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Well-posedness of the free surface problem on a Newtonian fluid between cylinders rotating at different speeds

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When a liquid fills the semi-infinite space between two concentric cylinders which rotate at different steady speeds, how about the shape of the free surface on top of the fluid? The different fluids will lead to a different shape. For the Newtonian fluid, the meniscus descends due to the centrifugal forces. However, for the certain non-Newtonian fluid, the meniscus climbs the internal cylinder. We want to explain the above phenomenon by a rigorous mathematical analysis theory. In the present paper, as the first step, we focus on the Newtonian fluid. This is a steady free boundary problem. We aim to establish the well-posedness of this problem. Furthermore, we prove the convergence of the formal perturbation series obtained by Joseph and Fosdick in Arch. Ration. Mech. Anal. 49 (1973), 321–380.

Keywords: Free boundary problem; Newtonian fluid; Well-posedness; Asymptotic expansion

2020 *Mathematics subject classification:* Primary: 35Q35; 35Q30

1. Introduction

We start from Joseph and Fosdick's paper [**11**, **12**], which studied a very interesting physical phenomenon for the non-Newtonian fluid, climbing effect or Weissenberg effect. When one considers a fluid in a vessel rotates as a rigid body, as mentioned in [**11**, **12**], § 1, it is well known that the free surface of the fluid is shaped by a balance of forces arising from centripetal accelerations, gravity and surface tension. In the absence of relative internal motion, the configuration of such a surface is independent of how the fluid responds to stresses, in particular the free surface of a fluid without surface tension has a paraboloidal shape. However, when the fluid is in internal motion, the situation is different, there will be a stress field which also affects the shape of the free surface. Thus, the shape which a surface assumes in the presence of relative internal motion is sensitive to the manner in which different liquids respond. In [**11**, **12**], the authors considered a most simple situation, that is the shape of the free surface of a liquid filling the semi-infinite space between two concentric cylinders which rotate at different steady speeds. Specifically, we consider a cylindrical container is filled with liquid, and a cylindrical rod is immersed in a

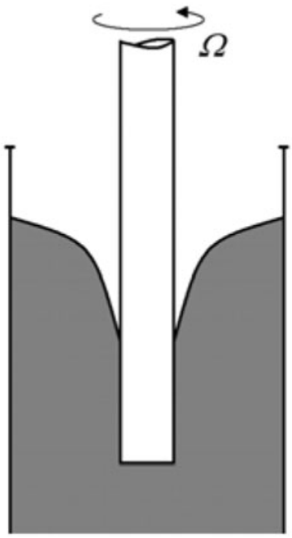

Figure 1. Newtonian.

container with its axis parallel to that of liquid. Next, the rod is rotated and kept at a constant angular velocity. Now the response of the free surface of a liquid to the rotation of the rod is dramatically different depending on the physical characteristic of a liquid. It will reach an equilibrium configuration which for a Newtonian liquid, like water, will move to the edges of the container, away from the rod, see figure 1, which comes from figure 4(a) on page 194 of [**5**], while for certain non-Newtonian liquids, like a concentrated polymer melt or solution, the equilibrium surface will climb the rod, see figure 2, which comes from figure 4(b) on page 194 of [**5**]. This effect is due to the normal stresses that the shear induces in the polymeric fluid: the shear stretches and orients the polymers. The difference in the normal components of the stress tensor that this anisotropy induces is such that the fluid is pulled inwards and climbs the rod, see [**4**].

In [**11**, **12**], Joseph and Fosdick developed a systematic construction in series for the shape of the free surface for the Newtonian fluid and non-Newtonian fluid. By analysing the main term of the perturbation series, one can explain the above climbing effect. However, Joseph and Fosdick's analysis is formal, we don't know if the formal perturbation series is convergent. This fact prompts us to seek a rigorous mathematical theory to explain the climbing effect. It is noticing that the non-Newtonian fluid model is relatively complicated, at the first stage, in the present paper we focus on the Newtonian fluid, which was considered by Joseph and Fosdick in [**12**], chapter I. By the domain perturbation method, they determined the shape of the free surface and give a formal series expansion of the solution. We aim to give a rigorous mathematical proof of convergence of the perturbation series in [**12**], § 4 (the difference is that we consider the case with the surface tension). To this end, we first give a mathematical description of the problem.

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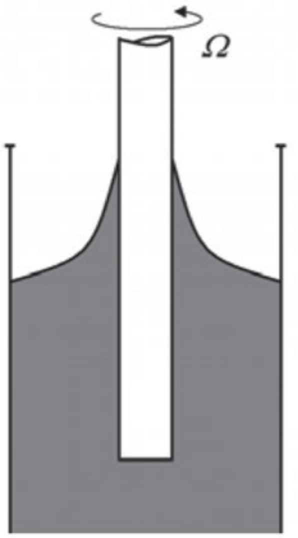

Figure 2. Non-Newtonian.

1.1. Formulation

We are concerned with a problem given by Joseph and Fosdick [**12**], § 4. It is stated as follows, see figure 3, which comes from figure 1 on page 326 of [**12**]: an incompressible Newtonian fluid initially occupies the space between two fixed concentric cylinders ($a \leqslant r \leqslant b$) and below the free surface. The inner and outer cylinders are then made to rotate about their common axis with angular velocities Ω and $\lambda\Omega$. The free surface of the rotating fluid cannot retain its static shape and its final steady shape $z = h(r, \theta; \Omega)$ is determined by a complex balance of central forces, normal stresses, surface tension and gravity, where (r, θ, z) stands for polar cylindrical coordinates. The problem is to seek a mathematical description of the shape of the free surface and of the fluid mechanics which determine this shape.

From the above statement on the problem, we have that the problem should satisfy the following system.

$$\begin{cases} -\nabla P + \operatorname{div} \mathbb{S}(v) = v \cdot \nabla v, & \text{in } \mathcal{V}_\Omega, \\ \operatorname{div} v = 0, & \text{in } \mathcal{V}_\Omega, \\ v \cdot N = N \cdot \mathbb{S}(v) \cdot e_\theta = N \cdot \mathbb{S}(v) \cdot T = 0, & \text{at } \Sigma = \{(r, \theta, z) : z = h(r, \theta; \Omega)\}, \\ N \cdot (-(P - p_a) I + \mathbb{S}(u)) \cdot N & \\ \quad = -gh + \dfrac{\sigma}{r} \left(\dfrac{r h'}{\sqrt{1 + h'^2}} \right)', & \text{on } \Sigma = \{(r, \theta, z) : z = h(r, \theta; \Omega)\}, \\ h'(a; \Omega) = \gamma_a, \ h'(b; \Omega) = \gamma_b, & \\ v = a \, \Omega \, e_\theta & \text{on } \Sigma_a = \{(a, \theta, z) : z < h(a, \theta; \Omega)\}, \\ v = \lambda b \, \Omega \, e_\theta & \text{on } \Sigma_b = \{(b, \theta, z) : z < h(b, \theta; \Omega)\}, \end{cases} \tag{1.1}$$

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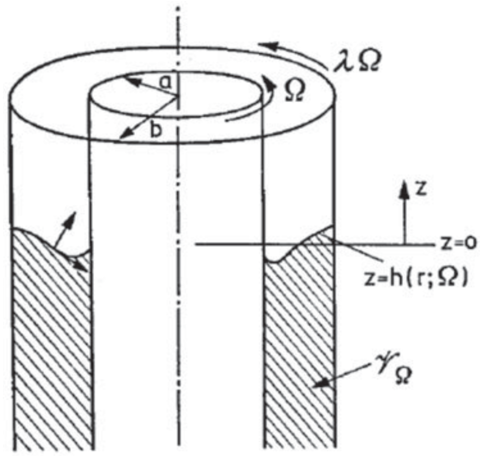

Figure 3. The free surface between rotating cylinders.

where v is the velocity field of the fluid, P is the pressure, p_a stands for the atmospheric pressure, $\mathbb{S}(v) = \nabla v + (\nabla v)^\top$ is the symmetric gradient of v, $\sigma > 0$ is the coefficient of surface tension, N is the outward-pointing unit normal of Σ, and T is the associated unit tangent. γ_a and γ_b are two constants, which determine the wetting angles. Throughout the whole paper, for convenience, we assume that $\gamma_a = \gamma_b = 0$.

It will be assumed throughout that the problem is axisymmetric so that v and h are independent of θ.

If we set

$$v(r,z) = v_r(r,z)e_r + v_\theta(r,z)e_\theta + v_z(r,z)e_z, \tag{1.2}$$

where

$$e_r = (\cos\theta, \sin\theta, 0)^\top, \quad e_\theta = (-\sin\theta, \cos\theta, 0)^\top, \quad e_z = (0,0,1)^\top, \tag{1.3}$$

then, we can write the above system as follows.

$$\begin{cases} -\partial_r P + \left(\partial_{rr} + \dfrac{\partial_r}{r} + \partial_{zz} - \dfrac{1}{r^2}\right) v_r = (v_r\partial_r + v_z\partial_z)v_r - \dfrac{v_\theta^2}{r} & \text{in } \mathcal{V}_\Omega^0, \\ \left(\partial_{rr} + \dfrac{\partial_r}{r} + \partial_{zz} - \dfrac{1}{r^2}\right) v_\theta = (v_r\partial_r + v_z\partial_z)v_\theta + \dfrac{v_r v_\theta}{r} & \text{in } \mathcal{V}_\Omega^0, \\ -\partial_z P + \left(\partial_{rr} + \dfrac{\partial_r}{r} + \partial_{zz}\right) v_z = (v_r\partial_r + v_z\partial_z)v_z & \text{in } \mathcal{V}_\Omega^0, \\ \partial_r v_r + \dfrac{v_r}{r} + \partial_z v_z = 0 & \text{in } \mathcal{V}_\Omega^0, \\ v_z - h'v_r = \partial_z v_\theta - h'\partial_r v_\theta = 2h'(\partial_z v_z - \partial_r v_r) & \\ \quad + (1-h'^2)(\partial_r v_z + \partial_z v_r) = 0 & \text{on } \Sigma^0, \\ -(P - p_a) + 2\partial_z v_z - h'(\partial_r v_z + \partial_z v_r) = -gh + \dfrac{\sigma}{r}\left(\dfrac{rh'}{\sqrt{1+h'^2}}\right)' & \text{on } \Sigma^0, \\ h'(a;\Omega) = h'(b;\Omega) = 0, & \\ v = a\,\Omega\, e_\theta & \text{on } \Sigma_a^0, \\ v = \lambda b\,\Omega\, e_\theta & \text{on } \Sigma_b^0, \end{cases} \tag{1.4}$$

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where

$$\begin{aligned} \mathcal{V}_\Omega^0 &= \{(r,z) : a < r < b,\ z < h(r;\Omega)\}, \\ \Sigma^0 &= \{(r,z) : z = h(r;\Omega)\}, \\ \Sigma_a^0 &= \{(a,z) : z < h(a)\}, \\ \Sigma_b^0 &= \{(b,z) : z < h(b)\}. \end{aligned} \tag{1.5}$$

Furthermore, we set

$$\bar\nabla = (\partial_r, \partial_z), \quad \bar\Delta = \partial_r^2 + \partial_z^2, \quad \overline{\mathrm{div}} = (\partial_r, \partial_z)(\cdot),$$

and

$$\bar v = (v_r, v_z)^\top, \quad \bar{\mathbb{S}}(v) = \begin{pmatrix} 2\partial_r v_r & \partial_z v_r + \partial_z v_r \\ \partial_z v_r + \partial_z v_r & 2\partial_z v_z \end{pmatrix},$$

then the system (1.4) can be rewritten as follows.

$$\begin{cases} \bar v \cdot \bar\nabla \bar v - \bar\nabla P + \bar\Delta \bar v + \dfrac{1}{r}\partial_r \bar v - \left(\dfrac{v_r}{r^2}, 0\right)^\top = -\left(\dfrac{v_\theta^2}{r}, 0\right)^\top & \text{in } \mathcal{V}_\Omega^0, \\ \bar\Delta v_\theta + \dfrac{1}{r}\partial_r v_\theta - \dfrac{v_\theta}{r^2} = \bar v \cdot \bar\nabla v_\theta + \dfrac{v_r v_\theta}{r} & \text{in } \mathcal{V}_\Omega^0, \\ \overline{\mathrm{div}}\, \bar v + \dfrac{v_r}{r} = 0 & \text{in } \mathcal{V}_\Omega^0, \\ \bar v \cdot \bar N = \bar N \cdot \bar\nabla v_\theta = \bar N \cdot \bar{\mathbb{S}}(v) \cdot \bar T = 0 & \text{on } \Sigma^0, \\ \bar N \cdot (-(P - p_a)\bar I + \bar{\mathbb{S}}(v)) \cdot \bar N = -gh + \dfrac{\sigma}{r}\left(\dfrac{rh'}{\sqrt{1+h'^2}}\right)' & \text{on } \Sigma^0, \\ h'(a;\Omega) = h'(b;\Omega) = 0, & \\ v = a\,\Omega\, e_\theta & \text{on } \Sigma_a^0, \\ v = \lambda b\,\Omega\, e_\theta & \text{on } \Sigma_b^0. \end{cases} \tag{1.6}$$

where $\bar N$ is the outward-pointing unit normal of Σ^0, $\bar T$ is the associated unit tangent.

We will solve the above equations as perturbations of a known static configuration ($\Omega = 0$).

When $\Omega = 0$, we know that $(v = 0, P = p_a, h = 0)$ solve the system (1.4). Now, we define

$$\mathcal{V}_0^0 = \{(r,z) : a < r < b,\ z < 0\}, \tag{1.7}$$

$$\Sigma_0^0 = \{(r,z) : a < r < b, z = 0\}. \tag{1.8}$$

and the mapping

$$\mathcal{V}_0^0 \in (r,z) \mapsto (r, z + \tilde h(r,z;\Omega)) := \Phi(r,z) \in \mathcal{V}_\Omega^0, \tag{1.9}$$

where $\tilde h(r,z;\Omega)$ is an extension of h such that $\tilde h(r,0;\Omega) = h(r)$, and set

$$\mathcal{A} := (\nabla_{r,z}\Phi^{-1})^\top = \begin{pmatrix} 1 & -\partial_r \tilde h(1+\partial_z \tilde h)^{-1} \\ 0 & (1+\partial_z \tilde h)^{-1} \end{pmatrix}, \tag{1.10}$$

and

$$w = v \circ \Phi, \;\; p = P \circ \Phi. \tag{1.11}$$

Then the above problem can be transformed to one on the fixed domain $\mathcal{V}_0^0$. In the new coordinates, the system (1.6) becomes

$$\begin{cases} -\bar{\nabla}_{\mathcal{A}} p + \bar{\Delta}_{\mathcal{A}} \bar{w} + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}} \bar{w})_r - \left(\frac{w_r}{r^2}, 0\right)^{\top} = \bar{w} \cdot \bar{\nabla}_{\mathcal{A}} \bar{w} - \left(\frac{w_\theta^2}{r}, 0\right)^{\top} & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}_{\mathcal{A}} w_\theta + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}} w_\theta)_r - \frac{w_\theta}{r^2} = \bar{w} \cdot \bar{\nabla}_{\mathcal{A}} w_\theta + \frac{w_r w_\theta}{r} & \text{in } \mathcal{V}_0^0, \\ \overline{\operatorname{div}}_{\mathcal{A}} \bar{w} + \frac{w_r}{r} = 0 & \text{in } \mathcal{V}_0^0, \\ \bar{w} \cdot \bar{\mathcal{N}} = \bar{\mathcal{N}} \cdot \bar{\nabla}_{\mathcal{A}} w_\theta = \bar{\mathcal{N}} \cdot \bar{\mathbb{S}}_{\mathcal{A}}(w) \cdot \bar{\mathcal{T}} = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}} \cdot (p\bar{I} + \bar{\mathbb{S}}_{\mathcal{A}}(w)) \cdot \bar{\mathcal{N}} = -gh + \frac{\sigma}{r}\left(\frac{rh'}{\sqrt{1+h'^2}}\right)' & \text{on } \Sigma_0^0, \\ h'(a;\Omega) = h'(b;\Omega) = 0, & \\ w = e_\theta \, \Omega \, a & \text{on } \Sigma_{a,0}^0, \\ w = e_\theta \, \lambda\Omega \, b & \text{on } \Sigma_{b,0}^0. \end{cases} \tag{1.12}$$

Here, for appropriate f and X, we write

$$(\bar{\nabla}_{\mathcal{A}} f)_i := \mathcal{A}_{ij} \bar{\partial}_j f, \;\; \overline{\operatorname{div}}_{\mathcal{A}} X = \mathcal{A}_{ij} \bar{\partial}_j X, \;\; \bar{\Delta}_{\mathcal{A}} f = \overline{\operatorname{div}}_{\mathcal{A}} \bar{\nabla}_{\mathcal{A}} f, \tag{1.13}$$

where

$$\bar{\partial}_1 := \partial_r, \quad \bar{\partial}_2 := \partial_z, \tag{1.14}$$

and

$$(u \cdot \bar{\nabla}_{\mathcal{A}} u)_i := u_j \mathcal{A}_{jk} \bar{\partial}_k u_i, \tag{1.15}$$

and

$$\bar{\mathcal{N}} := \left(\frac{-h'}{\sqrt{1+h'^2}}, \frac{1}{\sqrt{1+h'^2}}\right)^{\top}. \tag{1.16}$$

1.2. Formal asymptotic expansion

In [**12**], based on the domain perturbation method, Joseph and Fosdick obtained the formal perturbation series of problem (1.1).

Define

$$(\cdot)^{[n]} = \frac{\mathrm{d}^n(\cdot)}{\mathrm{d}\Omega^n}\Big|_{\Omega=0}, \tag{1.17}$$

where

$$\frac{\mathrm{d}(\cdot)}{\mathrm{d}\Omega} = \frac{\partial(\cdot)}{\partial\Omega} + \frac{\mathrm{d}\Phi}{\mathrm{d}\Omega} \cdot \nabla(\cdot), \tag{1.18}$$

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and

$$(\cdot)^{\{n\}} = \frac{\partial^n(\cdot)}{\partial\Omega^n}\,|_{\Omega=0}\ . \tag{1.19}$$

Formally, we have

$$\begin{pmatrix} v\circ\Phi \\ P\circ\Phi \\ h \end{pmatrix} = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} v^{[n]} \\ P^{[n]} \\ h^{[n]} \end{pmatrix}\Omega^n. \tag{1.20}$$

At zeroth order, i.e., $\Omega = 0$, note that $\gamma_a = \gamma_b = 0$, it is easy to get

$$v^{[0]} = h^{[0]} = 0, \quad P^{[0]} = p_a. \tag{1.21}$$

At first order problem, which is obtained by differentiating (1.1) with respect to Ω, i.e.,

$$\begin{cases} (v\cdot\nabla v)^{\{1\}} = -\nabla P^{\{1\}} + \Delta v^{\{1\}} & \text{in } \mathcal{V}_0^0, \\ \operatorname{div} v^{\{1\}} = 0 & \text{in } \mathcal{V}_0^0, \\ v^{\{1\}} = a\, e_\theta & \text{on } \Sigma_0^0, \\ v^{\{1\}} = \lambda\, b\, e_\theta & \text{on } \Sigma_0^0. \end{cases} \tag{1.22}$$

Noticing that

$$0 = (v\cdot\nabla v)^{\{1\}} = 0, \tag{1.23}$$

we have that

$$v^{\{1\}} = \left(Ar + \frac{B}{r}\right)e_\theta, \quad P^{\{1\}} = \text{constant} \tag{1.24}$$

is the solution of (1.22), where

$$A = \frac{b^2\lambda - a^2}{b^2 - a^2}, \quad B = \frac{a^2 b^2(1-\lambda)}{b^2 - a^2}. \tag{1.25}$$

Now, we have

$$v^{[1]} = v^{\{1\}} = \left(Ar + \frac{B}{r}\right)e_\theta, \quad P^{[1]} = P^{\{1\}} = \text{constant}. \tag{1.26}$$

Next, it follows from the condition

$$\begin{aligned} &-P^{\{1\}} + S_{zz}^{[1]} - (h' S_{rz})^{[1]} + g h^{[1]} = -P^{[1]} + g h^{[1]} = \frac{\sigma}{r}\left[\left(\frac{r h'}{\sqrt{1+h'^2}}\right)'\right]^{[1]} \\ &= \frac{\sigma}{r}\left[\left(r(h^{[1]})'\right)'\right], \end{aligned} \tag{1.27}$$

that

$$h^{[1]} = 0, \quad P^{[1]} = 0. \tag{1.28}$$

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At second order, we have

$$\begin{cases} (v\cdot\nabla v)^{\{2\}} = -2\left(Ar+\frac{B}{r}\right)^2 e_r/r = -\nabla P^{\{2\}} + \Delta v^{\{2\}} & \text{in } \mathcal{V}_0^0,\\ \operatorname{div} v^{\{2\}} = 0 & \text{in } \mathcal{V}_0^0,\\ v^{\{2\}} = 0 & \text{on } \Sigma_0^0,\\ v^{\{2\}} = 0 & \text{on } \Sigma_0^0. \end{cases} \tag{1.29}$$

and

$$v_z^{\{2\}} = \mathbb{S}_{z\theta}^{\{2\}} = \mathbb{S}_{rz}^{\{2\}} = 0 \text{ at } z=0, \tag{1.30}$$

and

$$-P^{\{2\}} + S_{zz}^{\{2\}} + gh^{[2]} = \frac{\sigma}{r}\left[\left(r(h^{[2]})'\right)'\right] \text{ at } z=0 \tag{1.31}$$

The solution of the above problem is

$$v^{[2]} = v^{\{2\}} = 0, \quad P^{[2]} = P^{\{2\}} = A^2r^2 + 4AB\log\frac{r}{b} - \frac{B^2}{r^2} + C_1, \tag{1.32}$$

and $h^{[2]}$ is determined by

$$-P^{[2]} + gh^{[2]} = \frac{\sigma}{r}\left[\left(r(h^{[2]})'\right)'\right] \tag{1.33}$$

with

$$(h^{[2]})'(a) = (h^{[2]})'(b) = 0, \tag{1.34}$$

where C_1 is fixed by

$$\int_a^b rh^{[2]} = 0. \tag{1.35}$$

Hence, formally, we have

$$\begin{aligned} v\circ(\Phi(r,z)) &= v^{[1]}\,\Omega + O(\Omega^3),\\ P\circ(\Phi(r,z)) &= p_a + \frac{1}{2}P^{[2]}\,\Omega^2 + O(\Omega^3),\\ h(r) &= \frac{1}{2}h^{[2]}\,\Omega^2 + O(\Omega^3). \end{aligned} \tag{1.36}$$

We want to give a rigorous proof of the above formal asymptotic expansion. To this end, we set

$$u = w - v^{[1]}\Omega.$$

Note that

$$\bar{\Delta}_{\mathcal{A}-\bar{I}} v_\theta^{[1]} = -\frac{1}{r}(\bar{\nabla}_{\mathcal{A}-\bar{I}} v_\theta^{[1]})_r = 0, \text{ and } \bar{\mathcal{N}}\cdot\nabla_{\mathcal{A}-\bar{I}} v_\theta^{[1]} = 0, \tag{1.37}$$

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and

$$\bar{\nabla}_{\mathcal{A}} P^{[2]} = \bar{\nabla} P^{[2]}. \tag{1.38}$$

By

$$\bar{\nabla} P^{[2]} = -2(v_\theta^{[1]})^2 e_r / r, \tag{1.39}$$

and

$$-P^{[2]} + g h^{[2]} = \frac{\sigma}{r}\left[\left(r(h^{[2]})'\right)'\right], \tag{1.40}$$

we can get

$$\begin{cases} -\bar{\nabla}_{\mathcal{A}}(p - p_a - \frac{\Omega^2}{2}P^{[2]}) + \bar{\Delta}_{\mathcal{A}}\bar{u} + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}}\bar{u})_r - (\frac{u_r}{r^2}, 0)^\top & \\ = \bar{u}\cdot\bar{\nabla}_{\mathcal{A}}\bar{u} - \dfrac{(u_\theta + v_\theta^{[1]}\Omega)^2 - (v_\theta^{[1]}\Omega)^2}{r} & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}_{\mathcal{A}} u_\theta + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}} u_\theta)_r - \frac{u_\theta}{r^2} = \bar{u}\cdot\bar{\nabla}_{\mathcal{A}}(u_\theta + v_\theta^{[1]}\Omega) + \dfrac{u_r(u_\theta + v_\theta^{[1]}\Omega)}{r} & \text{in } \mathcal{V}_0^0, \\ \overline{\mathrm{div}}_{\mathcal{A}}\bar{u} + \frac{u_r}{r} = 0, & \text{in } \mathcal{V}_0^0, \\ \bar{\mathcal{N}}\cdot\bar{\nabla}_{\mathcal{A}} u_\theta = \Omega(\bar{N} - \bar{\mathcal{N}})\cdot\nabla v_\theta^{[1]} & \text{on } \Sigma_0^0, \\ \bar{u}\cdot\bar{\mathcal{N}} = \bar{\mathcal{N}}\cdot\bar{\mathbb{S}}_{\mathcal{A}}(u)\cdot\bar{\mathcal{T}} = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}\cdot\left(\left(p - p_a - \frac{\Omega^2}{2}P^{[2]}\right)\bar{I} + \bar{\mathbb{S}}_{\mathcal{A}}(u)\right)\cdot\bar{\mathcal{N}} & \\ = -g\left(h - \frac{\Omega^2}{2}h^{[2]}\right) + \frac{\sigma}{r}\left(r\left(h - \frac{\Omega^2}{2}h^{[2]}\right)' + \mathcal{R}(0, h')\right)' & \text{on } \Sigma_0^0, \\ u = 0 & \text{on } \Sigma_{a,0}^0, \\ u = 0 & \text{on } \Sigma_{b,0}^0. \end{cases} \tag{1.41}$$

Finally, set $M = J^{-1}\nabla\Phi = (J\mathcal{A}^\top)^{-1}$ with $J = det\mathcal{A}$, and

$$\begin{aligned} &U_r = M_{11}\bar{u}_r + M_{12}\bar{u}_z, \quad U_z = M_{21}\bar{u}_r + M_{22}\bar{u}_z, \quad U_\theta = u_\theta, \\ &Q = q - p_a - \frac{\Omega^2}{2}P^{[2]}, \quad H = h - \frac{\Omega^2}{2}h^{[2]}, \end{aligned} \tag{1.42}$$

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then the above system can be rewritten as follows.

$$\begin{cases} -\bar{\nabla}_{\mathcal{A}}Q + \bar{\Delta}_{\mathcal{A}}(M^{-1}\bar{U}) + \dfrac{1}{r}(\bar{\nabla}_{\mathcal{A}}(M^{-1}\bar{U}))_r - \left(\dfrac{(M^{-1}U)_r}{r^2}, 0\right)^{\top} & \\ = M^{-1}\bar{U}\cdot\bar{\nabla}_{\mathcal{A}}(M^{-1}\bar{U}) - \dfrac{U_\theta(U_\theta + 2v_\theta^{[1]}\Omega)}{r} & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}_{\mathcal{A}}U_\theta + \dfrac{1}{r}(\bar{\nabla}_{\mathcal{A}})_r U_\theta - \dfrac{U_\theta}{r^2} = \bar{U}\cdot\bar{\nabla}_{\mathcal{A}}(U_\theta + v^{[1]}\Omega) + \dfrac{U_r(U_\theta + v^{[1]}\Omega)}{r} & \text{in } \mathcal{V}_0^0, \\ \overline{\operatorname{div}}\,\bar{U} + \dfrac{U_r}{r} = 0, & \text{in } \mathcal{V}_0^0, \\ \bar{U}\cdot\bar{N} = 0 & \text{on } \Sigma_0^0, \\ \bar{N}\cdot\bar{\nabla}_{\mathcal{A}}U_\theta = 0, & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}\cdot\bar{\mathbb{S}}_{\mathcal{A}}(M^{-1}U)\cdot\bar{\mathcal{T}} = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}\cdot(Q\bar{I} + \bar{\mathbb{S}}_{\mathcal{A}}(M^{-1}U))\cdot\bar{\mathcal{N}} = -gH & \\ \quad + \dfrac{\sigma}{r}\left(rH' + \mathcal{R}\left(0, \left(H + \dfrac{\Omega^2}{2}h^{[2]}\right)'\right)\right)' & \text{on } \Sigma_0^0, \\ U = 0 & \text{on } \Sigma_{a,0}^0, \\ U = 0 & \text{on } \Sigma_{b,0}^0, \end{cases} \tag{1.43}$$

1.3. Main results

We will prove the following theorem.

THEOREM 1.1. *Let $\delta_k \in (k + \delta_\omega, k+1)$ with $\omega = \frac{\pi}{2}$, where $\delta_\omega = \max\{0, 2 - \pi/\omega\} \in [0,1)$. There exists a universal smallness parameter $\epsilon > 0$ such that if*

$$\Omega \leqslant \epsilon, \tag{1.44}$$

then there exists a unique solution $(U, Q, H) \in W_{\delta_k}^{k+2}(\mathcal{V}_0^0) \times W_{\delta_k}^{k+1}(\mathcal{V}_0^0) \times W_{\delta_k}^{k+\frac{5}{2}}$ to (1.43), and

$$\|U\|^2_{W^{k+2}_{\delta_k}} + \|Q\|^2_{\mathring{W}^{k+1}_{\delta_k}} + \|H\|^2_{W^{k+\frac{5}{2}}_{\delta_k}} \leqslant C(k, a, b, \lambda, \sigma)\,\Omega^2, \tag{1.45}$$

where the spaces W_δ^k are defined by (2.3), and $C(k, a, b, \lambda, \sigma)$ is a constant depending on k, a, b, λ, σ. Hence, we have

$$\begin{aligned} \|U\|_{W^{k+2}_{\delta_k}} &\leqslant C(k, a, b, \lambda, \sigma)\Omega^3, \\ \|Q\|_{\mathring{W}^{k+1}_{\delta_k}} &\leqslant C(k, a, b, \lambda, \sigma)\Omega^3, \\ \|H\|_{W^{k+\frac{5}{2}}_{\delta_k}} &\leqslant C(k, a, b, \lambda, \sigma)\Omega^3. \end{aligned} \tag{1.46}$$

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which implies that

$$\begin{aligned} \|v\circ(\Phi(r,z)) - v^{[1]}\,\Omega\|_{W^{k+2}_{\delta_k}} &\leqslant C(k,a,b,\lambda,\sigma)\Omega^3, \\ \|p - p_a - P^{[2]}\,\Omega^2/2\|_{\mathring{W}^{k+1}_{\delta_k}} &\leqslant C(k,a,b,\lambda,\sigma)\Omega^3, \\ \|h(r) - h^{[2]}\,\Omega^2/2\|_{W^{k+\frac{5}{2}}_{\delta_k}} &\leqslant C(k,a,b,\lambda,\sigma)\Omega^3. \end{aligned} \tag{1.47}$$

REMARK 1.2. Throughout the whole paper, for convenience, we assume that $\gamma_a = \gamma_b = 0$, so the contact angle is $\frac{\pi}{2}$. However, our method is still valid when the contact angle is not $\frac{\pi}{2}$.

REMARK 1.3. For our case, that is the contact angle is $\frac{\pi}{2}$, by remark 2.1, following the proof of § 3, it is possible to obtain the following result. There exists a universal smallness parameter $\epsilon > 0$ such that if

$$\Omega \leqslant \epsilon, \tag{1.48}$$

then there exists a unique solution $(U,Q,H) \in W^{2,q} \times W^{1,q} \times W^{3-1/q,q}$ to (1.43), and

$$\|U\|^2_{W^{2,q}} + \|Q\|^2_{W^{1,q}} + \|H\|^2_{W^{3-1/q}} \leqslant C\,\Omega^2, \tag{1.49}$$

where $W^{k,q}$ is the usual Sobolev spaces, C is a generic constant. Hence, we have

$$\begin{aligned} \|U\|_{W^{2,q}} &\leqslant C\Omega^3, \\ \|Q\|_{W^{1,q}} &\leqslant C\Omega^3, \\ \|H\|_{W^{3-1/q,q}} &\leqslant C\Omega^3, \end{aligned} \tag{1.50}$$

which implies that

$$\begin{aligned} \|v\circ(\Phi(r,z)) - v^{[1]}\,\Omega\|_{W^{2,q}} &\leqslant C\Omega^3, \\ \|p - p_a - P^{[2]}\,\Omega^2/2\|_{W^{1,q}} &\leqslant C\Omega^3, \\ \|h(r) - h^{[2]}\,\Omega^2/2\|_{W^{3-1/q,q}} &\leqslant C\Omega^3. \end{aligned} \tag{1.51}$$

1.4. Some results on the free boundary problem

The free boundary problem is a very important problem in fluid mechanics. For the free boundary problem of the incompressible Navier-Stokes equations, i.e., the viscous surface wave problem, the reader can refer to [**1**–**3**] or [**6**–**8**] for the local/global well-posedness and large time behaviour of this problem. When one neglects the viscosity, the problem is the famous water wave problem, the reader can refer to [**17**, **18**] for local well-posedness, [**19**, **20**] for global well-posedness.

Our problem is a steady free boundary problem with contact points, i.e., we consider the stationary Navier-Stokes system with free, but unmoving boundary with contact points. Concerning this respect, Solonnikov [**15**] (see also [**16**]) proved the unique solvability of several 2D free surface problems describing a viscous flow

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with contact angle fixed at π. Jin [10] considered the stationary free boundary flow of a bounded 3D domain with angle $\pi/2$ of viscous incompressible fluid. Socolowsky [14] dealt with 2D coating problems with fixed contact angles.

1.5. Strategy of the proof

Compared with the original free boundary problem (1.1), we have reduced it to the fixed domain problem (1.43). However, since H needs to be determined, $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{T}$ are unknown, at the first step, we should fix these quantities. This fact motivates us to apply successive approximations to solve the problem. We first assume H is fixed, and study the following $\mathcal{A}$-Stokes system on the triple $(U, Q, H)^\top$:

$$\begin{cases} -\bar{\nabla}_{\mathcal{A}} Q + \bar{\Delta}_{\mathcal{A}}(M^{-1}\bar{U}) + \dfrac{1}{r}(\bar{\nabla}_{\mathcal{A}}(M^{-1}\bar{U}))_r & \\ \quad - \left(\dfrac{(M^{-1}U)_r}{r^2}, 0 \right)^\top = (G_r^1, G_z^1) & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}_{\mathcal{A}} U_\theta + \dfrac{1}{r}(\bar{\nabla}_{\mathcal{A}})_r U_\theta - \dfrac{U_\theta}{r^2} = G_\theta^1 & \text{in } \mathcal{V}_0^0, \\ \overline{\operatorname{div}}\, \bar{U} + \dfrac{U_r}{r} = 0 & \text{in } \mathcal{V}_0^0, \\ \bar{U} \cdot \bar{N} = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}} \cdot \bar{\nabla}_{\mathcal{A}} U_\theta = G^2 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}} \cdot \bar{\mathbb{S}}_{\mathcal{A}}(M^{-1}\bar{U}) \cdot \bar{\mathcal{T}} = G^3 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}} \cdot \mathbb{S}_{\mathcal{A}}(M^{-1}\bar{U}) \cdot \bar{\mathcal{N}} = -gH + \dfrac{\sigma}{r}\left(H' + G^4\right)' + G^5 & \text{on } \Sigma_0^0, \\ U = 0 & \text{on } \Sigma_{a,0}^0, \\ U = 0 & \text{on } \Sigma_{b,0}^0, \end{cases}$$

where $M = J^{-1}\nabla\Phi = (J\mathcal{A}^\top)^{-1}$ with $J = det\mathcal{A}$.

Next, we start to a triple $(U^{(0)}, Q^{(0)}, H^{(0)})$ to fix $\mathcal{A}$, $\mathcal{N}$, $\mathcal{T}$ and nonlinear terms in (1.43), and get a triple $(U^{(1)}, Q^{(1)}, H^{(1)})$ by solving the above linear system. Repeating this procedure, we can construct a sequence $(U^{(k)}, Q^{(k)}, H^{(k)})$. Finally, by taking an appropriate limit procedure, we can obtain a triple (U, Q, H) to solve the system (1.43). To obtain a limit, we must obtain higher-order elliptic estimates. However, we know the domain is piecewise C^2 but only Lipschitz because of the corners formed at the contact point. Hence, we can not use the usual elliptic regularity theory. To overcome this difficulty, we will use the weighted elliptic regularity theory to replace the standard elliptic regularity theory, which can avoid singularity at the point of the contact points and have used by [9] to study the stability of contact lines in fluids.

From the above analysis, we organize our paper as follows. In § 2, we focus on the $\mathcal{A}$-Stokes system, and establish the weighted elliptic estimates. In § 3, we will prove theorem 1.1.

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2. Estimates for the stokes problem

We first consider the following Stokes system in $\mathcal{V}_0^0$.

$$\begin{cases} -\bar{\nabla}Q + \bar{\Delta}\bar{U} + \frac{1}{r}\partial_r\bar{U} - \left(\frac{U_r}{r^2}, 0\right)^\top = (G_r^1, G_z^1)^\top & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}U_\theta + \frac{1}{r}\partial_r U_\theta - \frac{U_\theta}{r^2} = G_\theta^1 & \text{in } \bar{\mathcal{V}}_\Omega, \\ \overline{\operatorname{div}}\,\bar{U} + \frac{U_r}{r} = 0 & \text{in } \mathcal{V}_0^0, \\ \bar{U}\cdot\bar{N} = 0 & \text{on } \Sigma_0^0 \\ \bar{N}\cdot\bar{\nabla}U_\theta = G^2 & \text{on } \Sigma_0^0 \\ \bar{N}\cdot\bar{\mathbb{S}}(U)\cdot\bar{T} = G^3 & \text{on } \Sigma_0^0, \\ U = 0 & \text{on } \Sigma_{a,0}^0, \\ U = 0 & \text{on } \Sigma_{b,0}^0. \end{cases} \tag{2.1}$$

To this end, we introduce the spaces $W_\delta^k(\mathcal{V}_0^0)$, $W_\delta^{k-\frac{1}{2}}(\partial\mathcal{V}_0^0)$, $\mathring{W}_\delta^k(\mathcal{V}_0^0)$, which have been defined by [9]. Let

$$\mathcal{M} = \{(a, 0), (b, 0)\} \tag{2.2}$$

for the pair of the corner points of $\mathcal{V}_0^0$. For $0 < \delta < 1$ and $k \in \mathbb{N}$, let W_δ^k denote the space of functions such that $\|f\|_{W_\delta^k}^2 < \infty$, where

$$\|f\|_{W_\delta^k}^2 = \sum_{|\alpha|\leqslant k} \int_{\mathcal{V}_0^0} \operatorname{dist}(x, \mathcal{M})^{2\delta} |\partial^\alpha f(x)|^2 \,\mathrm{d}x. \tag{2.3}$$

The spaces $W_\delta^{k-\frac{1}{2}}(\partial\mathcal{V}_0^0)$ are defined as the trace spaces. We also define

$$\mathring{W}_\delta^k = \{U \in W_\delta^k(\mathcal{V}_0^0) : \int_{\mathcal{V}_0^0} U = 0\}, \tag{2.4}$$

for $k \geqslant 1$. We will establish the estimates of the system in the space X_δ^k for $0 < \delta < 1$. It is defined as follows:

$$(G^1, G^2, G^3) \in W_\delta^k(\mathcal{V}_0^0) \times W_\delta^{k+\frac{1}{2}}(\Sigma_0^0) \times W_\delta^{k+\frac{1}{2}}(\Sigma_0^0). \tag{2.5}$$

The weak solutions of (2.1) are defined as follows.

Definition 2.1. Assume that $(G^1, G^2, G^3) \in X_{\delta_0}^0$ for some $0 < \delta_0 < 1$. We say that a pair $(U, q) \in H^1(\mathcal{V}_0^0) \times H^0(\mathcal{V}_0^0)$ such that $\overline{\operatorname{div}}(U_r, U_z)^\top + \frac{U_r}{r} = 0$, $U \cdot \bar{N} = 0$ on Σ_0^0,

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and

$$\int_{\mathcal{V}_0^0} r\left[\bar{\nabla}U_r\cdot\bar{\nabla}w_r+\bar{\nabla}U_\theta\cdot\bar{\nabla}w_\theta+\bar{\nabla}U_z\cdot\bar{\nabla}w_z+\frac{U_rw_r}{r}+\frac{U_\theta w_\theta}{r}\right]$$
$$-q\left(\overline{\mathrm{div}}(w_r,w_z)^\top+\frac{w_r}{r}\right)$$
$$=\int_{\mathcal{V}_0^0} rG^1\cdot w+\int_{\Sigma_0^0} rG^2w_\theta+\int_{\Sigma_0^0} rG^3(w\cdot\bar{T}) \tag{2.6}$$

for all $w\in\{w\in H^1(\mathcal{V}_0^0): w\cdot\bar{N}=0 \text{ on } \Sigma_0^0, \text{ and } w=0, \text{ on } \Sigma_{a,0} \text{ and } \Sigma_{b,0}\}$ is a weak solution to (2.1). Here, $H^1(\mathcal{V}_0^0)$ stands for the usual Sobolev space, and we denote $H^0(\mathcal{V}_0^0)=L^2(\mathcal{V}_0^0)$, and $\mathring{H}^0(\mathcal{V}_0^0)=\{U\in H^0(\mathcal{V}_0^0):\int_{\mathcal{V}_0^0}U=0\}$.

Using the Riesz representation theorem, by a similar argument to theorem 5.4 of [9], we can obtain the following existence theorem of weak solutions.

Theorem 2.2. *Let $(G^1,G^2,G^3)\in X_{\delta_0}^0$ for some $0<\delta_0<1$. Then there exists a unique pair $(U,Q)\in H^1(\mathcal{V}_0^0)\times\mathring{H}^0(\mathcal{V}_0^0)$ that is a weak solution to (2.1). Moreover,*

$$\|U\|_{H^1}^2+\|Q\|_{\mathring{H}^0}^2\leqslant C(a,b)\left(\|G^1\|_{W_{\delta_0}^0}^2+\|G^2\|_{W_{\delta_0}^{\frac{1}{2}}}^2+\|G^3\|_{W_{\delta_0}^{\frac{1}{2}}}^2\right). \tag{2.7}$$

Now, we focus on the k-order regularity of the system (2.1). We have the following result.

Theorem 2.3. *Let $\omega\in(0,\pi)$ be the angle formed by Σ_0^0 at the corners of $\mathcal{V}_0^0$, $\delta_\omega=\max\{0,2-\pi/\omega\}\in[0,1)$, and $\delta_k\in(k+\delta_\omega,k+1)$. Let $(G^1,G^2,G^3)\in X_{\delta_k}^k$, and (U,q) be the weak solution to (2.1). Then $U\in W_{\delta_k}^{k+2}(\mathcal{V}_0^0)$, $Q\in\mathring{W}_{\delta_k}^{k+1}(\mathcal{V}_0^0)$, and*

$$\|U\|_{W_{\delta_k}^{k+2}}^2+\|Q\|_{\mathring{W}_{\delta_k}^{k+1}}^2\leqslant C(k,a,b)\left(\|G^1\|_{W_{\delta_k}^k}^2+\|G^2\|_{W_{\delta_k}^{k+\frac{1}{2}}}^2+\|G^3\|_{W_{\delta_k}^{k+\frac{1}{2}}}^2\right). \tag{2.8}$$

Proof. First, we set

$$K_\omega=\left\{x\in\mathbb{R}^2: R>0,\ \Theta\in(-\pi/2,-\pi/2+\omega)\right\}, \tag{2.9}$$

and

$$\Gamma_-=\{x\in\mathbb{R}^2: R>0,\ \Theta=-\pi/2\}, \tag{2.10}$$

and

$$\Gamma_+=\{x\in\mathbb{R}^2: R>0,\ \Theta=-\pi/2+\omega\}, \tag{2.11}$$

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where (R,Θ) are standard coordinates in $\mathbb{R}^2$. Now, we consider the following two problems:

$$\begin{cases} -\bar{\nabla} Q + \bar{\Delta}\bar{U} + \frac{1}{r}\partial_r \bar{U} - \left(\frac{u_r}{r^2}, 0\right)^\top = (G_r^1, G_z^1)^\top, & \text{in } K_\omega, \\ \overline{\operatorname{div}}\,\bar{U} + \frac{U_r}{r} = 0, & \text{in } K_\omega, \\ \bar{U}\cdot\bar{N} = 0, & \text{on } \Gamma_+ \\ \bar{N}\cdot\bar{\mathbb{S}}(U)\cdot\bar{T} = G^2, & \text{on } \Gamma_+, \\ U = 0, & \text{on } \Gamma_-, \end{cases} \tag{2.12}$$

and

$$\begin{cases} \bar{\Delta} u_\theta + \frac{1}{r}\partial_r U_\theta - \frac{U_\theta}{r^2} = G_\theta^1, & \text{in } K_\omega, \\ \bar{N}\cdot\bar{\nabla} U_\theta = G^3, & \text{on } \Sigma_+, \\ U = 0, & \text{on } \Sigma_-. \end{cases} \tag{2.13}$$

From theorem 5.2 of [**9**] or theorem 9.4.9 in [**13**] for the problem (2.12), and theorem 6.5.4 in [**13**] for the problem (2.13), we have

$$\|U\|^2_{W^{k+2}_{\delta_k}(K_\omega)} + \|Q\|^2_{\mathring{W}^{k+1}_{\delta_k}(K_\omega)} \leqslant C(k,a,b)\left(\|G^1\|^2_{W^k_{\delta_k}} + \|G^2\|^2_{W^{k+\frac{1}{2}}_{\delta_k}} + \|G^3\|^2_{W^{k+\frac{1}{2}}_{\delta_k}}\right). \tag{2.14}$$

Now, by means of the above estimates, we follow the arguments in [**9**] to obtain the estimates near the corners by constructing a diffeomorphism to transform the problem in $\mathcal{V}_0^0$ to K_ω, and get the estimates away from the corners by using the standard elliptic estimates. Finally, we can get the conclusion by collecting the estimates near the corners and away from the corners. □

Next, we let $\bar{H} \in W^{k+\frac{5}{2}}_{\delta_k}$ be given function with $\delta_k \in (0,1)$, which in turn determines $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{T}$, and start to study the following $\mathcal{A}$-Stokes system

$$\begin{cases} -\bar{\nabla}_{\mathcal{A}} Q + \bar{\Delta}_{\mathcal{A}}(M^{-1}\bar{U}) + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}}(M^{-1}\bar{U}))_r \\ \quad - \left(\frac{(M^{-1}U)_r}{r^2}, 0\right)^\top = (G_r^1, G_z^1)^\top & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}_{\mathcal{A}} U_\theta + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}})_r U_\theta - \frac{U_\theta}{r^2} = G_\theta^1 & \text{in } \mathcal{V}_0^0, \\ \overline{\operatorname{div}}\,\bar{U} + \frac{U_r}{r} = 0 & \text{in } \mathcal{V}_0^0, \\ \bar{U}\cdot\bar{N} = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}\cdot\bar{\nabla}_{\mathcal{A}} U_\theta = G^2 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}\cdot\bar{\mathbb{S}}_{\mathcal{A}}(M^{-1}\bar{U})\cdot\bar{\mathcal{T}} = G^3 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}\cdot\mathbb{S}_{\mathcal{A}}(M^{-1}\bar{U})\cdot\bar{\mathcal{N}} = -gH + \frac{\sigma}{r}\left(H' + G^4\right)' + G^5, & \text{on } \Sigma_0^0, \\ U = 0 & \text{on } \Sigma_{a,0}^0, \\ U = 0 & \text{on } \Sigma_{b,0}^0. \end{cases} \tag{2.15}$$

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The system (2.15) can be rewritten as follows.

$$\begin{cases} -\bar{\nabla} Q + \bar{\Delta}\bar{U} + \frac{1}{r}(\bar{\nabla}\bar{U})_r - \left(\frac{U_r}{r^2}, 0\right)^\top = (\tilde{G}^1_r, \tilde{G}^1_z)^\top & \text{in } \mathcal{V}^0_0, \\ \bar{\Delta} U_\theta + \frac{1}{r}\partial_r U_\theta - \frac{U_\theta}{r^2} = \tilde{G}^1_\theta & \text{in } \mathcal{V}^0_0, \\ \overline{\operatorname{div}}\, \bar{U} + \frac{U_r}{r} = 0 & \text{in } \mathcal{V}^0_0, \\ \bar{U} \cdot \bar{N} = 0 & \text{on } \Sigma^0_0, \\ \bar{\mathcal{N}} \cdot \bar{\nabla} U_\theta = \tilde{G}^2 & \text{on } \Sigma^0_0, \\ \bar{\mathcal{N}} \cdot \bar{\mathbb{S}}(U) \cdot \bar{\mathcal{T}} = \tilde{G}^3 & \text{on } \Sigma^0_0, \\ U = 0 & \text{on } \Sigma^0_{a,0}, \\ U = 0 & \text{on } \Sigma^0_{b,0}, \end{cases} \tag{2.16}$$

with

$$\bar{\mathcal{N}} \cdot \mathbb{S}_{\mathcal{A}}(M^{-1}\bar{U}) \cdot \bar{\mathcal{N}} = -gH + \frac{\sigma}{r}\left(H' + G^4\right)' + G^5 \quad \text{on } \Sigma^0_0, \tag{2.17}$$

where

$$\begin{aligned} &(\tilde{G}^1_r, \tilde{G}^1_z)^\top(U, q) \\ &\quad = (G^1_r, G^1_z)^\top + \bar{\nabla}_{\mathcal{A}-\bar{I}} q - \bar{\Delta}((M^{-1} - I)\bar{U}) - \bar{\Delta}_{\mathcal{A}-\bar{I}}(M^{-1}\bar{U}) \\ &\qquad + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}-\bar{I}}(M^{-1}\bar{U}))_r + \frac{1}{r}(\bar{\nabla}((M^{-1} - I)\bar{U}))_r, \end{aligned} \tag{2.18}$$

and

$$\tilde{G}^1_\theta(U, q) = G^1_\theta - \bar{\Delta}_{\mathcal{A}-\bar{I}} U_\theta + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}-\bar{I}} U_\theta)_r, \tag{2.19}$$

and

$$\tilde{G}^2(U, q) = G^2 + -(\bar{\mathcal{N}} - \bar{N}) \cdot \bar{\nabla}_{\mathcal{A}} U_\theta - \bar{N} \cdot \bar{\nabla}_{\mathcal{A}-\bar{I}} U_\theta, \tag{2.20}$$

and

$$\begin{aligned} \tilde{G}^3(U, q) = G^3 &+ -(\bar{\mathcal{N}} - \bar{N}) \cdot \bar{\mathbb{S}}_{\mathcal{A}}(M^{-1}U) \cdot \bar{\mathcal{T}} - \bar{N} \cdot \bar{\mathbb{S}}_{\mathcal{A}-\bar{I}}(M^{-1}U) \cdot \bar{\mathcal{T}} \\ &+ \bar{N} \cdot \bar{\mathbb{S}}(M^{-1}U) \cdot (\bar{\mathcal{T}} - \bar{T}) - \bar{\mathcal{N}} \cdot \bar{\mathbb{S}}((M^{-1} - I)U) \cdot \bar{\mathcal{T}}. \end{aligned} \tag{2.21}$$

Set

$$\tilde{G}(U, Q) = (\tilde{G}^1(U, q), \tilde{G}^2(U, Q), \tilde{G}^3(U, q)), \tag{2.22}$$

we can get that

$$\|\tilde{G}(U, Q)\|_{X^k_{\delta_k}} \leqslant \|(G^1, G^2, G^3)\|_{X^k_{\delta_k}} + P\left(\|\bar{H}\|_{W^{k+\frac{5}{2}}_{\delta_k}}\right)(\|U\|_{W^{k+2}_{\delta_k}} + \|Q\|_{\mathring{W}^{k+1}_{\delta_k}}), \tag{2.23}$$

and

$$\|\tilde{G}(U_1, Q_1) - \tilde{G}(U_2, Q_2)\|_{X^k_{\delta_k}} \leqslant P\left(\|\bar{H}\|_{W^{k+\frac{5}{2}}_{\delta_k}}\right)(\|U_1 - U_2\|_{W^{k+2}_{\delta_k}} + \|Q_1 - Q_2\|_{\mathring{W}^{k+1}_{\delta_k}}), \tag{2.24}$$

where $P(\cdot)$ is a polynomial with non-negative coefficients such that $P(0) = 0$. According to these two estimates, if $\|\bar{H}\|^2_{W^{k+\frac{5}{2}}_{\delta_k}}$ is small enough, then by contraction mapping principle and theorem 2.3, we can get that the system (2.16) is solvable, and

$$\|U\|^2_{W^{k+2}_{\delta_k}} + \|Q\|^2_{\mathring{W}^{k+1}_{\delta_k}} \leqslant C(k, a, b)\left(\|G^1\|^2_{W^k_{\delta_k}} + \|G^2\|^2_{W^{k+\frac{1}{2}}_{\delta_k}} + \|G^3\|^2_{W^{k+\frac{1}{2}}_{\delta_k}}\right). \tag{2.25}$$

Having obtained (U, H), by (2.17), we have

$$\|H\|^2_{W^{k+\frac{5}{2}}_{\delta_k}} \leqslant C(k, a, b)\left(\|U\|^2_{W^{k+2}_{\delta_k}} + \|(G^4)'\|^2_{W^{k+\frac{1}{2}}_{\delta_k}} + \|G^5\|^2_{W^{k+\frac{1}{2}}_{\delta_k}}\right). \tag{2.26}$$

Collecting (2.25) and (2.26), we have

THEOREM 2.4. *Set $\delta_\omega = \max\{0, 2 - \pi/\omega\} \in [0, 1)$, and $\delta_k \in (k + \delta_\omega, k + 1)$. Let $(G^1, G^2, G^3, G^4, G^5) \in X^k_{\delta_k}$. If $\|\bar{H}\|^2_{W^{k+\frac{5}{2}}_{\delta_k}}$ is small enough, then problem (2.15) is solvable. Moreover, we have*

$$\begin{aligned}\|U\|^2_{W^{k+2}_{\delta_k}} &+ \|Q\|^2_{\mathring{W}^{k+1}_{\delta_k}} + \|H\|^2_{W^{k+\frac{5}{2}}_{\delta_k}} \\ &\leqslant C(k, a, b)\Bigg(\|G^1\|^2_{W^k_{\delta_k}} + \|G^2\|^2_{W^{k+\frac{1}{2}}_{\delta_k}} + \|G^3\|^2_{W^{k+\frac{1}{2}}_{\delta_k}} \\ &\quad + \|(G^4)'\|^2_{W^{k+\frac{1}{2}}_{\delta_k}} + \|G^5\|^2_{W^{k+\frac{1}{2}}_{\delta_k}}\Bigg).\end{aligned} \tag{2.27}$$

REMARK 2.5. It follows from the argument of theorem 3.1 in [**10**], it is possible to obtain the following result:

Set $\omega = \pi/2$. Let $q > 3$ and $(G^1, G^2, G^3, G^4, G^5) \in L^q \times W^{1-1/q}_q \times W^{1-1/q,q} \times W^{1-1/q,q} \times W^{1-1/q,q}$, where $W^{k,q}$ is the usual Sobolev spaces. If $\|\bar{H}\|^2_{W^{3-1/q,q}}$ is small enough, then problem (2.15) is solvable. Moreover, we have

$$\begin{aligned}\|U\|^2_{W^{2,q}} &+ \|Q\|^2_{W^{1,q}} + \|H\|^2_{W^{3-1/q,q}} \\ &\leqslant C\left(\|G^1\|^2_{L^q} + \|G^2\|^2_{W^{1-1/q,q}} + \|G^3\|^2_{W^{1-1/q,q}}\right. \\ &\quad \left. + \|(G^4)'\|^2_{W^{1-1/q,q}} + \|G^5\|^2_{W^{1-1/q,q}}\right),\end{aligned} \tag{2.28}$$

where C is a generic constant.

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3. Proof of theorem 1.1

In this section, we focus on the proof of theorem 1.1. We solve the system (1.43) by successive approximations.

Let $(U^{(0)}, Q^{(0)}, H^{(0)})^\top = (0, p_a, 0)^\top$ and define $(U^{(l+1)}, Q^{(l+1)}, H^{(l+1)})^\top$ as the solution to the following system.

$$\begin{cases}
-\bar{\nabla}_{\mathcal{A}^{(l)}} Q^{(l+1)} + \bar{\Delta}_{\mathcal{A}^l}((M^{(l)})^{-1}\bar{U}^{(l+1)}) + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}\bar{U}^{(l+1)}))_r & \\
-\left(\frac{((M^{(l)})^{-1}U^{(l+1)})_r}{r^2}, 0\right)^\top = (M^{(l)})^{-1}\bar{U}^{(l)} \cdot \bar{\nabla}_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}\bar{U}^l) & \\
\quad -\frac{U_\theta^{(l)}(U_\theta^{(l)} + 2v_\theta^{[1]}\Omega)}{r} & \text{in } \mathcal{V}_0^0, \\
\bar{\Delta}_{\mathcal{A}^{(l)}} U_\theta^{(l+1)} + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}^{(l)}})_r U_\theta^{(l+1)} - \frac{U_\theta^{(l+1)}}{r^2} & \\
\quad = \bar{U}^{(l)} \cdot \bar{\nabla}_{\mathcal{A}^{(l)}}(U_\theta^{(l)} + v_\theta^{[1]}\Omega) + \frac{U_r^{(l)}(U_\theta^{(l)} + v_\theta^{[1]}\Omega)}{r} & \text{in } \mathcal{V}_0^0, \\
\overline{\operatorname{div}}\, \bar{U}^{(l+1)} + \frac{U_r^{(l+1)}}{r} = 0, & \text{in } \mathcal{V}_0^0, \\
\bar{U}^{(l+1)} \cdot \bar{\mathcal{N}}^{(l)} = 0, & \text{on } \Sigma_0^0, \\
\bar{\mathcal{N}}^{(l)} \cdot \bar{\nabla}_{\mathcal{A}^{(l)}} U_\theta^{(l+1)} = \Omega(\bar{N} - \bar{\mathcal{N}}^l) \cdot \nabla v_\theta^{[1]} & \text{on } \Sigma_0^0, \\
\bar{\mathcal{N}}^{(l)} \cdot \bar{\mathbb{S}}_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}U^{(l+1)}) \cdot \bar{\mathcal{T}}^{(l+1)} = 0 & \text{on } \Sigma_0^0, \\
\bar{\mathcal{N}}^{(l)} \cdot (Q^{(l+1)}\bar{I} + \bar{\mathbb{S}}_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}U^{(l+1)}) \cdot \bar{\mathcal{N}}^{(l)} & \\
= -gH^{(l+1)} + \frac{\sigma}{r}\left(r(H^{(l+1)})' + \mathcal{R}(0, (H^{(l)} + \frac{\Omega^2}{2}(h^{[2]}))')\right)' & \text{on } \Sigma_0^0, \\
U^{(l+1)} = 0 & \text{on } \Sigma_{a,0}^0, \\
U^{(l+1)} = 0 & \text{on } \Sigma_{b,0}^0.
\end{cases} \tag{3.1}$$

Here

$$\bar{\mathcal{N}}^{(l)} = \left(\frac{-(h^{(l)})'}{\sqrt{1 + (h^{(l)})'^2}}, \frac{1}{\sqrt{1 + (h^{(l)})'^2}}\right)^\top, \tag{3.2}$$

and

$$\mathcal{A}^{(l)} := \begin{pmatrix} 1 & -\partial_r \widetilde{h^{(l)}}(1 + \partial_z \widetilde{h^{(l)}})^{-1} \\ 0 & (1 + \partial_z \widetilde{h}^l)^{-1} \end{pmatrix}, \tag{3.3}$$

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and $M^{(l)} = (J^{(l)}(\mathcal{A}^{(l)})^{\top})^{-1}$ with $J^{(l)} = det\mathcal{A}^{(l)}$, where

$$h^{(l)} = H^{(l)} + \frac{\Omega^2}{2} h^{[2]}. \tag{3.4}$$

By theorem 2.4 and the following embedding theorem (see [**9**]),

$$W^2_\delta(\mathcal{V}^0_0) \hookrightarrow W^{2,q}(\mathcal{V}^0_0) \text{ with } 1 \leqslant q < \frac{2}{1+\delta}, \tag{3.5}$$

we have

$$\|U^{(1)}\|_{W^{k+2}_{\delta_k}} + \|Q^{(1)}\|_{\mathring{W}^{k+1}_{\delta_k}} + \|H^{(1)}\|_{W^{k+\frac{5}{2}}_{\delta_k}} \leqslant C_1(k,a,b,\sigma,\lambda)\Omega^3 \tag{3.6}$$

Choose $0 < \Omega < 1$ be sufficiently small $C_1(k,a,b,\sigma,\lambda)\Omega \leqslant 1$. Then

$$\|U^{(1)}\|_{W^{k+2}_{\delta_k}} + \|Q^{(1)}\|_{\mathring{W}^{k+1}_{\delta_k}} + \|H^{(1)}\|_{W^{k+\frac{5}{2}}_{\delta_k}} \leqslant \Omega^2. \tag{3.7}$$

Let Ω be sufficiently small such that $\|(M^{(1)})^{-1}\|_{L^\infty} \leqslant \frac{1}{4}$, and $\|H^{(1)}\|_{W^{k+\frac{5}{2}}_{\delta_k}}$ is small enough, then by theorem 2.4 and (3.5) again, we can get that

$$\begin{aligned}
&\|U^{(2)}\|_{W^{k+2}_{\delta_k}} + \|Q^{(2)}\|_{\mathring{W}^{k+1}_{\delta_k}} + \|H^{(2)}\|_{W^{k+\frac{5}{2}}_{\delta_k}} \\
&\quad \leqslant C_0(k,a,b,\sigma)\Big(\|(M^{(1)})^{-1}\bar{U}^{(1)} \cdot \bar{\nabla}_{\mathcal{A}^{(1)}}((M^{(1)})^{-1}\bar{U}^{(1)}) \\
&\qquad - r^{-1}\left[U^{(1)}_\theta(U^{(1)}_\theta + 2v^{[1]}_\theta\Omega)\right]\|_{W^k_{\delta_k}} \\
&\qquad + \|\bar{U}^{(1)} \cdot \bar{\nabla}_{\mathcal{A}}(U^{(1)}_\theta + v^{[1]}_\theta\Omega) + r^{-1}(U^{(1)}_r(U^{(1)}_\theta + v^{[1]}_\theta\Omega))\|_{W^k_{\delta_k}} \\
&\qquad + \|\Omega(\bar{N} - \bar{\mathcal{N}}^1) \cdot \nabla v^{[1]}_\theta\|_{W^k_{\delta_k}} + \|(\mathcal{R}(0,(h^{(1)})'))'\|_{W^{k+\frac{1}{2}}_{\delta_k}}\Big) \\
&\quad \leqslant C_2(k,a,b,\sigma)\left(\|U^{(1)}\|^2_{W^{k+2}_{\delta_k}} + \Omega\|U^{(1)}\|_{W^{k+2}_{\delta_k}} + \|H^{(1)}\|^3_{W^{k+\frac{5}{2}}_{\delta_k}} + \Omega^3\right) \\
&\quad \leqslant 4C_2(k,a,b,\sigma)\Omega^3.
\end{aligned} \tag{3.8}$$

Choose Ω be sufficiently small such that

$$4C_2(k,a,b,\sigma)\Omega \leqslant 1, \tag{3.9}$$

then

$$\|U^{(2)}\|_{W^{k+2}_{\delta_k}} + \|Q^{(2)}\|_{\mathring{W}^{k+1}_{\delta_k}} + \|H^{(2)}\|_{W^{k+\frac{5}{2}}_{\delta_k}} \leqslant \Omega^2. \tag{3.10}$$

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Hence we have

$$\begin{aligned}&\|U^{(3)}\|_{W^{k+2}_{\delta_k}}+\|Q^{(3)}\|_{\mathring{W}^{k+1}_{\delta_k}}+\|H^{(3)}\|_{W^{k+\frac{5}{2}}_{\delta_k}}\\&\quad\leqslant C_2(k,a,b,\sigma)\left(\|U^{(2)}\|^2_{W^{k+2}_{\delta_k}}+\Omega\|U^{(2)}\|_{W^{k+2}_{\delta_k}}+\|H^{(2)}\|^3_{W^{k+\frac{5}{2}}_{\delta_k}}+\Omega^3\right)\\&\quad\leqslant 4C_2(k,a,b,\sigma)\Omega^3.\end{aligned}\tag{3.11}$$

Repeating the above procedure, we can obtain for any $k\in\mathbb{N}$,

$$\begin{aligned}&\|U^{(l)}\|_{W^{k+2}_{\delta_k}}+\|Q^{(l)}\|_{\mathring{W}^{1}_{\delta_k}}+\|H^{(l)}\|_{W^{k+\frac{5}{2}}_{\delta_k}}\\&\quad\leqslant C_2(k,a,b,\sigma)\left(\|U^{(l-1)}\|^2_{W^{k+2}_{\delta_k}}+\Omega\|U^{(l-1)}\|_{W^{k+2}_{\delta_k}}+\|H^{(l-1)}\|^3_{W^{k+\frac{5}{2}}_{\delta_k}}+\Omega^3\right)\\&\quad\leqslant 4C_2(k,a,b,\sigma)\Omega^3.\end{aligned}\tag{3.12}$$

Next, we consider the system $(3.1)_{(l+1)}$–$(3.1)_{(l)}$, i.e.,

$$\begin{cases}-\bar\nabla_{\mathcal{A}^{(l)}}(Q^{(l+1)}-Q^{(l)})+\bar\Delta_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}(\bar U^{(l+1)}-\bar U^{(l)}))\\+\dfrac{1}{r}(\bar\nabla_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}(\bar U^{(l+1)}-\bar U^{(l)}))_r\\\quad-\left(\dfrac{U_r^{(l+1)}-U_r^{(l)}}{r^2},0\right)^\top=(G_r^{(l)},G_z^{(l)})^\top & \text{in } \mathcal{V}_0^0,\\ \bar\Delta_{\mathcal{A}^l}(U_\theta^{(l+1)}-U_\theta^{(l)})+\dfrac{1}{r}(\bar\nabla_{\mathcal{A}^{(l)}}(U_\theta^{(l+1)}-U_\theta^{(l)}))_r-\dfrac{U_\theta^{(l+1)}-U_\theta^{(l)}}{r^2}=G_\theta^{(l)} & \text{in } \mathcal{V}_0^0,\\ \overline{\mathrm{div}}(\bar U^{l+1}-\bar U^{(l)})+\dfrac{U_r^{(l+1)}-U_r^{(l)}}{r}=0 & \text{in } \mathcal{V}_0^0,\\ (\bar U^{(l+1)}-\bar U^{(l)})\cdot\bar N^{(l)}=0, & \text{on } \Sigma_0^0,\\ \bar{\mathcal{N}}^{(l)}\cdot\bar\nabla_{\mathcal{A}^{(l)}}(U_\theta^{(l+1)}-U_\theta^{(l)})=G^2 & \text{on } \Sigma_0^0,\\ \bar{\mathcal{N}}^{(l)}\cdot\bar{\mathbb{S}}_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}(U^{l+1}-U^{(l)}))\cdot\bar{\mathcal{T}}^{(l)}=G^3 & \text{on } \Sigma_0^0,\\ \bar{\mathcal{N}}^{(l)}\cdot((Q^{(l+1)}-Q^{(l)})\bar I+\bar{\mathbb{S}}_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}(U^{(l+1)}-U^{(l)}))\cdot\bar{\mathcal{N}}^{(l)}\\=-g(H^{(l+1)}-H^{(l)})+\dfrac{\sigma}{r}\left(r(H^{(l+1)}-H^{(l)})'+G^4\right)'+G^5 & \text{on } \Sigma_0^0,\\ (H^{(l+1)}-H^{(l)})'(a;\Omega)=0,\quad (H^{(l+1)}-H^{(l)})'(b;\Omega)=0,\\ U^{(l+1)}-U^{(l)}=0 & \text{on } \Sigma_{a,0}^0,\\ U^{(l+1)}-U^{(l)}=0 & \text{on } \Sigma_{b,0}^0,\end{cases}\tag{3.13}$$

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where

$$
\begin{aligned}
(G_r^{(l)}, G_z^{(l)}) = {} & (M^{(l)})^{-1}\bar{U}^{(l)} \cdot \bar{\nabla}_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}\bar{U}^{l}) - \frac{U_\theta^{(l)}(U_\theta^{(l)} + 2v_\theta^{[1]}\Omega)^2)}{r} \\
& - (M^{(l-1)})^{-1}\bar{U}^{(l-1)} \cdot \bar{\nabla}_{\mathcal{A}^{(l-1)}}((M^{(l-1)})^{-1}\bar{U}^{l-1}) \\
& + \frac{U_\theta^{(l-1)}(U_\theta^{(l-1)} + 2v_\theta^{[1]}\Omega)}{r} \\
& + \bar{\nabla}_{\mathcal{A}^{(l)}-\mathcal{A}^{(l-1)}}(Q^{(l)}) - \bar{\Delta}_{\mathcal{A}^{(l)}-\mathcal{A}^{(l-1)}}((M^{(l-1)})^{-1}(\bar{U}^{(l)})) \\
& - \bar{\Delta}_{\mathcal{A}^{(l-1)}}((M^{(l)} - M^{(l-1)})^{-1}(\bar{U}^{(l)})) \\
& - \frac{1}{r}(\bar{\nabla}_{\mathcal{A}^{(l)}-\mathcal{A}^{(l-1)}}((M^{(l-1)})^{-1}(\bar{U}^{(l)}))) \\
& - \frac{1}{r}(\bar{\nabla}_{\mathcal{A}^{(l-1)}}((M^{(l)} - M^{(l-1)})^{-1}(\bar{U}^{(l)}))), \qquad (3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
G_\theta^1 = {} & \bar{U}^{(l)} \cdot \bar{\nabla}_{\mathcal{A}}(U_\theta^{(l)} + v_\theta^{[1]}\Omega) + \frac{U_r^{(l)}(U_\theta^{(l)} + v_\theta^{[1]}\Omega)}{r} \\
& - \bar{U}^{(l-1)} \cdot \bar{\nabla}_{\mathcal{A}}(U_\theta^{(l-1)} + v_\theta^{[1]}\Omega) - \frac{U_r^{(l-1)}(U_\theta^{(l-1)} + v_\theta^{[1]}\Omega)}{r} \\
& - \bar{\Delta}_{\mathcal{A}^{(l)}-\mathcal{A}^{(l-1)}}U_\theta^{(l)} + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}^{(l)}-\mathcal{A}^{(l-1)}})_r U_\theta^{(l)}, \qquad (3.15)
\end{aligned}
$$

and

$$
G^2 = \bar{\mathcal{N}}^{(l-1)} \cdot \bar{\nabla}_{\mathcal{A}^{(l-1)}}U_\theta^{(l)} - \bar{\mathcal{N}}^{(l)} \cdot \bar{\nabla}_{\mathcal{A}^{(l)}}U_\theta^{(l)}, \qquad (3.16)
$$

and

$$
G^3 = \bar{\mathcal{N}}^{(l-1)} \cdot \bar{\mathbb{S}}_{\mathcal{A}^{(l-1)}}((M^{(l-1)})^{-1}U^{(l)}) \cdot \bar{\mathcal{T}}^{(l)} - \bar{\mathcal{N}}^{(l)} \cdot \bar{\mathbb{S}}_{\mathcal{A}^{(l)}}((M^{l})^{-1}U^{(l)}) \cdot \bar{\mathcal{T}}^{(l)}, \qquad (3.17)
$$

and

$$
G^4 = \mathcal{R}(0, (h^{(l)})') - \mathcal{R}(0, (h^{(l-1)})'), \qquad (3.18)
$$

and

$$
\begin{aligned}
G^5 = {} & \bar{\mathcal{N}}^{(l-l)} \cdot (Q^{(l)}\bar{I} + \bar{\mathbb{S}}_{\mathcal{A}^{(l-1)}}(M^{(l-1)})^{-1}U^{(l)}) \cdot \bar{\mathcal{N}}^{(l-1)} \\
& - \bar{\mathcal{N}}^{(l)} \cdot (Q^{(l)}\bar{I} + \bar{\mathbb{S}}_{\mathcal{A}^{(l)}}((M^{(l)})^{-1}U^{(l)}) \cdot \bar{\mathcal{N}}^{(l)}. \qquad (3.19)
\end{aligned}
$$

By theorem 2.4 and the embedding result (3.5), it follows from (3.12) that

$$
\begin{aligned}
& \|U^{(l+1)} - U^{(l)}\|_{W_{\delta_k}^{k+2}} + \|Q^{(l+1)} - Q^{(l)}\|_{\mathring{W}_{\delta_k}^{k+1}} + \|H^{(l+1)} - H^{(l)}\|_{W_{\delta_k}^{k+\frac{5}{2}}} \\
& \quad \leqslant C_0(a, b, \lambda, \sigma)\Big(\|U^{(l+1)}\|_{W_{\delta_k}^{k+2}} + \|U^{(l)}\|_{W_{\delta_k}^{2}} + \|U^{(l-1)}\|_{W_{\delta_k}^{k+2}} + \|H^{(l+1)}\|_{W^{k+\frac{5}{2}}}
\end{aligned}
$$

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$$
\begin{aligned}
&+\|H^{(l)}\|_{W^{k+\frac{5}{2}}}+\|H^{(l-1)}\|_{W^{k+\frac{5}{2}}}+\|Q^{(l)}\|_{\mathring{W}^{k+1}_{\delta_k}}+\Omega\Big)\\
&\times\left(\|U^{(l)}-U^{(l-1)}\|_{W^{k+2}_{\delta_k}}+\|Q^{(l)}-Q^{(l-1)}\|_{\mathring{W}^{k+1}_{\delta_k}}+\|H^{(l)}-H^{(l-1)}\|_{W^{k+\frac{5}{2}}_{\delta_k}}\right)\\
\leqslant{}& 3C_0(a,b,\lambda,\sigma)\Omega\left(\|U^{(l)}-U^{(l-1)}\|_{W^{k+2}_{\delta_k}}+\|Q^{(l)}-Q^{(l-1)}\|_{\mathring{W}^{k+1}_{\delta_k}}\right.\\
&\left.+\|H^{(l)}-H^{(l-1)}\|_{W^{k+\frac{5}{2}}_{\delta_k}}\right). \qquad (3.20)
\end{aligned}
$$

Let $3C_0(a,b,\lambda,\gamma_a,\gamma_b,\sigma)\Omega\leqslant\frac{1}{2}$, then

$$
\begin{aligned}
&\|U^{(l+1)}-U^{(l)}\|_{W^{k+2}_{\delta_k}}+\|Q^{(l+1)}-Q^{(l)}\|_{\mathring{W}^{k+1}_{\delta_k}}+\|H^{(l+1)}-H^{(l)}\|_{W^{k+\frac{5}{2}}_{\delta_k}}\\
&\leqslant\frac{1}{2}\left(\|U^{(l)}-U^{(l-1)}\|_{W^{k+2}_{\delta_k}}+\|Q^{(l)}-Q^{(l-1)}\|_{\mathring{W}^{k+1}_{\delta_k}}+\|H^{(l)}-H^{(l-1)}\|_{W^{k+\frac{5}{2}}_{\delta_k}}\right).
\end{aligned} \tag{3.21}
$$

Now, it follows from (3.12) and (3.21) that $(U^{(l)},Q^{(l)},H^{(l)})$ is a cauchy sequence in $W^{k+2}_{\delta}\times W^{k+1}_{\delta_k}\times W^{k+\frac{5}{2}}_{\delta_k}$. Hence, there exists a subsequence $(U^{(l_m)},Q^{(l_m)},H^{(l_m)})$, and $(U,Q,H)\in W^{k+2}_{\delta_k}\times W^{k+1}_{\delta_k}\times W^{k+\frac{5}{2}}_{\delta_k}$ such that as $m\to\infty$,

$$
(U^{(l_m)},Q^{(l_m)},H^{(l_m)})\to(U,Q,H),\quad \text{in } W^{k+2}_{\delta_k}\times W^{k+1}_{\delta_k}\times W^{k+\frac{5}{2}}_{\delta_k}. \tag{3.22}
$$

Now, let $m\to\infty$ for equations $(3.1)_{l_m}$, we have (U,Q,H) satisfy

$$
\begin{cases}
-\bar\nabla_{\mathcal{A}}Q+\bar\Delta_{\mathcal{A}}(M^{-1}\bar U)+\dfrac{1}{r}(\bar\nabla_{\mathcal{A}}(M^{-1}\bar U))_r-\left(\dfrac{(M^{-1}U)_r}{r^2},0\right)^{\top} & \\
=M^{-1}\bar U\cdot\bar\nabla_{\mathcal{A}}(M^{-1}\bar U)-\dfrac{(U_\theta+v^{[1]}_\theta\Omega)^2-(v^{[1]}_\theta\Omega)^2}{r} & \text{in } \mathcal{V}^0_0,\\
\bar\Delta_{\mathcal{A}}U_\theta+\dfrac{1}{r}(\bar\nabla_{\mathcal{A}})_rU_\theta-\dfrac{U_\theta}{r^2}=\bar U\cdot\bar\nabla_{\mathcal{A}}(U_\theta+w^0_\theta)+\dfrac{U_r(U_\theta+v^{[1]}_\theta\Omega)}{r} & \text{in } \mathcal{V}^0_0,\\
\overline{\operatorname{div}}\,\bar U+\dfrac{U_r}{r}=0 & \text{in } \mathcal{V}^0_0,\\
\bar U\cdot\bar N=0 & \text{on } \Sigma^0_0,\\
\bar N\cdot\bar\nabla_{\mathcal{A}}U_\theta=0 & \text{on } \Sigma^0_0,\\
\bar{\mathcal{N}}\cdot\bar{\mathbb{S}}_{\mathcal{A}}(M^{-1}U)\cdot\bar{\mathcal{T}}=0 & \text{on } \Sigma^0_0,\\
\bar{\mathcal{N}}\cdot(Q\bar I+\bar{\mathbb{S}}_{\mathcal{A}}(M^{-1}U))\cdot\bar{\mathcal{N}}=-gH+\dfrac{\sigma}{r}\left(rH'+\mathcal{R}(0,h')\right)' & \text{on } \Sigma^0_0,\\
U=0 & \text{on } \Sigma^0_{a,0},\\
U=0 & \text{on } \Sigma^0_{b,0},
\end{cases} \tag{3.23}
$$

and

$$\|U\|_{W^{k+2}_{\delta_k}} + \|Q\|_{\mathring{W}^{k+1}_{\delta_k}} + \|H\|_{W^{k+\frac{5}{2}}_{\delta_k}} \leqslant 4C_2(k,a,b,\sigma)\Omega^3. \tag{3.24}$$

Thus, we have completed the proof of theorem 1.1.

Finally, we turn to the uniqueness. Assume that (U^1, Q^1, H^1) and (U^2, Q^2, H^2) are the solution of the system (1.43), that is,

$$\begin{cases} -\bar{\nabla}_{\mathcal{A}^1} Q^1 + \bar{\Delta}_{\mathcal{A}^1}((M^1)^{-1}\bar{U}^1) + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}^1}((M^1)^{-1}\bar{U}^1))_r - \left(\frac{((M^1)^{-1}U^1)_r}{r^2}, 0\right)^\top & \\ = (M^1)^{-1}\bar{U}^1 \cdot \bar{\nabla}_{\mathcal{A}^1}((M^1)^{-1}\bar{U}^1) - \frac{((U^1)_\theta + v_\theta^{[1]}\Omega)^2 - (v_\theta^{[1]}\Omega)^2}{r} & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}_{\mathcal{A}^1} U_\theta^1 + \frac{1}{r}(\bar{\nabla}(\mathcal{A}^1)_r U_\theta^1 - \frac{U_\theta^1}{r^2} = \bar{U}^1 \cdot \bar{\nabla}_{\mathcal{A}^1}(U_\theta^1 + w_\theta^0) + \frac{U_r^1(U_\theta^1 + v_\theta^{[1]}\Omega)}{r} & \text{in } \mathcal{V}_0^0, \\ \overline{\operatorname{div}}\, \bar{U}^1 + \frac{U_r^1}{r} = 0 & \text{in } \mathcal{V}_0^0, \\ \bar{U}^1 \cdot \bar{N}^1 = 0 & \text{on } \Sigma_0^0, \\ \bar{N}^1 \cdot \bar{\nabla}_{\mathcal{A}^1} U_\theta^1 = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}^1 \cdot \bar{\mathbb{S}}_{\mathcal{A}^1}((M^1)^{-1}U^1) \cdot \bar{\mathcal{T}}^1 = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}^1 \cdot (Q^1 \bar{I} + \bar{\mathbb{S}}_{\mathcal{A}^1}((M^1)^{-1}U^1)) \cdot \bar{\mathcal{N}}^1 = -gH^1 + \frac{\sigma}{r}\left(r(H^1)' + \mathcal{R}(0,(h^1)')\right)' & \text{on } \Sigma_0^0, \\ U^1 = 0 & \text{on } \Sigma_{a,0}^0, \\ U^1 = 0 & \text{on } \Sigma_{b,0}^0, \end{cases} \tag{3.25}$$

and

$$\begin{cases} -\bar{\nabla}_{\mathcal{A}^2} Q^2 + \bar{\Delta}_{\mathcal{A}^2}((M^2)^{-1}\bar{U}^2) + \frac{1}{r}(\bar{\nabla}_{\mathcal{A}^2}((M^2)^{-1}\bar{U}^2))_r - \left(\frac{((M^2)^{-1}U^2)_r}{r^2}, 0\right)^\top & \\ = (M^2)^{-1}\bar{U}^2 \cdot \bar{\nabla}_{\mathcal{A}^2}((M^2)^{-1}\bar{U}^2) - \frac{((U^2)_\theta + v_\theta^{[1]}\Omega)^2 - (v_\theta^{[1]}\Omega)^2}{r} & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}_{\mathcal{A}^2} U_\theta^2 + \frac{1}{r}(\bar{\nabla}_(\mathcal{A}^2)_r U_\theta^2 - \frac{U_\theta^1}{r^2} = \bar{U}^2 \cdot \bar{\nabla}_{\mathcal{A}^2}(U_\theta^2 + w_\theta^0) + \frac{U_r^2(U_\theta^1 + v_\theta^{[1]}\Omega)}{r} & \text{in } \mathcal{V}_0^0, \\ \overline{\operatorname{div}}\, \bar{U}^2 + \frac{U_r^2}{r} = 0 & \text{in } \mathcal{V}_0^0, \\ \bar{U}^2 \cdot \bar{N}^2 = 0 & \text{on } \Sigma_0^0, \\ \bar{N}^2 \cdot \bar{\nabla}_{\mathcal{A}^2} U_\theta^2 = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}^2 \cdot \bar{\mathbb{S}}_{\mathcal{A}^2}((M^2)^{-1}U^2) \cdot \bar{\mathcal{T}}^2 = 0 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}^2 \cdot (Q^2 \bar{I} + \bar{\mathbb{S}}_{\mathcal{A}^2}((M^2)^{-1}U^2)) \cdot \bar{\mathcal{N}}^2 = -gH^2 + \frac{\sigma}{r}\left(r(H^2)' + \mathcal{R}(0,(h^2)')\right)' & \text{on } \Sigma_0^0, \\ U^2 = 0 & \text{on } \Sigma_{a,0}^0, \\ U^2 = 0 & \text{on } \Sigma_{b,0}^0, \end{cases} \tag{3.26}$$

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Consider the system (3.26)–(3.25), we have

$$\begin{cases} -\bar{\nabla}_{\mathcal{A}^2}(Q^2-Q^1)+\bar{\Delta}_{\mathcal{A}^2}((M^2)^{-1}(\bar{U}^2-\bar{U}^1)) \\ +\frac{1}{r}(\bar{\nabla}_{\mathcal{A}^2}((M^2)^{-1}(\bar{U}^2-\bar{U}^1))_r-\left(\frac{U_r^2-U_r^1}{r^2},0\right)^\top=(G_r^1,G_z^1)^\top & \text{in } \mathcal{V}_0^0, \\ \bar{\Delta}_{\mathcal{A}^2}(U_\theta^2-U_\theta^1)+\frac{1}{r}(\bar{\nabla}_{\mathcal{A}^2}(U_\theta^2-U_\theta^1))_r-\frac{U_\theta^2-U_\theta^1}{r^2}=G_\theta^1 & \text{in } \mathcal{V}_0^0, \\ \overline{\mathrm{div}}(\bar{U}^2-\bar{U}^1)+\frac{U_r^2-U_r^1}{r}=0 & \text{in } \mathcal{V}_0^0, \\ (\bar{U}^2-\bar{U}^1)\cdot\bar{\mathcal{N}}^2=0, & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}^2\cdot\bar{\nabla}_{\mathcal{A}^2}(U_\theta^2-U_\theta^1)=G^2 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}^2\cdot\bar{\mathbb{S}}_{\mathcal{A}^2}((M^2)^{-1}(U^2-U^1))\cdot\bar{\mathcal{T}}^2=G^3 & \text{on } \Sigma_0^0, \\ \bar{\mathcal{N}}^2\cdot((Q^2-Q^1)\bar{I}+\bar{\mathbb{S}}_{\mathcal{A}^2}((M^2)^{-1}(U^2-U^1))\cdot\bar{\mathcal{N}}^2 \\ =-g(H^2-H^1)+\frac{\sigma}{r}\left(r(H^2-H^1)'+G^4\right)'+G^5 & \text{on } \Sigma_0^0, \\ (H^2-H^1)'(a;\Omega)=0, \quad (H^2-H^1)'(b;\Omega)=0, \\ U^2-U^1=0 & \text{on } \Sigma_{a,0}^0, \\ U^2-U^1=0 & \text{on } \Sigma_{b,0}^0, \end{cases} \tag{3.27}$$

where

$$\begin{aligned}(G_r^1,G_z^1)&=(M^2)^{-1}\bar{U}^2\cdot\bar{\nabla}_{\mathcal{A}^2}((M^2)^{-1}\bar{U}^2)-\frac{U_\theta^2(U_\theta^2+2v_\theta^{[1]}\Omega)^2)}{r}\\&\quad-(M^1)^{-1}\bar{U}^1\cdot\bar{\nabla}_{\mathcal{A}^1}((M^1)^{-1}\bar{U}^1)+\frac{U_\theta^1(U_\theta^1+2v_\theta^{[1]}\Omega)}{r}\\&\quad+\bar{\nabla}_{\mathcal{A}^2-\mathcal{A}^1}(Q^1)-\bar{\Delta}_{\mathcal{A}^2}((M^2-M^1)^{-1}(\bar{U}^2))\\&\quad-\bar{\Delta}_{\mathcal{A}^2-\mathcal{A}^2}((M^1)^{-1}(\bar{U}^1))-\frac{1}{r}(\bar{\nabla}_{\mathcal{A}^2}((M^2-M^1)^{-1}(\bar{U}^2))_r\\&\quad-\frac{1}{r}(\bar{\nabla}_{\mathcal{A}^2-\mathcal{A}^1}((M^1)^{-1}(\bar{U}^1))_r,\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}G_\theta^1&=\bar{U}^2\cdot\bar{\nabla}_{\mathcal{A}}(U_\theta^2+v_\theta^{[1]}\Omega)+\frac{U_r^2(U_\theta^2+v_\theta^{[1]}\Omega)}{r}\\&\quad-\bar{U}^1\cdot\bar{\nabla}_{\mathcal{A}}(U_\theta^1+v_\theta^{[1]}\Omega)-\frac{U_r^1(U_\theta^1+v_\theta^{[1]}\Omega)}{r}\\&\quad-\bar{\Delta}_{\mathcal{A}^2-\mathcal{A}^1}(U_\theta^1)+\frac{1}{r}(\bar{\nabla}_{\mathcal{A}^2-\mathcal{A}^1}(U_\theta^1))_r,\end{aligned} \tag{3.29}$$

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and

$$G^2 = \bar{\mathcal{N}}^1 \cdot \bar{\nabla}_{\mathcal{A}^1} U_\theta^1 - \bar{\mathcal{N}}^2 \cdot \bar{\nabla}_{\mathcal{A}^2} U_\theta^1, \tag{3.30}$$

and

$$G^3 = \bar{\mathcal{N}}^1 \cdot \bar{\mathbb{S}}_{\mathcal{A}^1}((M^1)^{-1}U^1) \cdot \bar{\mathcal{T}}^1 - \bar{\mathcal{N}}^2 \cdot \bar{\mathbb{S}}_{\mathcal{A}^2}((M^2)^{-1}U^1) \cdot \bar{\mathcal{T}}^2, \tag{3.31}$$

and

$$G^4 = \mathcal{R}(0, (h^2)') - \mathcal{R}(0, (h^1)'), \tag{3.32}$$

and

$$\begin{aligned} G^5 = {} & \bar{\mathcal{N}}^2 \cdot (Q^1 \bar{I} + \bar{\mathbb{S}}_{\mathcal{A}^2}(M^2)^{-1}U^1) \cdot \bar{\mathcal{N}}^2 \\ & - \bar{\mathcal{N}}^1 \cdot (Q^1 \bar{I} + \bar{\mathbb{S}}_{\mathcal{A}^1}((M^1)^{-1}U^1) \cdot \bar{\mathcal{N}}^1. \end{aligned} \tag{3.33}$$

Similar to the arguments of (3.20) and (3.21), by theorem 2.4 and the embedding result (3.5), assume that Ω is sufficiently small, then we can obtain that

$$\begin{aligned} & \|U^2 - U^1\|_{W^{k+2}_{\delta_k}} + \|Q^2 - Q^1\|_{\mathring{W}^{k+1}_{\delta_k}} + \|H^2 - H^1\|_{W^{k+\frac{5}{2}}_{\delta_k}} \\ & \quad \leqslant \frac{1}{2}\left(\|U^2 - U^1\|_{W^{k+2}_{\delta_k}} + \|Q^2 - Q^1\|_{\mathring{W}^{k+1}_{\delta_k}} + \|H^2 - H^1\|_{W^{k+\frac{5}{2}}_{\delta_k}}\right), \end{aligned} \tag{3.34}$$

which implies $\|U^2 - U^1\|_{W^{k+2}_{\delta_k}} + \|Q^2 - Q^1\|_{\mathring{W}^{k+1}_{\delta_k}} + \|H^2 - H^1\|_{W^{k+\frac{5}{2}}_{\delta_k}} = 0$, that is, $(U^1, Q^1, H^1) = (U^2, Q^2, H^2)$.

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