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Adiabatic eigenflows in a vertical porous channel

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The existence of an infinite class of buoyant flows in a vertical porous channel with adiabatic and impermeable boundary walls, called adiabatic eigenflows, is discussed. A uniform heat source within the saturated medium is assumed, so that a stationary state is possible with a net vertical through-flow convecting away the excess heat. The simple isothermal flow with uniform velocity profile is a special adiabatic eigenflow if the power supplied by the heat source is zero. The linear stability analysis of the adiabatic eigenflows is carried out analytically. It is shown that these basic flows are unstable. The only exception, when the power supplied by the heat source is zero, is the uniform isothermal flow, which is stable. The existence of adiabatic eigenflows and their stability analysis is extended to the case of spanwise lateral confinement, *viz.* in the case of a vertical rectangular channel. A generalisation of this study to a vertical channel with an arbitrary cross-sectional shape is also presented.

Key words: Bénard convection, buoyancy-driven instability, convection in porous media

1. Introduction

A well-established aspect of fluid mechanics is that stationary solutions of the governing equations for fluid flow may exist that are unlikely to be observed in a laboratory experiment. The reason is that such solutions can be unstable to external perturbations, at least in a given parametric domain. A wealth of scientific papers and textbooks discuss this point, with reference either to purely hydrodynamic instability or to thermal instability (see e.g. Drazin & Reid 2004).

The issue of thermal instability in saturated porous media is widely discussed in §§ 6 and 7 of Nield & Bejan (2013), as well as in Rees (2000), Tyvand (2002) and Barletta (2011). In the special case of vertical plane channels, Gill (1969) proved that the cellular stationary flow, arising when a temperature difference is maintained between the impermeable and isothermal boundary walls, is always stable. Later on, the same conclusion was retrieved by other authors (Rees 1988; Straughan 1988; Lewis, Bassom & Rees 1995; Rees 2011; Scott & Straughan 2013), who extended Gill's analysis by including effects such as a finite value of the Prandtl–Darcy number, or the lack of local thermal equilibrium between fluid and solid phases.

The stable nature of flow in a vertical plane porous channel changes dramatically if the temperature boundary conditions are turned from isothermal to isoflux. With a

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net heat supply, a non-vanishing vertical temperature gradient arises in the basic flow state. This situation has been studied by Barletta (2013), and the results have been extended to a vertical porous channel with a circular cross-section (Barletta 2014). The basic solution, in these cases, entails necessarily a net flow rate in order to ensure convection of the excess heat supplied through the boundaries under stationary conditions. Two regimes exist for the basic flow: one termed buoyancy-assisted flow, and one termed buoyancy-opposed flow. While the former regime yields a positive temperature gradient in the upward direction, the latter yields a negative one and, hence, a potentially unstable thermal stratification. In fact, the conclusion drawn by Barletta (2013, 2014) is that buoyancy-assisted basic flows are linearly stable, while buoyancy-opposed basic flows are unstable.

Another porous channel set-up that implies an unstable thermal stratification is one where an internal heat source in the saturated medium exists and the boundary walls of the vertical channel are thermally insulated. As in the case of isoflux boundaries, the only chance for a stationary state is that the excess heat is convected away, in the vertical streamwise direction, by a prescribed flow rate. This paper studies such a set-up. The nature of the basic stationary regime is more complicated than in previous studies, as there are infinitely many allowed stationary flows. Among them, when the internal heat source is absent, the simplest case is the isothermal flow with a uniform velocity profile. The governing equations and boundary conditions lead to a differential problem for the basic state having the nature of an eigenvalue problem, and this motivates the term used in the title of this paper: adiabatic eigenflows. A linear stability analysis of adiabatic eigenflows is carried out, revealing that they are unstable in any case. When the internal heat source is absent, the only exception is the uniform and isothermal flow, which is stable. Then, this result is extended to the case where a lateral confinement exists in the spanwise direction, so that the plane channel is in fact a rectangular channel. Finally, the existence and the instability of adiabatic eigenflows are investigated for a vertical porous channel having an arbitrary cross-sectional shape. We mention that Hewitt, Neufeld & Lister (2013) recently carried out a two-dimensional stability analysis where the basic solution has strong mathematical analogies with the adiabatic eigenflows studied here. These authors were in fact interested in the buoyancy-driven convection in an unbounded porous medium with no internal heat sources. Hewitt *et al.* (2013) denoted their basic solution as 'heat-exchanger flow'.

The basic physical argument behind the forthcoming study is the following. We know that a net (uniform) heat supply from the boundary walls in a vertical porous channel can be steadily convected away by a parallel flow, and that this flow may be stable if the flow direction is buoyancy-assisted. This problem has been investigated by Barletta (2013, 2014). If the heat supply is provided by a uniform internal heat source, and the heat transfer through the boundary walls is blocked, is it then possible to convect away the excess heat by a stationary (and stable) parallel flow? The answer to this basic, but non-trivial, question is the main physical motivation of our work. Several possible applicative contexts where seepage flows with an internal heat source emerge can be mentioned. An example is the design of packed bed reactors where the heat source is a result of exothermic chemical reactions. The influence of radioactive heat sources on convection processes in the Earth's mantle and lithosphere is another example.

2. Problem formulation

Our aim is the analysis of buoyant flow in a vertical porous channel with plane, impermeable and adiabatic walls separated by a distance ℓ. As shown in figure 1, the

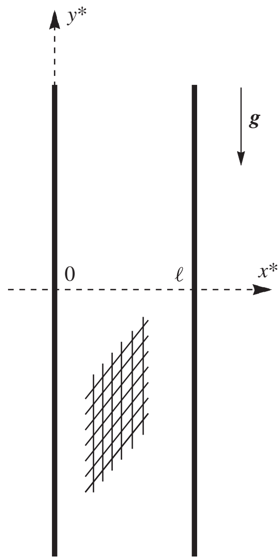

FIGURE 1. Sketch of the porous channel.

x^*-axis is horizontal, while the y^*-axis is vertical. We consider the porous medium as homogeneous and isotropic, assuming that the saturating fluid is in local thermal equilibrium with the solid. Internal heating takes place within the saturated porous medium, caused by a uniformly distributed source that supplies a power Q per unit volume.

The physical scheme adopted here to model the buoyant flow is based on Darcy's law and on the Oberbeck–Boussinesq approximation.

2.1. *Governing equations*

The governing equations are expressed as

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1a}$$

$$\nabla \times \boldsymbol{u} = \nabla \times (T\hat{\boldsymbol{e}}_y), \tag{2.1b}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \nabla^2 T + R, \tag{2.1c}$$

where $\boldsymbol{u}$ is the dimensionless velocity, T is the dimensionless temperature, t is the dimensionless time, and R is the Rayleigh number associated with the internal heat source. The dimensionless velocity components along the dimensionless coordinate axes (x, y, z) are denoted by (u, v, w), respectively. Equation (2.1b) is the vorticity formulation of Darcy's law including the buoyancy body force, obtained by taking the curl of both sides of the local momentum balance equation.

Here, the dimensionless quantities have been defined from the corresponding dimensional (starred) quantities as

$$\left.\begin{aligned} &(x^*, y^*, z^*)\frac{1}{\ell} = (x, y, z), \quad t^*\frac{\varkappa}{\sigma \ell^2} = t, \\ &(u^*, v^*, w^*)\frac{\ell}{\varkappa} = (u, v, w), \quad (T^* - T_0)\frac{g\beta K \ell}{\nu \varkappa} = T, \end{aligned}\right\} \tag{2.2}$$

where $\varkappa$ is the average thermal diffusivity, σ is the ratio between the average heat capacity of the saturated porous medium and the heat capacity of the fluid, K is the permeability, g is the modulus of the gravitational acceleration $\boldsymbol{g}$, β is the thermal expansion coefficient of the fluid, and ν is its kinematic viscosity. The Rayleigh number is defined as

$$R = \frac{g\beta Q K \ell^3}{\nu \varkappa \lambda}, \tag{2.3}$$

where λ is the average thermal conductivity of the porous medium.

The boundary conditions are given by

$$x = 0, 1: \quad u = 0, \quad \frac{\partial T}{\partial x} = 0. \tag{2.4a,b}$$

2.2. *Adiabatic eigenflows*

A time-independent basic solution of (2.1) and (2.4) can be expressed as

$$\nabla T_b = F(x)\hat{\boldsymbol{e}}_x - \alpha^2 \hat{\boldsymbol{e}}_y, \quad \boldsymbol{u}_b = G(x)\hat{\boldsymbol{e}}_y, \tag{2.5a,b}$$

where 'b' stands for 'basic solution'.

Equation (2.5) describes a stationary parallel flow along the vertical y-axis, endowed with a uniform temperature gradient in the streamwise direction. One may conceive this vertical temperature change as partly induced by the internal heat source, and partly caused by a temperature difference imposed between the far-away upstream and downstream regions. The latter cause is that giving rise to a non-vanishing α^2 in the special limiting case where the heat source is switched off, $R \to 0$. Further details on this point are given in appendix A.

We obtain from the governing equations (2.1),

$$G'(x) = F(x), \quad -\alpha^2 G(x) = F'(x) + R, \tag{2.6a,b}$$

where primes denote derivatives with respect to the argument of the function. On combining the two equations, we get

$$F''(x) + \alpha^2 F(x) = 0. \tag{2.7}$$

Thus, the solution satisfying the boundary conditions (2.4), namely

$$F(0) = F(1) = 0, \tag{2.8}$$

is given by

$$\left.\begin{aligned} &F(x) = A\alpha \sin(\alpha x), \quad G(x) = -A\cos(\alpha x) - \frac{R}{\alpha^2}, \\ &\alpha = n\pi, \quad n = 1, 2, \ldots . \end{aligned}\right\} \tag{2.9}$$

The nature of (2.7) and (2.8) discloses the reason why, in (2.5), we assumed $\partial T_b/\partial y \leqslant 0$. A positive value of $\partial T_b/\partial y$ would have in fact precluded the possibility of satisfying (2.1) and (2.4).

It must be mentioned that, if we set $R = 0$, there exists an extra solution. It is the solution with $n = 0$. It yields, on account of (2.9), $F(x) = 0$ and $G(x) = -A$. As can be inferred from (2.5), it is the stationary uniform velocity profile with a uniform temperature field, in short, the uniform isothermal flow. For every non-vanishing and, nonetheless, arbitrarily small R, the uniform isothermal flow is not allowed. Hereafter, this very special solution will be tacitly ignored except in § 3.2, where we will show that the uniform isothermal flow is linearly stable, as expected.

The average velocity through the channel cross-section, on account of (2.9), is given by

$$\int_0^1 v_b \, \mathrm{d}x = \int_0^1 G(x) \, \mathrm{d}x = -\frac{R}{\alpha^2}. \tag{2.10}$$

Therefore, for any prescribed $R > 0$, we have an infinite class of parallel flows with a downward-oriented average velocity. Each flow is labelled by the integer n, and by the real parameter A.

Hereafter, the dependence on n is explicitly indicated, with the notation $\{F_n(x), G_n(x)\}$.

The physical meaning of A is related to the average kinetic energy $\mathscr{E}_n$ within the channel,

$$\mathscr{E}_n = \frac{1}{2} \int_0^1 G_n(x)^2 \, \mathrm{d}x = \frac{A^2}{4} + \frac{R^2}{2\pi^4 n^4}. \tag{2.11}$$

Since these basic parallel flows are actually obtained by the solution of an eigenvalue problem, (2.7) and (2.8), they can be called adiabatic eigenflows. Adiabatic eigenflows exist for every choice of the governing parameters (n, A, R). They can exist even in the special case $R \to 0$ when the internal heat source is switched off. Another special case is $A = 0$ with $R > 0$, where the basic velocity profiles $G_n(x)$ are uniform, for every n, but the vertical temperature gradient is non-zero.

The existence of adiabatic eigenflows entails the vertical temperature gradient assuming discrete special values, and likewise for the average flow velocity as well as for the average kinetic energy. The mathematical analogy with what happens in a quantum mechanical system, say an electron in an infinite potential well (see e.g. Gasiorowicz 2003), is striking. We also mention that the possible existence of adiabatic eigenflows was previously predicted by Barletta *et al.* (2008).

A physical discussion of the adiabatic eigenflows, for the special case $R = 0$, is provided in appendix A, where these flows are regarded as the mid-region of Rayleigh–Bénard cells in a tall rectangular cavity.

The peculiar nature of adiabatic eigenflows, being endowed with a negative temperature gradient in the upward direction, suggests that these flows could be hardly observable in a real laboratory experiment, owing to their possible instability. An important information to this end comes from the stability analysis versus small-amplitude perturbations.

As explicitly declared from the beginning, our concern is to check the existence of stationary parallel flows. However, a very simple unsteady solution of (2.1) and (2.4) is worth mentioning. This solution is with $T = Rt + \text{const.}$ and any uniform, constant and parallel velocity field $\boldsymbol{u}$ arbitrarily oriented in the (y, z)-plane. Hereafter, this solution is not further considered in this paper.

The basic solution given by (2.5) and (2.9) is quite similar to the basic flow state considered by Hewitt *et al.* (2013) and termed 'heat-exchanger flow'. The system studied by Hewitt *et al.* (2013) is in fact an unbounded two-dimensional porous medium, where no heat source is present. Thus, their basic solution mathematically coincides with (2.5) and (2.9), provided that $R = 0$ and that the eigenvalue α has a continuous spectrum.

3. Linear stability analysis

We now perturb the adiabatic eigenflows, (2.5) and (2.9), by arbitrary disturbances modulated by a small parameter, $\varepsilon \ll 1$. Thus, we write

$$T = T_b + \varepsilon \tilde{T}, \quad \boldsymbol{u} = \boldsymbol{u}_b + \varepsilon \tilde{\boldsymbol{u}}. \tag{3.1a,b}$$

Here, $\tilde{T}$ is the temperature perturbation, and $\tilde{\boldsymbol{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$ is the velocity perturbation. Substitution of (3.1) into (2.1) and (2.4) yields

$$\nabla \cdot \tilde{\boldsymbol{u}} = 0, \tag{3.2a}$$

$$\nabla \times \tilde{\boldsymbol{u}} = \nabla \times (\tilde{T}\hat{\boldsymbol{e}}_y), \tag{3.2b}$$

$$\frac{\partial \tilde{T}}{\partial t} + F_n(x)\tilde{u} - n^2\pi^2\tilde{v} + G_n(x)\frac{\partial \tilde{T}}{\partial y} = \nabla^2 \tilde{T}, \tag{3.2c}$$

$$x = 0,\, 1: \quad \tilde{u} = 0, \quad \frac{\partial \tilde{T}}{\partial x} = 0. \tag{3.2d}$$

In (3.2), terms $O(\varepsilon^2)$ have been neglected and use has been made of (2.5) and (2.9). Equation (3.2*b*) is identically satisfied by assuming the existence of a pressure perturbation field, $\tilde{P}$, such that

$$\tilde{\boldsymbol{u}} = \tilde{T}\hat{\boldsymbol{e}}_y - \nabla \tilde{P}. \tag{3.3}$$

By employing (3.3), one may reformulate (3.2*a*), (3.2*c*) and (3.2*d*) as

$$\nabla^2 \tilde{P} = \frac{\partial \tilde{T}}{\partial y}, \tag{3.4a}$$

$$\nabla^2 \tilde{T} = \frac{\partial \tilde{T}}{\partial t} - F_n(x)\frac{\partial \tilde{P}}{\partial x} - n^2\pi^2\left(\tilde{T} - \frac{\partial \tilde{P}}{\partial y}\right) + G_n(x)\frac{\partial \tilde{T}}{\partial y}, \tag{3.4b}$$

$$x = 0,\, 1: \quad \frac{\partial \tilde{P}}{\partial x} = 0, \quad \frac{\partial \tilde{T}}{\partial x} = 0. \tag{3.4c}$$

3.1. *Normal modes*

The analysis of the linearised perturbation equations (3.4) is carried out by a normal mode expansion of the perturbations, where the normal modes are independent of each other and given by

$$\begin{Bmatrix} \tilde{P}(x, y, z, t) \\ \tilde{T}(x, y, z, t) \end{Bmatrix} = \begin{Bmatrix} f(x) \\ h(x) \end{Bmatrix} \mathrm{e}^{\eta t}\, \mathrm{e}^{\mathrm{i}(k_y y + k_z z)}, \tag{3.5}$$

where k_y and k_z are real parameters expressing the y-component and the z-component of the dimensionless wavevector, respectively. The complex parameter η is such that its real part, $\mathrm{Re}(\eta)$, is the growth rate and marks the unstable nature of the normal mode when it is positive, while $\mathrm{Re}(\eta) < 0$ means stability and $\mathrm{Re}(\eta) = 0$ neutral stability. On the other hand, the imaginary part, $\mathrm{Im}(\eta)$, defines the dimensionless angular frequency as $\omega = -\mathrm{Im}(\eta)$.

Each normal mode must satisfy (3.4), so that we obtain the system of ordinary differential equations

$$f'' - k^2 f - \mathrm{i}\gamma k h = 0, \tag{3.6a}$$

$$h'' - [k^2 + \eta - n^2\pi^2 + \mathrm{i}\gamma k G_n(x)]h + F_n(x) f' - \mathrm{i}\gamma k n^2\pi^2 f = 0, \tag{3.6b}$$

with the boundary conditions

$$f'(0) = f'(1) = 0, \quad h'(0) = h'(1) = 0. \tag{3.7a,b}$$

Here, the wavenumber k and the parameter γ are defined as

$$k = (k_y^2 + k_z^2)^{1/2}, \quad \gamma = \frac{k_y}{k}. \tag{3.8a,b}$$

Longitudinal modes are the normal modes with $k_y = 0$ (or, equivalently, $\gamma = 0$); transverse modes are the normal modes with $k_z = 0$; and oblique modes are the general normal modes defined by (3.5).

3.2. *Uniform isothermal flow*

In § 2.2, we mentioned the very special basic solution with $R = 0$, such that $n = 0$, $G_0(x) = -A$ and $F_0(x) = 0$. In this case, (3.6) and (3.7) run into a dramatic simplification, namely

$$f'' - k^2 f - \mathrm{i}\gamma k h = 0, \tag{3.9a}$$

$$h'' - (k^2 + \eta - \mathrm{i}\gamma k A)h = 0, \tag{3.9b}$$

$$f'(0) = f'(1) = 0, \quad h'(0) = h'(1) = 0. \tag{3.9c}$$

These equations can be solved analytically. The solution is given by

$$f(x) = -\frac{\mathrm{i}\gamma k}{k^2 + m^2\pi^2}\cos(m\pi x), \quad h(x) = \cos(m\pi x) \quad m = 0, 1, 2, \ldots, \tag{3.10a,b}$$

subject to the dispersion relation

$$\eta = -k^2 - m^2\pi^2 + \mathrm{i}\gamma k A. \tag{3.11}$$

Equation (3.11) allows one to obtain

$$\mathrm{Re}(\eta) = -k^2 - m^2\pi^2, \quad \omega = -\mathrm{Im}(\eta) = -\gamma k A. \tag{3.12a,b}$$

The conclusion that may be immediately drawn from (3.12) is that $\mathrm{Re}(\eta)$ cannot be positive, so that the uniform isothermal flow is linearly stable.

3.3. *Longitudinal modes*

If we set $\gamma = 0$, (3.6) and (3.7) are drastically simplified, namely

$$f'' - k^2 f = 0, \tag{3.13a}$$

$$h'' - (k^2 + \eta - n^2\pi^2)h + F_n(x) f' = 0, \tag{3.13b}$$

$$f'(0) = f'(1) = 0, \quad h'(0) = h'(1) = 0. \tag{3.13c}$$

Equations (3.13) can be solved analytically, thus obtaining

$$f(x) = 0, \quad h(x) = \cos(m\pi x), \quad \text{with } m = 0, 1, 2, \ldots, \tag{3.14a,b}$$

subject to the dispersion relation

$$\eta = (n^2 - m^2)\pi^2 - k^2. \tag{3.15}$$

It is evident from (3.15) that, for the establishment of instability ($\eta > 0$), the most effective among the longitudinal m modes defined by (3.5) and (3.14) are those corresponding to the lowest m, namely $m = 0$. This reasoning leads us to the instability condition

$$k^2 < n^2\pi^2. \tag{3.16}$$

We recall that $n = 1, 2, 3, \ldots$, so that (3.16) can be satisfied for all adiabatic eigenflows, provided that the wavenumber k is small enough.

The conclusion is that all adiabatic eigenflows are unstable to the longitudinal modes with $m = 0$. In fact, every real positive wavenumber k lies within the spectrum of a random small-amplitude perturbation produced in a laboratory experiment and, hence, also those wavenumbers k satisfying inequality (3.16).

3.3.1. *Lateral confinement*

The above stated conclusion is based on the assumption that the spanwise width (*viz.* the width along the z-axis) is infinite. If this assumption is perfectly reasonable on theoretical grounds, it may not be perfectly reliable when comparing the mathematical predictions about stability with experimental observations. In a practical case, the vertical channel is not infinitely wide in the spanwise direction, even if its spanwise width may be large enough if compared to the distance between the parallel bounding walls. Thus, a more realistic assumption may be a finite aspect ratio, s, between the spanwise width and the distance between the bounding walls. Following this assumption, the transverse cross-section of the channel becomes a two-dimensional rectangular domain,

$$x \in [0, 1], \quad z \in [0, s], \tag{3.17}$$

with the lateral walls at $z = 0$, and $z = s$ assumed to be adiabatic and impermeable inasmuch as the walls at $x = 0$ and $x = 1$. In other words, one must satisfy the additional conditions

$$z = 0, s\!: \quad \frac{\partial \tilde{P}}{\partial z} = 0, \quad \frac{\partial \tilde{T}}{\partial z} = 0. \tag{3.18a,b}$$

The consequence is that the whole procedure described in §§ 3.1 and 3.3 is still valid provided that the factor $\exp(\mathrm{i}k_z z)$ in (3.5) is replaced by $\cos(k_z z)$, where $k_z = p\pi/s$ with $p = 0, 1, 2, \ldots$.

Now, the consequences of inequality (3.16) have to be reassessed. In fact, this inequality has to be rewritten as

$$\frac{p^2}{s^2} < n^2, \tag{3.19}$$

which is satisfied for every adiabatic eigenflow with $n = 1, 2, 3, \ldots,$ if we set $p = 0$ or, equivalently, $k_z = 0$.

Even in the presence of a lateral confinement in the spanwise direction, we conclude again that all adiabatic eigenflows are unstable to longitudinal modes. This conclusion is independent of the aspect ratio s, as (3.15) can always be satisfied with $p = 0$. The preferred mode of instability is one with $m = 0$ and $p = 0$, meaning, on account of (3.5) and (3.14), a perturbation given by

$$\begin{Bmatrix} \tilde{P}(x, y, z, t) \\ \tilde{T}(x, y, z, t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \mathrm{e}^{\eta t}, \tag{3.20}$$

or equivalently, on account of (3.3),

$$\begin{Bmatrix} \tilde{u}(x, y, z, t) \\ \tilde{v}(x, y, z, t) \\ \tilde{w}(x, y, z, t) \\ \tilde{T}(x, y, z, t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} \mathrm{e}^{\eta t}. \tag{3.21}$$

Equation (3.21) implies a uniform time growth of the vertical velocity component, $v_b + \varepsilon \tilde{v}$, and of the temperature, $T_b + \varepsilon \tilde{T}$. It is worth mentioning that, in (3.20), one could equally well write $\tilde{P}(x, y, z, t) = \text{const.} \times \mathrm{e}^{\eta t}$, where the constant is arbitrary. The reason is that (3.13a) and (3.13c) can be satisfied with any constant value of f,

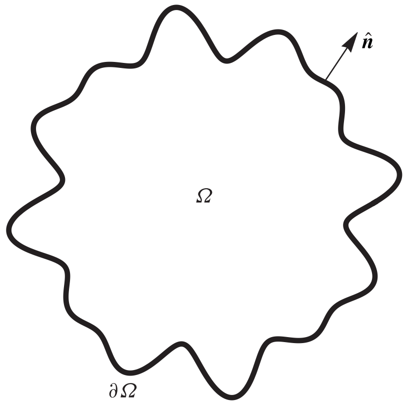

FIGURE 2. Arbitrary cross-sectional shape of the channel.

when $k_z = k = 0$. This evidence has no effect whatsoever on the validity of (3.21), as $(\tilde{u}, \tilde{v}, \tilde{w})$ are defined, by (3.3), only through the spatial derivatives of $\tilde{P}$.

We have established that, with or without lateral confinement, there exist normal modes leading to the instability of all adiabatic eigenflows, for every assignment of (n, R, A). These normal modes are of the longitudinal type. However, other oblique modes may actually be preferred when instability develops. An analysis of this issue is beyond the scope of the present study, even if it is an interesting opportunity for future research. As pointed out by Hewitt *et al.* (2013), transverse modes may display the highest growth rate at sufficiently large values of parameter A. This feature, relative to an unbounded porous domain, could possibly emerge also in a channel with adiabatic walls. A definitive pronouncement on this matter may be obtained only by a numerical solution of the governing equations for the disturbances. The response from this analysis does not alter in any case the conclusion we have reached from the analysis of longitudinal rolls: all adiabatic eigenflows are unstable. Thus, in the following, we will not carry out the analysis of the reaction to oblique modes. On the other hand, we will generalise our investigation by considering a vertical porous channel with an arbitrary cross-section.

4. Channel with an arbitrary cross-sectional shape

If we now release the assumption of plane channel walls, the channel cross-section may be considered as a bounded region Ω in the (x, z)-plane. The boundary of Ω may be considered as any sufficiently smooth, or piecewise smooth, closed curve $\partial\Omega$. Let $\hat{\boldsymbol{n}}$ be the two-dimensional vector in the (x, z)-plane that indicates the unit outward normal to $\partial\Omega$ (see figure 2). The vertical y-axis is still regarded as the streamwise channel direction.

The governing equations (2.1) are now endowed with the boundary conditions

$$(x, z) \in \partial\Omega: \quad \hat{\boldsymbol{n}} \cdot \boldsymbol{u} = 0, \quad \hat{\boldsymbol{n}} \cdot \nabla T = 0. \tag{4.1a,b}$$

The basic stationary solution satisfies a generalised formulation of (2.5), namely

$$T_b = \varphi(x, z) - \alpha^2 y + \text{const.}, \quad \boldsymbol{u}_b = \left[\varphi(x, z) - \frac{R}{\alpha^2}\right] \hat{\boldsymbol{e}}_y, \tag{4.2a,b}$$

where the constant in the expression of T_b is left undetermined by the governing equations and boundary conditions. Equations (2.1*a*) and (2.1*b*) are satisfied for every choice of function $\varphi(x, z)$, and of the constant α^2. Eventually, (2.1*c*) can be rewritten as

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} + \alpha^2\varphi = 0, \tag{4.3}$$

with the boundary conditions, (4.1), given by

$$(x, z) \in \partial\Omega: \quad \hat{\boldsymbol{n}} \boldsymbol{\cdot} \boldsymbol{\nabla}\varphi = 0. \tag{4.4}$$

Equations (4.3) and (4.4) define an eigenvalue problem that generalises that given by (2.7) and (2.8), for the plane channel. Equation (4.3) is the two-dimensional Helmholtz equation, so that the eigenvalues α^2 can be identified with the eigenfrequencies of a membrane with a free boundary $\partial\Omega$. The eigenvalues α^2 are non-negative and form a discrete increasing sequence, where the lowest eigenvalue is zero (see e.g. Pólya 1961; Benguria 2014),

$$0 = \alpha_0^2 < \alpha_1^2 < \alpha_2^2 < \cdots < \alpha_n^2 < \cdots . \tag{4.5}$$

A well-known fact about the eigenvalue problem (4.3) and (4.4) is that $\alpha_n^2 \to \infty$ when $n \to \infty$, and that the eigenfunction φ_0 corresponding to $\alpha_0^2 = 0$ is an arbitrary constant (Benguria 2014). Thus, only for the special case $R = 0$ is the eigenflow with $n = 0$ allowed as a generalisation of the uniform isothermal flow discussed in § 2.2, as can be inferred from (4.2). The case $n = 0$ does not arise as an acceptable solution whenever $R > 0$.

For the adiabatic eigenflows with $n \geqslant 1$, (4.3) and (4.4) imply an explicit constraint on $\varphi_n(x, z)$, namely

$$\int_\Omega \varphi_n(x, z)\, \mathrm{d}x\, \mathrm{d}z = 0. \tag{4.6}$$

This result can be proved through an integration by parts of (4.3), taking into account the boundary condition (4.4).

The constraint (4.6), on account of (4.2), implies that (with $n \geqslant 1$) the average velocity in a cross-section is still given by $-R/\alpha_n^2$ as predicted by (2.10) relative to a plane-parallel channel. The eigenfunctions $\varphi_n(x)$ and the eigenvalues α_n^2 depend on the geometry of the domain Ω and can be determined analytically only for a few special cases, while in general their evaluation can be obtained only by a numerical solution of (4.3) and (4.4).

Numerical values of α_n/π, for $1 \leqslant n \leqslant 10$, are reported in table 1. Four sample cross-sectional shapes are considered: square, circle, ellipse and equilateral triangle. The reference length ℓ is the side (square, triangle), the diameter (circle) and the major axis (ellipse). For the ellipse, the ratio between the minor and major axes is 1/2. As can be reckoned from the discussion carried out in § 3.3.1, the values of α_n/π reported in table 1 for the square are obtained as the lowest 10 values of the expression $\sqrt{p^2 + q^2}$ with p and q non-negative integers. For the special case of the circle, discussed in the forthcoming § 4.1, we evaluated α_n/π by searching the first 10 positive roots of (4.8). The solution of the Neumann eigenvalue problem based on the Helmholtz equation for the ellipse and for the triangle is found numerically by employing a finite-element solver. The numerical solution has been actually developed within the software environment Comsol© Multiphysics 3.5. An unstructured mesh for the domain Ω with triangular elements and increasing refinements is adopted, so that convergence of the first 10 eigenvalues within five significant figures is achieved.

n	Square	Circle	Ellipse	Triangle
1	1	1.1721	1.1928	1.3333
2	$\sqrt{2} \approx 1.4142$	1.9444	2.1766	2.3094
3	2	2.4393	2.2507	2.6667
4	$\sqrt{5} \approx 2.2361$	2.6746	2.9547	3.5277
5	$2\sqrt{2} \approx 2.8284$	3.3853	3.1379	4.0000
6	3	3.3941	3.7148	4.6188
7	$\sqrt{10} \approx 3.1623$	4.0843	4.0858	4.8074
8	$\sqrt{13} \approx 3.6056$	4.2693	4.3037	5.3333
9	4	4.4663	4.5130	5.8119
10	$\sqrt{17} \approx 4.1231$	4.7755	5.0111	6.1101

TABLE 1. Values of α_n/π, with $n \geqslant 1$, for channels where the cross-section is: a square with side ℓ; a circle with diameter ℓ; an ellipse with major axis ℓ and minor axis $\ell/2$; and an equilateral triangle with side ℓ.

4.1. *An example: the circular duct*

If we consider a vertical circular duct, the domain Ω can be chosen as a circle with unit diameter and centred at the origin. In this case, the solution of (4.3) and (4.4) is given by

$$\varphi(r,\theta) = -A\mathrm{J}_p(\alpha_{p,q} r)\cos(p\theta), \quad \text{with } 0 \leqslant r \leqslant 1/2,\ 0 \leqslant \theta < 2\pi, \tag{4.7}$$

where (p, q) is any pair of non-negative integers, A is an arbitrary constant, while (r, θ) are the polar coordinates in the (x, z)-plane, and J_p is the Bessel function of the first kind and order p. On account of (4.4), each eigenvalue $\alpha^2_{p,q}$ is obtained by determining the qth non-negative root of

$$\mathrm{J}'_p(\alpha/2) = 0. \tag{4.8}$$

We note that these roots are labelled by the pairs (p, q). When they are put in increasing order, the correspondence between each pair (p, q) and the index n of the eigenvalue, employed in the general equation (4.5), comes out naturally. A discussion on this ordering can be found, for instance, in Beattie (1958), Barletta & Storesletten (2013) and Barletta (2014).

4.2. *Linear stability*

By defining a perturbation of the basic state, defined again by (3.1), and by employing the definition of $\tilde{P}$, (3.3), the linearised equations (3.4) are now written for the general case as

$$\nabla^2\tilde{P} = \frac{\partial \tilde{T}}{\partial y}, \tag{4.9a}$$

$$\nabla^2\tilde{T} = \frac{\partial \tilde{T}}{\partial t} - \frac{\partial \varphi_n(x,z)}{\partial x}\frac{\partial \tilde{P}}{\partial x} - \alpha_n^2\left(\tilde{T} - \frac{\partial \tilde{P}}{\partial y}\right) - \frac{\partial \varphi_n(x,z)}{\partial z}\frac{\partial \tilde{P}}{\partial z} + \left[\varphi_n(x,z) - \frac{R}{\alpha_n^2}\right]\frac{\partial \tilde{T}}{\partial y}, \tag{4.9b}$$

$$(x,z) \in \partial\Omega: \quad \hat{\boldsymbol{n}}\boldsymbol{\cdot}\nabla\tilde{P} = 0, \quad \hat{\boldsymbol{n}}\boldsymbol{\cdot}\nabla\tilde{T} = 0. \tag{4.9c}$$

The normal modes are expressed as

$$\begin{Bmatrix} \tilde{P}(x, y, z, t) \\ \tilde{T}(x, y, z, t) \end{Bmatrix} = \begin{Bmatrix} f(x, z) \\ h(x, z) \end{Bmatrix} \mathrm{e}^{\eta t}\, \mathrm{e}^{\mathrm{i}ky}, \tag{4.10}$$

so that, by substituting (4.10) into (4.9), we obtain

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} - k^2 f - \mathrm{i}kh = 0, \tag{4.11a}$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - \left\{ k^2 + \eta - \alpha_n^2 + \mathrm{i}k \left[\varphi_n(x, z) - \frac{R}{\alpha_n^2} \right] \right\} h + \frac{\partial \varphi_n(x, z)}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial \varphi_n(x, z)}{\partial z}\frac{\partial f}{\partial z} - \mathrm{i}k\alpha_n^2 f = 0, \tag{4.11b}$$

$$(x, z) \in \partial\Omega: \quad \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla f = 0, \quad \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla h = 0. \tag{4.11c}$$

4.2.1. *Uniform isothermal flow*

In the absence of any internal heat source ($R = 0$), we can allow for the special case $n = 0$. In this case, the eigenfunction is a constant, $\varphi_0(x, z) = \text{const.} = -A$, and $\alpha_0^2 = 0$. Hence, (4.11) yield

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} - k^2 f - \mathrm{i}kh = 0, \tag{4.12a}$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - (k^2 + \eta - \mathrm{i}kA)h = 0, \tag{4.12b}$$

$$(x, z) \in \partial\Omega: \quad \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla f = 0, \quad \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla h = 0. \tag{4.12c}$$

The solution of the eigenvalue problem (4.12) is thus given by

$$f(x, z) = -\frac{\mathrm{i}k}{k^2 + \alpha_m^2}\varphi_m(x, z), \quad h(x, z) = \varphi_m(x, z) \quad m = 0, 1, 2, \ldots, \tag{4.13a,b}$$

with the dispersion relation

$$\eta = -k^2 - \alpha_m^2 + \mathrm{i}kA. \tag{4.14}$$

Equation (4.14) can be rewritten as

$$\mathrm{Re}(\eta) = -k^2 - \alpha_m^2, \quad \omega = -\mathrm{Im}(\eta) = -kA, \tag{4.15a,b}$$

which implies that $\mathrm{Re}(\eta)$ is always non-positive, on account of (4.5). This fact enables a generalisation of the conclusion drawn in § 3.2: the uniform isothermal flow is linearly stable.

4.2.2. *Longitudinal modes*

Longitudinal modes are such that $k = 0$. By considering the adiabatic eigenflows with $n \geqslant 1$, (4.11) can be simplified to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} = 0, \tag{4.16a}$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} - (\eta - \alpha_n^2)h + \frac{\partial \varphi_n(x, z)}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial \varphi_n(x, z)}{\partial z}\frac{\partial f}{\partial z} = 0, \tag{4.16b}$$

$$(x, z) \in \partial\Omega: \quad \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla f = 0, \quad \hat{\boldsymbol{n}} \boldsymbol{\cdot} \nabla h = 0. \tag{4.16c}$$

Equation (4.16*a*) subject to the Neumann boundary condition (4.16*c*) can be satisfied only by a uniform field $f(x, z)$, so that the solution of (4.16*c*) can be expressed as

$$f(x, z) = \text{const.}, \quad h(x, z) = \varphi_m(x, z), \quad \text{with } m = 0, 1, 2, \ldots. \tag{4.17a,b}$$

The dispersion relation, in this case, is written as

$$\eta = \alpha_n^2 - \alpha_m^2, \quad \text{with } m = 0, 1, 2, \ldots \text{ and } n = 1, 2, 3, \ldots. \tag{4.18}$$

As a consequence of (4.5), one infers from (4.18) that $\omega = -\mathrm{Im}(\eta) = 0$. Instability, $\mathrm{Re}(\eta) > 0$, arises for the adiabatic eigenflows with $n \geqslant 1$ according to the following scheme:

- the eigenflow with $n = 1$ is unstable to longitudinal rolls with $m = 0$;
- the eigenflow with $n = 2$ is unstable to longitudinal rolls with $m = 0$ and $m = 1$;
  ...
- the eigenflow with $n = N$ is unstable to longitudinal rolls with $m = 0$, $m = 1, \ldots$ and $m = N - 1$;
  ....

Therefore, all adiabatic eigenflows with $n \geqslant 1$ are unstable to the longitudinal mode of perturbation with $m = 0$. This mode of perturbation evolves in time according to (3.21), which predicts a spatially uniform growth in time of the basic temperature field, as well as of the y-component of the basic velocity.

The result of the analysis carried out for a channel with an arbitrary cross-sectional shape extends the same conclusion stated at the end of § 3.3.1: all adiabatic eigenflows are linearly unstable, with the only exception being the uniform isothermal flow ($n = 0$), which is linearly stable. Hence, this conclusion turns out to be independent of the shape of the channel cross-section.

5. Conclusions

The emergence of non-trivial stationary buoyant flows within a vertical plane porous channel with adiabatic walls and a uniform internal heat source has been investigated. These flows may be allowed only if the vertical temperature gradient takes on special discrete values forming an infinite increasing sequence. The eigenvalue character of the governing equations for such flows motivated the name of adiabatic eigenflows. A Darcy–Rayleigh number, $R > 0$, proportional to thermal power per unit volume supplied by the internal heat source, is defined. Adiabatic eigenflows exist with every value of R; they give rise to a net downward flow except for the case $R = 0$ where the net flow rate is zero. In the case $R = 0$, the adiabatic eigenflows can be physically interpreted as the mid-region of Rayleigh–Bénard cells in a very tall vertical porous cavity.

The linear stability of adiabatic eigenflows has been studied by the classical normal mode expansion of the disturbances. The main result of this analysis is that all adiabatic eigenflows are unstable. Among the longitudinal modes, independent of the vertical coordinate, the most unstable is the horizontally uniform mode, or mode $m = 0$. There is just an exception, with $R = 0$, corresponding to the special case where the vertical temperature gradient is also zero. For this special case, the basic state is one with uniform velocity and uniform temperature and is termed uniform isothermal flow. The uniform isothermal flow, as expected, is stable.

The stability analysis for the vertical plane channel has been extended to include the effects of lateral confinement through adiabatic walls. In this case, the infinite horizontal length in the spanwise direction becomes finite, so that the wavenumber spectrum of longitudinal modes becomes discrete instead of continuous. Lateral confinement actually means that the plane channel becomes a rectangular channel with an arbitrary aspect ratio s. It has been proved that the instability of adiabatic eigenflows is unaffected by lateral confinement.

Finally, the analysis of adiabatic eigenflows and their instability has been generalised to the case of a vertical channel having an arbitrary cross-sectional shape. It has been shown that the discrete values allowed for the vertical temperature gradient can be computed through the solution of the Neumann problem for the two-dimensional Helmholtz equation. In other words, the eigenvalues characteristic of the adiabatic eigenflows are the eigenfrequencies of a membrane with a free boundary. It has been proved that, for every possible shape of the channel cross-section, adiabatic eigenflows are allowed, but they are all unstable to longitudinal modes of perturbation. As for the case of the plane channel, the only exception is the uniform isothermal flow, which is stable.

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Appendix A. Rayleigh–Bénard cell in a vertical porous slot

In the special case $R=0$, let us consider the vertical channel studied in § 2.2 as the mid-region of an extremely tall two-dimensional porous cavity with adiabatic vertical boundaries (see figure 3). The far-away upper and lower horizontal walls are isothermal and kept at different temperatures so that a heating-from-below condition is set up.

We adopt a stationary two-dimensional formulation based on the stream function, ψ, so that the Cartesian components of the velocity are written as

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x}. \tag{A 1a,b}$$

Then, (2.1*b*) and (2.1*c*) with $R=0$ are rewritten as

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial T}{\partial x}=0, \tag{A 2a}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}-\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=0. \tag{A 2b}$$

As sketched in figure 3, in the mid-region, the streamlines are approximately straight and vertical, so that one may assume $u=\partial\psi/\partial y\approx 0$. Moreover, the vertical component of the temperature gradient, set up through the heating from below, is approximately a constant, namely $\partial T/\partial y\approx-\alpha^2$. With this ansatz, (A 2) undergo a drastic simplification, namely

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial T}{\partial x}=0, \tag{A 3a}$$

$$\frac{\partial^2 T}{\partial x^2}-\alpha^2\frac{\partial\psi}{\partial x}=0. \tag{A 3b}$$

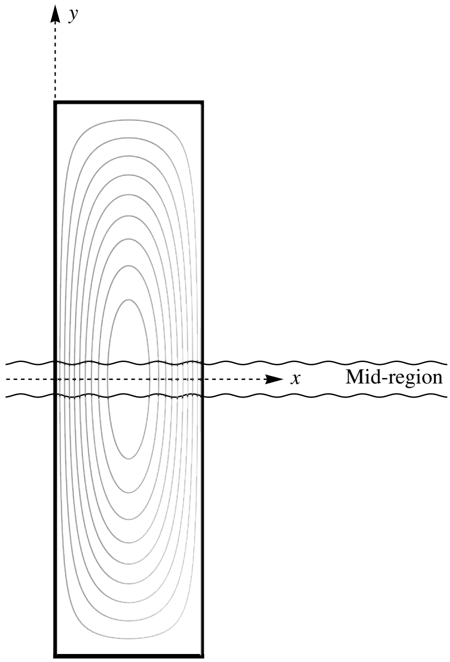

FIGURE 3. Rayleigh–Bénard cell in a vertical porous slot: qualitative sketch of the streamlines.

Combination of (A 3a) and (A 3b) and an explicit statement of the adiabatic boundary conditions at $x=0$, 1 leads to

$$\frac{\partial^3 T}{\partial x^3} + \alpha^2 \frac{\partial T}{\partial x} = 0, \tag{A 4a}$$

$$x = 0, 1: \quad \frac{\partial T}{\partial x} = 0. \tag{A 4b}$$

Setting $\partial T/\partial x = F(x)$ implies that (A 4) are perfectly coincident with the eigenvalue problem, (2.7) and (2.8), discussed in § 2.2. Solution of this eigenvalue problem implies that α^2 can assume only special discrete values, and this is here interpreted as a consequence of the stretching undergone by the Rayleigh–Bénard cell (or cells) in order to fit the space between the vertical adiabatic walls. Incidentally, figure 3 sketches just the simplest case where a single tall cell is allotted within the cavity, and this corresponds to the lowest non-vanishing eigenvalue, $\alpha = \alpha_1 = \pi$.

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