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CONTINUITY OF THE VARIATIONAL EIGENVALUES OF THE p-LAPLACIAN WITH RESPECT TO p

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Abstract

In this note it is shown that a result of Champion and De Pascale ['Asymptotic behavior of nonlinear eigenvalue problems involving p-Laplacian type operators', *Proc. Roy. Soc. Edinburgh Sect. A* **137** (2007), 1179–1195] implies that the variational eigenvalues of the p-Laplacian are continuous with respect to p.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. We consider the following boundary value problem:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < +\infty$, $\lambda \in \mathbb{R}$ and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian. A number $\lambda \in \mathbb{R}$ is called an *eigenvalue* if there exists a function $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ (called an *eigenfunction*) such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \lambda \int_\Omega |u|^{p-2} u v$$

for every $v \in W_0^{1,p}(\Omega)$. One can prove (see [3]) the existence of a sequence of eigenvalues $\{\lambda_k(p; \Omega)\}_{k=1}^{+\infty}$ such that

$$0 < \lambda_1(p; \Omega) < \lambda_2(p; \Omega) \leq \cdots \leq \lambda_k(p; \Omega) \to +\infty$$

as $k \to +\infty$. These eigenvalues are characterized by (2.1) below and are often referred to as *variational eigenvalues*. In the literature one can find investigations of the behaviour of the variational eigenvalues for varying p; the asymptotic behaviour for $p \to 1$ was studied in [7, 9], while the case $p \to +\infty$ was considered in [5, 6].

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In this note we consider the issue of the continuity of the variational eigenvalues with respect to p. In [4] it was shown that $\lambda_1(p;\,\Omega)$ and $\lambda_2(p;\,\Omega)$ depend continuously on p, but the continuity of the remaining eigenvalues was stated as an open problem. As we will see in Section 3, this can actually be shown by means of the more general results of [2].

2. Definitions and preliminary results

2.1. Variational eigenvalues. Let $A \subset W_0^{1,p}(\Omega)$. The *Krasnoselskii genus* of A is defined as

$$\gamma(A) := \min\{m \in \mathbb{N} \mid \exists \varphi : A \to \mathbb{R}^m \setminus \{0\},\ \varphi \text{ continuous and odd}\}.$$

For $k \in \mathbb{N}$ we define

$$\begin{aligned}\Gamma_k := \{A \subset W_0^{1,p}(\Omega) \cap \{\|v\|_p = 1\} \mid\ & A \text{ symmetric, compact in } W_0^{1,p}(\Omega),\\ & \text{nonempty},\ \gamma(A) \geq k\}.\end{aligned}$$

Then, the numbers

$$\lambda_k(p;\,\Omega) := \inf_{A \in \Gamma_k} \sup_{v \in A} \int_\Omega |\nabla v|^p \tag{2.1}$$

are eigenvalues of the p-Laplacian satisfying

$$0 < \lambda_1(p;\,\Omega) < \lambda_2(p;\,\Omega) \leq \cdots \leq \lambda_k(p;\,\Omega) \to +\infty$$

as $k \to +\infty$ (see for instance [8, Corollary 3.1]). It is not known whether other eigenvalues exist, unless $p = 2$ or $n = 1$ where the answer is negative; in any case, there does not exist any eigenvalue between $\lambda_1(p;\,\Omega)$ and $\lambda_2(p;\,\Omega)$ (see [1]).

2.2. Γ-convergence. Let X be a metric space. We say that a sequence of functionals $F_j : X \to [-\infty, +\infty]$ Γ-*converges* to $F_\infty : X \to [-\infty, +\infty]$ for $j \to \infty$ if for every $x \in X$ we have the following.

(i) (*liminf inequality*) For every sequence $\{x_j\}_{j=1}^\infty$ converging to x we have

$$F_\infty(x) \leq \liminf_{j \to +\infty} F_j(x_j).$$

(ii) (*limsup inequality*) There exists a sequence $\{x_j\}_{j=1}^\infty$ converging to x (called a *recovery sequence*) such that

$$F_\infty(x) \geq \limsup_{j \to +\infty} F_j(x_j).$$

The function F_∞ is called the Γ-*limit* of $\{F_j\}$, and we write

$$F_\infty = \Gamma - \lim_{j \to +\infty} F_j.$$

A family $\{F_\varepsilon\}_{\varepsilon > 0}$ of functionals Γ-converges to a functional F_0 for $\varepsilon \to 0$ if

$$F_0 = \Gamma - \lim_{j \to +\infty} F_{\varepsilon_j}$$

for every subsequence $\varepsilon_j \to 0$.

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2.3. A convergence result. For the sake of completeness, we recall here the main results of [2]. Let $\{F_\varepsilon\}_{\varepsilon>0}$ be a family of functionals $F_\varepsilon : L^1(\Omega) \to [0, +\infty]$ such that the following hold.

(i) For every $\varepsilon > 0$, F_ε is convex and 1-homogeneous.

(ii) There exist $\beta > \alpha > 0$ such that for every $\varepsilon > 0$ there exists $p_\varepsilon \in [1, +\infty]$ for which

$$\begin{cases} \alpha\|\nabla v\|_{p_\varepsilon} \le F_\varepsilon(v) \le \beta\|\nabla v\|_{p_\varepsilon} & \text{if } v \in W_0^{1,p_\varepsilon}(\Omega), \\ F_\varepsilon(v) = +\infty & \text{otherwise.} \end{cases}$$

(iii) The family $\{p_\varepsilon\}_{\varepsilon>0}$ converges to some $p_0 \in [1, +\infty]$, and the family $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges in $L^{p_0}(\Omega)$ to some functional F_0.

Under these hypotheses, one has from [2, Theorem 3.3, Corollary 3.6] that the numbers

$$\lambda_\varepsilon^k := \inf_{A\in\Gamma_k} \sup_{v\in A} F_\varepsilon(v),$$

defined for $\varepsilon \ge 0$, satisfy

$$\lambda_\varepsilon^k \to \lambda_0^k$$

as $\varepsilon \to 0$.

We define now for $q > 1$

$$F_q(u) := \begin{cases} \|\nabla u\|_q & \text{for } u \in W_0^{1,q}(\Omega), \\ +\infty & \text{for } u \in L^1(\Omega) \setminus W_0^{1,q}(\Omega). \end{cases} \tag{2.2}$$

It is clear that the family $\{F_q\}$ satisfies conditions (i) and (ii) above, and that

$$\lambda_q^k := \inf_{A\in\Gamma_k} \sup_{v\in A} F_q(v) = \lambda_k(q; \Omega)^{1/q}.$$

If we manage to prove that the functionals F_q Γ-converge in $L^p(\Omega)$ to F_p for $q \to p$, which means that also condition (iii) is satisfied, then it would follow that $\lambda_k(p; \Omega)$ is continuous with respect to p because $\lambda_q^k \to \lambda_p^k$ as $q \to p$ (see Theorem 3.2). The Γ-convergence of F_q to F_p for $q \to p$ is in fact the content of Proposition 3.1.

3. Main results

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $p, q \in (1, +\infty)$. Define*

$$F_q(u) := \begin{cases} \|\nabla u\|_q & \textit{for } u \in W_0^{1,q}(\Omega), \\ +\infty & \textit{for } u \in L^1(\Omega) \setminus W_0^{1,q}(\Omega). \end{cases}$$

Then the functionals F_q Γ-converge in $L^p(\Omega)$ to the functional F_p for $q \to p$.

PROOF. We distinguish two cases: the case $q \to p^+$ and the case $q \to p^-$.

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The case $q \to p^+$. liminf inequality. Let $u_q \to u$ in $L^p(\Omega)$ for $q \to p^+$; if $\liminf_{q\to p^+} F_q(u_q) = +\infty$ there is nothing to prove. If $\liminf_{q\to p^+} F_q(u_q) = c < +\infty$ then the u_q are uniformly bounded in $W_0^{1,p}(\Omega)$ by Hölder's inequality; hence there exists a sequence u_{q_k} such that $q_k \to p^+$ as $k \to +\infty$, $\lim_{k\to+\infty} F_{q_k}(u_{q_k}) = c$ and $u_{q_k} \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. From the weak lower semicontinuity of the norm it follows that

$$\begin{aligned}
\int_\Omega |\nabla u|^p &\le \liminf_{k\to+\infty} \int_\Omega |\nabla u_{q_k}|^p \\
&\le \liminf_{k\to+\infty} \left(\int_\Omega |\nabla u_{q_k}|^{q_k} \right)^{p/q_k} |\Omega|^{(q_k-p)/q_k} \\
&\le \liminf_{k\to+\infty} \left(\int_\Omega |\nabla u_{q_k}|^{q_k} \right)^{p/q_k} \cdot \limsup_{k\to+\infty} |\Omega|^{(q_k-p)/q_k} \\
&= \liminf_{k\to+\infty} \left(\int_\Omega |\nabla u_{q_k}|^{q_k} \right)^{p/q_k}
\end{aligned}$$

so that

$$F_p(u) \le \liminf_{k\to+\infty} F_{q_k}(u_{q_k}) = \liminf_{q\to p^+} F_q(u_q).$$

limsup inequality. Let $\{q_k\}_{k=1}^{+\infty}$ be an arbitrary sequence such that $q_k \to p^+$ as $k \to +\infty$. If $u \notin W_0^{1,p}(\Omega)$, there is nothing to prove. Let us suppose $u \in W_0^{1,p}(\Omega)$; if $u = 0$, simply take $u_k = 0$. If $u \neq 0$, we can find a sequence of functions u_k in $C_c^\infty(\Omega)$ (and hence in $W_0^{1,\infty}(\Omega)$) such that $u_k \to u$ in the $W^{1,p}$-norm. It follows that

$$\begin{aligned}
\left(\int_\Omega |\nabla u|^p \right)^{1/p} &= \lim_{k\to+\infty} \left(\int_\Omega |\nabla u_k|^p \right)^{1/p} \\
&= \lim_{k\to+\infty} \|\nabla u_k\|_\infty \left(\int_\Omega \frac{|\nabla u_k|^p}{\|\nabla u_k\|_\infty^p} \right)^{1/p} \\
&\ge \limsup_{k\to+\infty} \|\nabla u_k\|_\infty \left(\int_\Omega \frac{|\nabla u_k|^{q_k}}{\|\nabla u_k\|_\infty^{q_k}} \right)^{1/p} \\
&\ge \limsup_{k\to+\infty} (\|\nabla u_k\|_\infty)^{(p-q_k)/p} \left(\int_\Omega |\nabla u_k|^{q_k} \right)^{1/p} \\
&\ge \liminf_{k\to+\infty} (\|\nabla u_k\|_\infty)^{(p-q_k)/p} \cdot \limsup_{k\to+\infty} \left(\int_\Omega |\nabla u_k|^{q_k} \right)^{1/p}.
\end{aligned}$$

If $\liminf_{k\to+\infty} \|\nabla u_k\|_\infty = c > 0$, we obtain

$$\left(\int_\Omega |\nabla u|^p \right)^{1/p} \ge \limsup_{k\to+\infty} \left(\int_\Omega |\nabla u_k|^{q_k} \right)^{1/p} = \limsup_{k\to+\infty} \left(\int_\Omega |\nabla u_k|^{q_k} \right)^{1/q_k}$$

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which is the claim. If $\liminf_{k\to+\infty} \|\nabla u_k\|_\infty = 0$, we would have, by the liminf inequality,

$$\left(\int_\Omega |\nabla u|^p\right)^{1/p} \le \liminf_{k\to+\infty}\left(\int_\Omega |\nabla u_k|^{q_k}\right)^{1/q_k} \le \liminf_{k\to+\infty} \|\nabla u_k\|_\infty \cdot |\Omega|^{1/q_k} = 0$$

and thus $u = 0$, a case which we ruled out.

The case $q \to p^-$. liminf inequality. Let $u_q \to u$ in $L^p(\Omega)$ for $q \to p^-$ and fix $\varepsilon > 0$; if $\liminf_{q\to p^-} F_q(u_q) = +\infty$ there is nothing to prove. If $\liminf_{q\to p^-} F_q(u_q) = c < +\infty$ then the u_q are uniformly bounded in $W_0^{1,p-\varepsilon}(\Omega)$ by Hölder's inequality; hence there exists a sequence u_{q_k} such that $q_k \to p^-$ as $k \to +\infty$, $\lim_{k\to+\infty} F_{q_k}(u_{q_k}) = c$ and $u_{q_k} \rightharpoonup u$ weakly in $W_0^{1,p-\varepsilon}(\Omega)$. From the weak lower semicontinuity of the norm it follows that

$$\begin{aligned}
\int_\Omega |\nabla u|^{p-\varepsilon} &\le \liminf_{k\to+\infty} \int_\Omega |\nabla u_{q_k}|^{p-\varepsilon} \\
&\le \liminf_{k\to+\infty}\left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{(p-\varepsilon)/q_k} |\Omega|^{(q_k-p+\varepsilon)/q_k} \\
&\le \liminf_{k\to+\infty}\left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{(p-\varepsilon)/q_k} \cdot \limsup_{k\to+\infty} |\Omega|^{(q_k-p+\varepsilon)/q_k} \\
&= |\Omega|^{\varepsilon/p} \liminf_{k\to+\infty}\left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{(p-\varepsilon)/q_k}
\end{aligned}$$

so that

$$F_{p-\varepsilon}(u) \le |\Omega|^{\varepsilon/p(p-\varepsilon)} \liminf_{k\to+\infty} F_{q_k}(u_{q_k}) = |\Omega|^{\varepsilon/p(p-\varepsilon)} \liminf_{q\to p^-} F_q(u_q).$$

Notice that the value $\liminf_{q\to p^-} F_q(u_q)$ depends neither on the choice of the particular subsequence, nor on the choice of ε. Letting ε tend to 0, we obtain

$$F_p(u) \le \liminf_{q\to p^-} F_q(u_q).$$

limsup inequality. Set $q_k \to p^-$. If $u \notin W_0^{1,p}(\Omega)$, there is nothing to prove. If $u \in W_0^{1,p}(\Omega)$, then it belongs in particular to $W_0^{1,q_k}(\Omega)$ for every k and so we can simply consider the constant sequence $u_k := u$ for every k; then of course

$$F_p(u) = \lim_{k\to+\infty} F_{q_k}(u_k).$$

This concludes the proof. □

THEOREM 3.2. *The variational eigenvalues $\lambda_k(p;\Omega)$ are continuous functions with respect to p.*

PROOF. By Proposition 3.1 the functionals F_q defined in (2.2) satisfy conditions (i), (ii) and (iii) in Section 2.3 with $\varepsilon = |q - p|$, $\alpha = \beta = 1$, $p_\varepsilon = q$ and $p_0 = p$. Define

$$\lambda_q^k := \inf_{A \in \Gamma_k} \sup_{v \in A} F_q(v).$$

It is clear that $\lambda_q^k = \lambda_k(q; \Omega)^{1/q}$. From [2, Theorem 3.3] we obtain

$$\lambda_q^k \to \lambda_p^k$$

as $q \to p$, and hence the claim. □

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