that $\alpha|U|^p = \beta|V|^q$ or $E[|U|^p] = 0$ or $E[|V|^q] = 0$ (see, for example, [**13**]). For $0 < x < y$, we set $p = y/x$ and set $U = Z^x$ and $V = 1$. It is not hard to see that in this case Hölder's inequality is strict. Hence, we obtain that

$$E[Z^x]^{1/x} < E[Z^y]^{1/y},$$

which can be restated as follows.

Theorem 6.1. *For $0 < x < y$, we have $B(x)^{1/x} < B(y)^{1/y}$.*

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