A BOGOMOLOV UNOBSTRUCTEDNESS THEOREM FOR LOG-SYMPLECTIC MANIFOLDS IN GENERAL POSITION

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Abstract   We consider compact Kählerian manifolds X of even dimension 4 or more, endowed with a log-symplectic holomorphic Poisson structure Π which is sufficiently general, in a precise linear sense, with respect to its (normal-crossing) degeneracy divisor $D(\Pi)$. We prove that (X, Π) has unobstructed deformations, that the tangent space to its deformation space can be identified in terms of the mixed Hodge structure on H^2 of the open symplectic manifold $X \setminus D(\Pi)$, and in fact coincides with this H^2 provided the Hodge number $h_X^{2,0} = 0$, and finally that the degeneracy locus $D(\Pi)$ deforms locally trivially under deformations of (X, Π).

We consider here holomorphic Poisson manifolds (X, Π), i.e., complex manifolds X (generally, compact and Kählerian) endowed with a holomorphic Poisson structure Π. We say that (X, Π) is *log-symplectic* if X has even dimension $2n$ and the degeneracy or Pfaffian divisor $D(\Pi)$ of Π, i.e., the divisor of the section $\Pi^n \in H^0(X, \wedge^{2n} T_X) = H^0(X, -K_X)$, is a reduced divisor with (local) normal crossings (NB: there are other notions of log-symplectic in the literature). (X, Π) is *simply log-symplectic* if moreover $D(\Pi)$ has simple normal crossings, i.e., is a transverse union of smooth components. We note that Lima and Pereira [11] have shown that if (X, Π) is simply log-symplectic and X is a Fano manifold of dimension 4 or more with cyclic Picard group, then X is $\mathbb{P}^{2n}$ with a standard (toric) Poisson structure

$$\Pi = \sum a_{ij} x_i x_j \, \partial / \partial x_i \, \partial / \partial x_j,$$

and x_i homogeneous coordinates (and consequently deformations of (X, Π) are of the same kind, at least set-theoretically).

A natural question about Poisson manifolds is that of understanding deformations of the pair (X, Π); in particular whether (X, Π) has unobstructed deformations, and whether such deformations change (e.g., smooth out) the singularities of the degeneracy locus $D(\Pi)$ or whether, on the contrary, $D(\Pi)$ deforms locally trivially. When (X, Π) is

symplectic, i.e., $D(\Pi) = 0$, unobstructedness is of course well known (Bogomolov, Tian, and Todorov). In [19] we proved unobstructedness and local triviality when Π satisfies a special condition called P-normality which states that D is smooth at every point where the corank of Π is exactly 2. This condition is equivalent to saying that Π has the largest possible corank, viz., $2m$, at a point of given multiplicity m on D (or equivalently, the smallest multiplicity for a given corank). P-normality is also equivalent to (X, Π) being decomposable locally as the product of Poisson surfaces with smooth or empty degeneracy divisor.

Our interest here is in Poisson structures with a property that goes in the opposite direction to, and indeed excludes, P-normality. This property, that we call r-general position and need just for $r = 2$, states that Π is 'general' in a neighborhood of D, in the sense that the columns of its matrix with respect to a coordinate system adapted to D satisfy a linear general position property. The 2-general position property implies that (X, Π) cannot split off locally a factor equal to a Poisson surface with singular (normal-crossing) degeneracy divisor. Our main result (Theorem 8) states that if Π is in 2-general position (which implies that $2n = \dim(X) \geqslant 4$), then (X, Π) has unobstructed deformations and moreover, these deformations induce locally trivial deformations of the degeneracy divisor $D(\Pi)$. In fact, the tangent space to the deformation space can be identified locally in terms of the (mixed) Hodge theory of the open symplectic variety $U = X \setminus D(\Pi)$, namely it equals $F^1 H^2(U, \mathbb{C})$. Whenever $h_X^{2,0} = 0$, the latter coincides with $H^2(U, \mathbb{C})$. For $X = \mathbb{P}^{2n}$ and Π a toric Poisson structure, this is due on the set-theoretic level to Lima and Pereira [11]. Here we give an extension of the Lima–Pereira result to the toric case: we will determine the deformations of 2-general log-symplectic toric Poisson structures on smooth projective toric varieties, showing in particular that they remain toric (cf. Corollary 10).

The local triviality conclusion is surprising because it is decidedly false in dimension 2. Though in [19] we proved an analogous local triviality result for deformations of P-normal Poisson structures, the latter certainly does hold in dimension 2 (where the degeneracy locus must be smooth). While the 2-general position – or something like it – seems necessary for local triviality, it is unclear whether it is necessary for mere unobstructedness. In the case of Poisson surfaces, i.e., surfaces endowed with an effective anticanonical divisor, deformations are unobstructed whenever the divisor is reduced and has normal crossings, but can be obstructed when the divisor is non-reduced (see [19]).

The strategy for the proof is simple. Poisson deformations are 'controlled' by the Poisson–Schouten differential graded Lie algebra (dgla) $(T_X^\bullet, [\,.\,,\Pi])$. As shown in [19], using only that D has normal crossings, the sub-dgla $T_X^\bullet\langle -\log D\rangle$ has unobstructed deformations thanks to its isomorphism with the dgla of log forms $(\Omega_X^\bullet\langle \log D\rangle, d)$. We extend this isomorphism to an isomorphism between $T_X^\bullet$ and a certain dgla of meromorphic forms denoted by $\Omega_X^\bullet\langle \log{}^+D\rangle$, the 'log-plus forms', which contains the log forms. We identify the quotient complex $\Omega_X^\bullet\langle \log{}^+D\rangle/\Omega_X^\bullet\langle \log D\rangle$ and prove – under the 2-general position hypothesis – that it is exact in degrees $\leqslant 2$.

Other unobstructedness results of 'Bogomolov–Tian–Todorov type' were obtained by Kontsevich *et al.* [9, 10]. Earlier, Hitchin [8] and Fiorenza and Manetti [4] had proven unobstructedness for certain deformation directions, namely, those corresponding to the

Kähler class itself. Also, Pym [16] has introduced the notion of elliptic Poisson structures and more generally, Pym and Schedler [18] then introduced the notion of *holonomic* Poisson manifolds, where the degeneracy divisor is reduced but not necessarily normal crossings, and have studied their deformations focusing on questions of local finite dimensionality.

The referee points out the work of Mǎrcut and Osorno Torres [12] in the real case, which considers deformations of real C^∞ Poisson structures with *smooth* degeneracy locus (of real codimension 1) . Though the setting is different, their methods, especially in their §3, are somewhat related to those of our §2. The analogue of their quasi-isomorphism result (Lemma 3) also holds in the holomorphic case when the degeneracy divisor is smooth.

In this paper we work in the holomorphic category and use the complex topology unless otherwise mentioned.

1. Basics

See [3], [1], or [17] for basic facts on Poisson manifolds and [5] (especially the appendix), [6], or [13], and references therein for information on deformations of Poisson complex structures.

1.1. Log vector fields, duality

Consider a log-symplectic $2n$-dimensional Poisson manifold (X, Π) with degeneracy or Pfaffian divisor $D = [\Pi^n]$. We denote by $\Pi^\sharp$ the interior multiplication map

$$\Pi^\sharp : \Omega^1_X \to T_X.$$

As $\Pi^\sharp$ is generically an isomorphism, it extends to an isomorphism on meromorphic differentials. The following result is apparently well known and was stated in [19, Lemma 6] under the needlessly strong hypothesis that Π is P-normal.

Lemma 1. *If D has local normal crossings, then $\Pi^\sharp$ yields an isomorphism*

$$\Pi^\sharp : \Omega^1_X \langle \log D \rangle \to T_X \langle -\log D \rangle. \tag{1}$$

Proof. The assertion is obvious at points where Π is nondegenerate. As smooth points of D, it follows easily from Weinstein's normal form. Thus, $\Pi^\sharp$ is a morphism of locally free sheaves on X which is an isomorphism off a codimension-2 subset, viz., $\mathrm{sing}(D)$. Therefore $\Pi^\sharp$ is an isomorphism. $\qquad\square$

1.2. Log deformations

Note that Π can be viewed as a section of

$$T^2_X \langle -\log D \rangle := \wedge^2 T_X \langle -\log D \rangle,$$

i.e., as 'Poisson structure for the Lie algebra sheaf $T_X \langle -\log D \rangle$'; one can think of the lemma as saying that Π is *nondegenerate as such*. Note that a $T_X \langle -\log D \rangle$-deformation, say over a local Artin algebra R, consists of a flat deformation of X, $\tilde{X}/R$ plus a compatible

locally trivial deformation of D, $\tilde{D}/R$. On the other hand, a $T_X^\bullet\langle -\log D\rangle$-deformation over R consists of a flat deformation of X, $\tilde{X}/R$, a compatible locally trivial deformation of D, $\tilde{D}/R$, plus a compatible deformation $\tilde{\Pi}$ of Π, that is, a section of $T_{\tilde{X}}^2\langle -\log \tilde{D}\rangle$ extending Π. In that case clearly the degeneracy locus $D(\tilde{\Pi}) = \tilde{D}$. Thus the natural forgetful map

$$T_X^\bullet\langle -\log D\rangle \to T_X\langle -\log D\rangle$$

corresponds to mapping

$$\tilde{\Pi} \to D(\tilde{\Pi}).$$

We record this for future reference:

Lemma 2. *Let (X, Π) be a log-symplectic manifold with degeneracy locus $D = D(\Pi)$ and let R be a local Artin algebra. Then an R-valued $T_X^\bullet\langle -\log D\rangle$-deformation induces*

 (i) *a flat deformation $\tilde{X}/R$;*
 (ii) *an R-linear extension $\tilde{\Pi}$ of Π to $\tilde{X}$, such that the degeneracy locus $D(\tilde{\Pi})$ is a locally trivial deformation of D.*

Remark 3. *Any deformation of (X, Π) induces a deformation of $D(\Pi)$ but the latter deformation need not be locally trivial in general. This is already clear when X is 2-dimensional, so Π corresponds to an effective anticanonical divisor which can be singular and deform to a nonsingular one. For example, when $X = \mathbb{P}^2$, log-symplectic structures Π are in 1–1 correspondence with nodal or smooth cubic curves $C = D(\Pi)$.*

It also follows from Lemma 1, and was also noted in [19], that $\Pi^\sharp$ extends to an isomorphism of differential graded algebras

$$\Pi^\sharp : (\Omega_X^{\leqslant\bullet}\langle\log D\rangle, d) \to (T_X^\bullet\langle -\log D\rangle, [., \Pi]) \tag{2}$$

where $\Omega_X^{\leqslant\bullet}\langle\log D\rangle$ denotes the nonnegative complex with i-forms in degree $i-1$ for $i > 0$; in other words, $F^1\Omega^\bullet\langle\log D\rangle[1]$ where F denotes the Hodge or 'stupid' filtration. We will denote the graded isomorphism inverse to $\Pi^\sharp$ by $\Pi^\flat$. Working locally in a neighborhood of a point, let $D_1, \ldots, D_m$ be the components of D, with respective local equations $x_1, \ldots, x_m$ which are part of a local coordinate system $x_1, \ldots, x_{2n}$. Thus, denoting $\partial_i = \partial / \partial x_i$, we have a local basis for $T_X\langle -\log D\rangle$ of the form

$$v_i := \begin{cases} x_i\,\partial_i, & i = 1, \ldots, m, \\ \partial_i, & i = m+1, \ldots, 2n, \end{cases}$$

with the dual basis

$$\eta_i := \begin{cases} dx_i/x_i, & i = 1, \ldots, m, \\ dx_i, & i = m+1, \ldots, 2n \end{cases}$$

being a local basis for $\Omega_X^1\langle\log D\rangle$. Let $A = (a_{ij})$ be the matrix of $\Pi^\sharp$ with respect to these bases and $B = (b_{ij}) = A^{-1}$. Thus A and B are skew-symmetric. Then we have

$$\Pi = \sum_{1\leqslant i < j \leqslant 2n} a_{ij} v_i v_j,$$

$$\Phi := \Pi^{-1} = \sum_{1 \leqslant i < j \leqslant 2n} b_{ij} \eta_i \eta_j.$$

Note that the map $\Pi^\flat$ is just interior multiplication by Φ.

1.3. General position

A pair of $k \times k$ matrices M, N with entries in a ring R are said to be in *t-relative general position* if every collection of t columns of M, together with the corresponding columns of N, generates a free and cofree rank-$2t$ submodule of R^k; equivalently, for every such 'matched' collection of $2t$ columns from M and N, there exists a collection of $2t$ rows such that the corresponding $2t \times 2t$ minor is a unit (we will henceforth abbreviate the latter conclusion by simply saying that the columns in question are linearly independent). Clearly this condition is never satisfied if $2t > k$ while if $2t \leqslant k$ it is satisfied if N is nonsingular and M is general relative to N. M is said to be in (standard) *t-general position* if M, I_k are in relative t-general position. The log-symplectic structure Π as above is said to be in t-general position if, locally at each point of the degeneracy locus, there is a local coordinate system as above such that the matrix A above is in (standard) t-general position. In general, if A is 'generic', Π will be in n-general position. On the other hand, if (X, Π) decomposes as a product structure with a factor of dimension $2k$ then Π is not in $(2k)$-general position.

2. The Log-plus complex

2.1. Definition of log-plus complex

With notations as above, we also denote by $\Omega^1_X(*D)$ the sheaf of meromorphic forms with arbitrary-order pole along D:

$$\Omega^1_X(*D) = \bigcup_{k=1}^\infty \Omega^1_X(kD).$$

Then $\Pi^\flat$ extends to an injective map $T_X \to \Omega^1_X(*D)$, whose image we denote by $\Omega^1_{X,\Pi}\langle \log^+ D \rangle$. As subsheaf of $\Omega^1_X(*D)$, this sheaf depends on Π, but we will often abuse notation and denote it simply by $\Omega^1_X\langle \log^+ D \rangle$. Thus, $\Pi^\flat$ yields an isomorphism

$$\Pi^\flat : T_X \to \Omega^1_X\langle \log^+ D \rangle.$$

Also set

$$\Omega^k_X\langle \log^+ D \rangle = \wedge^k \Omega^1_X\langle \log^+ D \rangle.$$

Note that the isomorphism of complexes

$$\Pi^\flat : (\wedge^\bullet T_X\langle -\log D \rangle, [\,.\,, \Pi]) \to (\Omega^{\leqslant \bullet}_X\langle \log D \rangle, d)$$

now extends to an isomorphism of complexes

$$\Pi^\flat : (\wedge^\bullet T_X, [\,.\,, \Pi]) \to (\Omega^{\leqslant \bullet}_X\langle \log^+ D \rangle, d). \tag{3}$$

The log-plus complex has appeared before, in a different context, in the work of Polishchuk [15].

2.2. Comparison with log complex

Denote by $\Omega^1_X\langle\log{}^{++}D\rangle$ the subsheaf of $\Omega^1_X(*D)$, generated by $\Omega^1_X\langle\log D\rangle$ and $\eta_j/x_i, i \leqslant m, j \neq i$. Then using, e.g., (1), note that $\Omega^1_X\langle\log{}^+D\rangle$ is a subsheaf of $\Omega^1_X\langle\log{}^{++}D\rangle$. Also set

$$\Omega^k_X\langle\log{}^{++}D\rangle = \wedge^k\Omega^1_X\langle\log{}^{++}D\rangle.$$

Now let $(\phi_\bullet) = \Pi^\flat(\partial_\bullet)$ be the local basis of $\Omega^1_X\langle\log{}^+D\rangle$ corresponding to the basis $\partial_1, \ldots, \partial_{2n}$ of T_X. Thus

$$\phi_i = \sum_{j=1}^{m} b_{ij}\, dx_j/x_i x_j + \sum_{j=m+1}^{2n} b_{ij}\, dx_j/x_i, \quad i = 1, \ldots, m$$

while $\phi_{m+1}, \ldots, \phi_{2n} \in \Omega^1_X\langle\log D\rangle$. Note that for $i = 1, \ldots, m$, $x_i\phi_i$ corresponds to the log vector field $v_i = x_i\,\partial_i$, which is automatically locally Hamiltonian off the degeneracy locus; hence $x_i\phi_i$ is a closed form. Therefore

$$d\phi_i = \phi_i \wedge dx_i/x_i, i = 1, \ldots, m.$$

More generally, for a multi-index $I = (i_1 < \cdots < i_k)$, setting

$$\phi_I = \phi_{i_1} \wedge \cdots \wedge \phi_{i_k},$$

we have

$$d\phi_I = \sum_{j=1}^{k}((-1)^j\, dx_{i_j}/x_{i_j})\phi_I. \tag{4}$$

This implies immediately that $(\Omega^{\leqslant\bullet}_X\langle\log{}^+D\rangle, d)$ is a complex, i.e., that $d\Omega^k_X\langle\log{}^+D\rangle \subset \Omega^{k+1}_X\langle\log{}^+D\rangle$. Moreover, $\Omega^{\leqslant\bullet}_X\langle\log{}^+D\rangle$ admits an ascending filtration $\mathcal{F}_\bullet$ by subcomplexes (and $\Omega^\bullet_X\langle\log D\rangle$-modules under exterior product) defined by:

– $\mathcal{F}_0 = \Omega^{\leqslant\bullet}_X\langle\log D\rangle$;

– for each $i \geqslant 0$, $\mathcal{F}_{i+1}$ is generated by $\mathcal{F}_i$ and $\Omega^1_X\langle\log{}^+D\rangle\mathcal{F}_i$.

Thus, $\mathcal{F}_i$ is generated over $\Omega^\bullet_X\langle\log D\rangle$ by $\Omega^j_X\langle\log{}^+D\rangle$, $j = 1, \ldots, i$. In particular, the quotient $\mathcal{G}_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ is generated over $\Omega^\bullet_X\langle\log D\rangle$ by the images of $\phi_I, |I| = i$, which we denote by $\bar{\phi}_I$. Note that $\mathrm{Ann}_{\mathcal{O}_X}(\bar{\phi}_I) = (x_r : r \in I)$. Let D_I be the corresponding stratum. Then $\mathcal{G}_i$ decomposes at a direct sum of complexes of locally free $\mathcal{O}_{D_I}$-modules:

$$\mathcal{G}_i = \bigoplus_{|I|=i} \bar{\phi}_I\, \Omega^\bullet_X\langle\log D\rangle[-i] \otimes \mathcal{O}_{D_I}. \tag{5}$$

The directness of the sum follows from the corresponding assertion for $\wedge^\bullet T_X\langle-\log D\rangle$, which is obvious.

We denote the summand above that corresponds to I by $Q^\bullet_I$, a complex in degrees $\geqslant i$. Note that

$$\Omega^1_X\langle\log D\rangle \otimes \mathcal{O}_{D_I} = \Omega^1_{D_I}\langle\log \bar{D}_I\rangle \oplus \bigoplus_{r\in I}(dx_r/x_r)\mathcal{O}_{D_I}$$

where $\bar{D}_I$ is the normal-crossing divisor on D_I induced by D (i.e., by the components of D not containing D_I). This decomposition induces an analogous one on the exterior power $\Omega_X^k\langle \log D\rangle \otimes \mathcal{O}_{D_I}$. We now introduce the hypothesis that Π is in t-general position. This implies that for any index set $I \subset \underline{m}$ with $|I| \leqslant t$, any set of $2t$ log forms among $\{dx_i/x_i, x_i\phi_i : \in I\}$ is linearly independent. Equivalently, $x_i\phi_i, i \in I$ pull back to a collection of nonvanishing linearly independent log forms on D_I provided $|I| \leqslant t$. Note also that

$$\left(\sum_{r=1}^{m} \mathcal{O}_X \phi_r\right) \cap \Omega_X^1\langle \log D\rangle = \sum_{r=1}^{m} \mathcal{O}_X x_i \phi_r$$

(just apply the isomorphism $\Pi^\sharp$ to both sides). Consequently, we have, *provided Π is in t-general position*, that

$$Q_I^{|I|+k} \simeq \bigoplus_{j=0}^{k} \Omega_{D_I}^{k-j}\langle \log \bar{D}_I\rangle \left(\bigoplus_{\substack{|J|=j \\ J \subset I}} d\log(x)_J\right), \quad |I|+k \leqslant t, \tag{6}$$

where

$$d\log(x)_J := \wedge_{i \in J} dx_i/x_i.$$

t-general position is needed to ensure that any collection of t distinct dx_j/x_j-s and $x_i\phi_i$-s is independent, i.e., spans a free and cofree submodule of $\Omega_X^1\langle \log D\rangle$, and consequently for every $d\log(x)_J$ appearing above, $d\log(x)_J \wedge \phi_I$ is nonvanishing. Note that in terms of the above identification, the differential on $\mathbb{Q}_I^\bullet$ is given by

$$d(\psi_J \, d\log(x)_J) = (d\psi_J) \, d\log(x)_J + \sum_{i \in I\setminus J} \psi_J(-1)^i \, d\log(x)_J \, dx_i/x_i. \tag{7}$$

Of course $d\log(x)_J$ and $d\log(x)_J \, dx_i/x_i$ are just placeholders, so dropping these, we get more concretely the formula for the 'twisted' differential as

$$\tilde{d}(\psi_J) = (d\psi_J, -\psi_J)_{i \in I\setminus J}. \tag{8}$$

Thus, for elements ψ_J in the component of type (k, J) (so that ψ_J is a differential of degree $k - |J|$), the differential has components of type $(k+1, J)$ (namely $d\psi_J$) and type $(k+1, J \cup \{i\})$ (namely $\pm\psi_J$), for all $i \in I \setminus J$, and they are given as above.

Note that such a complex is automatically exact except if it consists of a single group: a cocycle of the form (ψ, γ) such that $\psi = d\gamma$ equals $\tilde{d}(\pm\gamma.0)$. Consequently, we conclude (where we recall that by definition the Q complexes have i forms on degree $i - 1$) the following.

Lemma 4. *If Π is in r-general position, then for each I, $Q_I^\bullet$ is exact in degrees $\leqslant r - 1$.*

Now from the local-to-global spectral sequence for the hypercohomology we conclude

$$\mathbb{H}^i(\Omega_X^{\leqslant\bullet}\langle \log {}^+D\rangle)/(\Omega_X^{\leqslant\bullet}\langle \log D\rangle) = 0, \quad \forall i < r.$$

Then the long exact sequence of hypercohomology yields the following corollary.

Corollary 5. *If Π is in r-general position, then the natural map*

$$\mathbb{H}^i(\Omega_X^{<\bullet}\langle\log D\rangle) \to \mathbb{H}^i(\Omega_X^{<\bullet}\langle\log{}^+ D\rangle)$$

is an isomorphism for $i \leqslant r-1$ and an injection for $r = i$.

Remark 6. *In the real C^∞ case with smooth degeneracy locus, Mărcut and Osorno Torres (see [12, §3]) show that the analogous inclusion map is a quasi-isomorphism. In the holomorphic case with smooth degeneracy locus, the analogous fact is true as well: the inclusion map from the log complex to the log-plus complex is a quasi-isomorphism; no general position hypotheses are needed. This follows easily from computations as above, given Weinstein's normal form [1].*

Remark 7. *A 2-dimensional log-symplectic Poisson structure Π with* singular *degeneracy divisor $D(\Pi)$ is never in 2-general position and in fact the complex $\mathcal{G}_2$ above is not locally acyclic in degree 1 (i.e., on 2-forms): for the Poisson structure $\frac{dx_1\,dx_2}{x_1 x_2}$, the 2-form $\frac{dx_1\,dx_2}{x_1^2 x_2^2}$ is closed but not exact in the log-plus complex.*

Naturally the same applies whenever the log-symplectic manifold (X, Π) splits as a (Poisson) product with a 2-dimensional factor (X_1, Π_1) such that $D(\Pi_1)$ is singular. Thus such a manifold cannot be in 2-general position.

3. Main theorem

3.1. Main result

Our main result is the following.

Theorem 8. *Let (X, Π) be a log-symplectic compact Kählerian Poisson manifold of dimension $2n \geqslant 4$ with degeneracy divisor D. Assume (X, Π) is in 2-general position. Then*

(i) *(X, Π) has unobstructed deformations;*

(ii) *the tangent space to the deformation space of (X, Π) coincides with the Hodge-level subspace $F^1\mathbb{H}^2(X \setminus D, \mathbb{C})$;*

(iii) *any deformation of (X, Π) induces a locally trivial deformation of the degeneracy locus $D(\Pi)$.*

Proof. As is well known, the deformation theory of (X, Π) is the deformation theory of the dgla sheaf $(T_X^\bullet, [\,.\,, \Pi])$. The latter is isomorphic as dgla to $(\Omega_X^{<\bullet}\langle\log{}^+ D\rangle, d)$, so it suffices to prove that obstructions vanish for the latter. By Corollary 5, the inclusion map

$$\Omega_X^{<\bullet}\langle\log D\rangle \to \Omega_X^{<\bullet}\langle\log{}^+ D\rangle$$

induces an isomorphism on $\mathbb{H}^1$ and an injection on $\mathbb{H}^2$. However, it was already observed in [19], based on Hodge theory for the log complex and the isomorphism $\Pi^\sharp : \Omega_X^{<\bullet}\langle\log D\rangle \to T_X^\bullet\langle-\log D\rangle$, that the dgla $\Omega_X^{<\bullet}\langle\log D\rangle$ has vanishing obstructions. Since these obstructions reside in $\mathbb{H}^2$, it follows that $\Omega_X^{<\bullet}\langle\log{}^+ D\rangle$ has vanishing obstructions as well. This proves (i).

Now $\Omega_X^{\leq \bullet}\langle \log D \rangle$ is none other than a shift of the first level F^1 of the usual Hodge filtration on the log complex. Therefore (ii) follows from basic mixed Hodge theory, specifically the fact that the log complex of a divisor with normal crossings computes the cohomology of the complement (see [2, 14], Chapter 4 or [7, §5, esp. Proposition 5.14]), while (iii) follows from the fact observed above that $T_X^{\bullet}\langle -\log D \rangle$-deformations induce locally trivial deformations of D. □

Corollary 9. *With notations as above, assume additionally that $h_X^{2,0} = 0$. Then the tangent space to the deformation space of (X, Π) coincides with $H^2(X \setminus D, \mathbb{C})$.*

Proof. Our assumption implies that $H^2(\mathcal{O}_X) = H^2(\Omega_X^0 \langle \log D \rangle) = 0$. Hence by Deligne's mixed Hodge theory as referenced above (e.g., [14, Theorem 4.2]), $F^1 H^2(X \setminus D, \mathbb{C}) = H^2(X \setminus D, \mathbb{C})$. □

3.2. Toric Poisson structures

By way of example, we will apply Theorem 8 to toric Poisson structures on toric varieties. We begin with some general remarks on toric varieties. Let X be a smooth projective toric variety of dimension d with open orbit $U \simeq (\mathbb{C}^*)^d$ and boundary $D = X \setminus U$, a divisor with normal crossings. Then the invariant differentials dx_i/x_i pulled back from the $\mathbb{C}^*$ factors extend to log differentials on X, and in particular,

$$h^0(\Omega_X^i \langle \log D \rangle) \geqslant \binom{d}{i}. \tag{9}$$

On the other hand, note that

$$U \simeq (\mathbb{C}^*)^d \sim_{\text{homotopy}} (S^1)^d;$$

hence $H^i(U, \mathbb{C})$ is $\binom{d}{i}$-dimensional. But by basic mixed Hodge theory (see [2], [7], or [14]), the latter vector space has a filtration with quotients $H^{i-j}(\Omega_X^j \langle \log D \rangle)$. Comparing with (9) we conclude

$$h^i(\Omega_X^j \langle \log D \rangle) = \begin{cases} 0, & i > 0; \\ \binom{d}{j}, & i = 0. \end{cases} \tag{10}$$

Thus,

$$H^i(U, \mathbb{C}) \simeq H^0(\Omega_X^i \langle \log D \rangle). \tag{11}$$

Now let $v_1, \ldots, v_d$ be vector fields on X corresponding to the $(\mathbb{C}^*)^d$-action. They are dual to dx_i/x_i. Then for any skew-symmetric $d \times d$ matrix $A = (a_{ij})$ we get a $(\mathbb{C}^*)^d$-invariant Poisson structure on X:

$$\Pi_A = \sum a_{ij} v_i \wedge v_j. \tag{12}$$

Now assume $d = 2n$ even, and that A is nonsingular and in 2-general position. Then (X, Π_A) is log-symplectic in 2-general position. Note that

$$H^0(T_X^2 \langle -\log D \rangle) \simeq H^0(\Omega_X^2 \langle \log D \rangle)$$

is generated by $v_i \wedge v_j$. Therefore by applying Theorem 8 we conclude the following.

Corollary 10. *With notations as above, and assuming A is nonsingular in 2-general position, $(X, \Pi)_A$ has unobstructed deformations parametrized by a germ of $H^0(X, T^2_X \langle -\log D \rangle)$, all of the form $(X, \Pi_{A'})$ obtained by deforming A, and the tangent space to the deformation space coincides with $H^2(X \setminus D, \mathbb{C})$ and is $\binom{2n}{2}$-dimensional. In particular, all deformations of (X, Π) are toric.*

When $n = 1$, no 2-dimensional Poisson structure is in 2-general position as has been noted in §1.3. For $n > 1$, taking (a_{ij}) general, or e.g., specifically $a_{ij} = i - j$ yields 2-general position, while A corresponding to a direct sum of hyperbolic planes does not (see Remark 7).

In the case $X = \mathbb{P}^{2n}$ a similar result (for A generic) was obtained by Lima and Pereira [11].

References

1. N. CICCOLI, From Poisson to quantum geometry, Notes taken by P. Witkowski, available on http://toknotes.mimuw.edu.pl/sem4/files/Ciccoli_fpqg.pdf.
2. P. DELIGNE, Théorie de Hodge II, *Publ. Math. Inst. Hautes Études Sci.* (40) (1972), 5–57.
3. J.-P. DUFOUR AND N. T. ZUNG, *Poisson Structures and Their Normal Forms*, Prog. Math., Volume 242 (Birkhauser, Basel–Boston–Berlin, 2005).
4. D. FIORENZA AND M. MANETTI, Formality of Koszul brackets and deformations of holomorphic Poisson manifolds, *Homology, Homotopy Appl.* **14** (2012), 63–75, arXiv:1109.4309v3.
5. V. GINZBURG AND D. KALEDIN, Poisson deformations of symplectic quotient singularities, *Adv. Math.* **186** (2004), 1–57, arXiv:0212279v5.
6. R. GOTO, Rozanski–Witten invariants of log-symplectic manifolds, in *Integrable Systems, Topology, and Physics, Tokyo 2000*, Contemporary Mathematics, Volume 309 (American Mathematical Society, Providence, RI, 2002).
7. P. GRIFFITHS AND W. SCHMID, *Recent Developments in Hodge Theory, Discr. Subgr. Lie Groups & Appl. to Moduli, Proc. Int. Colloq. Bombay*, pp. 31–127 (Oxford University Press, 1973).
8. N. HITCHIN, Deformations of holomorphic Poisson manifolds, Preprint, 2011, arXiv:1105.4775v1.
9. L. KATZARKOV, M. KONTSEVICH AND T. PANTEV, Bogomolov–Tian–Todorov theorems for Landau–Ginzburg models, Preprint, 2014, arXiv:1409.5996.
10. M. KONTSEVICH, Generalized Tian–Todorov theorems, in *Proceedings of Kinosaki Conference* (Kyoto University, 2008). https://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/215060/1/2008-02.pdf.
11. R. LIMA AND J. V. PEREIRA, A characterization of diagonal Poisson structures, *Bull. Lond. Math. Soc.* **46** (2014), 1203–1217.
12. I. MĂRCUT AND B. OSORNO TORRES, Deformations of log symplectic structures, *J. Lond. Math. Soc. (2)* **90** (2014), 197–212.

13. Y. Namikawa, *Flops and Poisson Deformations of Symplectic Varieties*, Publications of the Research Institute for Mathematical Sciences, Volume 44, pp. 259–314 (Kyoto University, Kyoto, Japan, 2008).

14. C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge Structures*, Ergebnisse der Math. und ihrer Grenzgebiete 3. Folge (Springer, 2008).

15. A. Polishchuk, Algebraic geometry of Poisson brackets, *J. Math. Sci.* **84** (1997), 1413–1444.

16. B. Pym, Elliptic singularities on log symplectic manifolds and Feigin–Odeskii Poisson brackets, Preprint, 2015, arXiv:1507.05668.

17. B. Pym, Constructions and classifications of projective Poisson varieties, Preprint, 2017, arXiv:1701.08852v1.

18. B. Pym and T. Schedler, Holonomic Poisson manifolds and deformations of elliptic algebras, Preprint, 2017, arXiv:1707.06035v1.

19. Z. Ran, Deformations of holomorphic pseudo-symplectic Poisson manifolds, *Adv. Math.* **304** (2017), 1156–1175, arXiv:1308.2442.