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The joint law of the last zeros of Brownian motion and of its Lévy transform

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Abstract. The joint study of functionals of a Brownian motion B and its Lévy transform $\beta = |B| - L$, where L is the local time of B at zero, is motivated by the conjectured ergodicity of the Lévy transform.

Here, we compute explicitly the covariance of the last zeros before time one of B and β, which turns out to be strictly positive.

1. *Motivation and main results*

1.1. Let $(B_t, t \geq 0)$ be a one-dimensional Brownian motion starting from zero, and $(L_t, t \geq 0)$ its local time at zero. There has been quite some interest during the last decade in the so-called perturbed Brownian motions $(B_t^{(\mu)} \stackrel{\text{def}}{=} |B_t| - \mu L_t, t \geq 0)$ which have a number of interesting properties, for example: the time spent by $(B_t^{(\mu)}, t \leq 1)$ below zero is beta distributed, a generalization of the arcsine law due to F. Petit; see, more generally, the last chapter of Yor [**18**] for a number of recent studies.

As is well known, in the particular case $\mu = 1$, $(B_t^{(1)}, t \geq 0)$, the Lévy transform of B, is a Brownian motion, and some interest in the pair $(B, B^{(1)})$ stems from the open question:

$$\text{is the Lévy transform } T : (B_t) \longrightarrow (B_t^{(1)}) \text{ ergodic?}$$

For more details about the ergodicity problem for Lévy's transform, see Dubins *et al* [**5**], Dubins and Smorodinsky [**6**] and Malric [**11**].

1.2. As a step towards the study of the ergodicity of T, Smorodinsky [**15**] asked one of us to describe the joint law of the pair (g, γ), where

$$g \stackrel{\text{def}}{=} \sup\{t < 1 : B_t = 0\} \quad \text{and} \quad \gamma \stackrel{\text{def}}{=} \sup\{t < 1 : |B_t| - L_t = 0\},$$

which, as is well known, has arcsine distributed marginals.

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From [**15**], we infer that certain properties of this joint law might lead to a density property of the sequence $g_n \overset{\text{def}}{=} g(T^n(B))$, $n \in \mathbb{N}$, of the last zeros of the iterates of B under T. This, in turn, could lead to the ergodicity of T; however, we leave the exploitation of the results in our paper to ergodic experts.

1.3. The apparently simple and quite natural question of describing the joint law of (g, γ) turns out to necessitate in fact the use of most of the present knowledge about the decomposition of the Brownian path $(B_t, t \leq 1)$ before and after g.

Below, we shall express the law of (g, γ) in terms of the following independent variables:

(a) g;

(b) $S_m \overset{\text{def}}{=} \sup_{u \leq 1} m_u$, where $(m_u, u \leq 1)$ is a standard Brownian meander;

(c) $(\widetilde{\theta}, \widetilde{m}_1)$, where $(\widetilde{m}_u, u \leq 1)$ is a standard Brownian meander process and $\widetilde{\theta} \overset{\text{def}}{=} \inf\{t > 0 : \widetilde{m}_t = \widetilde{m}_1\}$;

(d) (U_b, S_b), where $(b(u), u \leq 1)$ is a standard Brownian bridge process, $S_b \overset{\text{def}}{=} \sup_{u \leq 1} b(u)$, and U_b is the almost surely unique location of the maximum of b.

Here are our main results.

THEOREM 1.1. *For every Borel function* $f : [0, 1]^2 \longrightarrow \mathbb{R}_+$, *one has*

$$\mathbb{E}[f(g, \gamma)] = \tfrac{1}{2}\mathbb{E}[f(gU_b, g)] + \tfrac{1}{2}\mathbb{E}[f(gU_b, g)\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}S_b\}}] + \mathbb{E}[f(g, g(1 - \widetilde{\theta}))\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}\widetilde{m}_1\}}]. \tag{1.1}$$

THEOREM 1.2. *The covariance of* g *and* γ *is given by*

$$\operatorname{cov}(g, \gamma) = \mathbb{E}(g\gamma) - \mathbb{E}(g)\mathbb{E}(\gamma) = \mathbb{E}(g\gamma) - \tfrac{1}{4},$$

and

$$\mathbb{E}(g\gamma) = \tfrac{1}{8} + \tfrac{1}{24}\pi^2 - \tfrac{7}{32}\zeta(3) \approx 0.273\ldots,$$

where $\zeta(\cdot)$ *is the Riemann zeta function. Consequently,* $\operatorname{cov}(g, \gamma) > 0$.

1.4. Our paper is organized as follows: in §2, we prove Theorem 1.1; in §3, we give a precise description of the law of $(\widetilde{\theta}, \widetilde{m}_1)$ found in (c) above; in §4, we present some probabilistic discussions of the formula (1.2) below, and give a few extensions.

In §5, we prove Theorem 1.2, which hinges on the key formula: for $a > 0$, and C_a a Cauchy variable with parameter a,

$$\mathbb{E}\left[\frac{C_a}{\sinh(C_a)}\right] = 1 + 2\sum_{k=1}^{\infty}(-1)^k \frac{a}{a + k\pi}, \tag{1.2}$$

and some of its consequences. As the reader will soon realize, we shall often use formula (1.2) together with the identity in law:

$$|C_1| \overset{\text{law}}{=} \sqrt{\frac{g}{1 - g}}. \tag{1.3}$$

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Originally, we found formula (1.2) in Gradshteyn and Ryzhik [**7**, p. 348], in the analytical form:

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{b^2+x^2}\frac{x}{\sinh(ax)} = \frac{1}{2ab} + \sum_{k=1}^\infty \frac{(-1)^k}{ab+k\pi}. \tag{1.4}$$

However, in §3, we offer a discussion of (1.4), emphasizing in particular how closely related this formula is to the classical Kolmogorov–Smirnov result:

$$\sup_{u\le 1} |b(u)|^2 \overset{\text{law}}{=} T^{(3)}_{\pi/2},$$

where $T_c^{(3)} \overset{\text{def}}{=} \inf\{t > 0 : R_t = c\}$ with R a three-dimensional Bessel process starting from 0.

In §6, we explicitly compute the martingale representations of $\mathbb{E}(g \mid \mathcal{F}_t)$ and $\mathbb{E}(\gamma \mid \mathcal{F}_t)$, where $(\mathcal{F}_t)_{t\ge 0}$ denotes the natural filtration of B. This would shed light on the dependence structure of (g, γ).

Finally, in §§7 and 8, we discuss some related questions.

2. *Proof of Theorem 1.1*

We recall a decomposition theorem for the Brownian sample paths. For any stochastic process $(X_t, 0 \le t \le 1)$, and any random times a and b with $0 \le a < b \le 1$, define

$$X^{[a,b]} \overset{\text{def}}{=} \left(\frac{1}{\sqrt{b-a}} X_{a+t(b-a)}, 0 \le t \le 1\right).$$

Then, $b \overset{\text{def}}{=} B^{[0,g]}$ is a standard Brownian bridge; $m \overset{\text{def}}{=} |B|^{[g,1]}$ is a Brownian meander; b, m and g are independent.

Another representation of the meander process is given by [**1**, **3**]:

$$(|b_u| + \ell_u, u \le 1),$$

where (ℓ_u) is the local time at zero of the bridge b.

We also recall the following representation of the meander [**1**; **14**, Exercise XII.4.25]:

$$\left(2\sup_{s\le u} b_s - b_u, u \le 1\right).$$

We now study the joint law of (g, γ) on two disjoint events: $\{g > \gamma\}$ and $\{g < \gamma\}$.

2.1. *First situation: $g > \gamma$.*

Observe that on $\{g > \gamma\}$,

$$\begin{aligned}\gamma &= \sup\{t \le g : |B_t| = L_t\}\\ &= g \sup\{u \le 1 : |b_u| = \ell_u\},\end{aligned}$$

where (ℓ_u) denotes as before the local time process at zero of the Brownian bridge b. Therefore, on $\{g > \gamma\}$,

$$\gamma = g(1 - \inf\{v \le 1 : |b_{1-v}| + (\ell_1 - \ell_{1-v}) = \ell_1\}).$$

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Write $\widetilde{b}_v \stackrel{\text{def}}{=} b_{1-v}$, which is again a Brownian bridge, whose local time at zero is $\widetilde{\ell}_v = \ell_1 - \ell_{1-v}$. Thus,

$$\gamma = g(1 - \inf\{v < 1 : |\widetilde{b}_v| + \widetilde{\ell}_v = \widetilde{\ell}_1\}).$$

Define $\widetilde{m}_t \stackrel{\text{def}}{=} |\widetilde{b}_t| + \widetilde{\ell}_t$, which is a Brownian meander. Accordingly,

$$\gamma = g(1 - \widetilde{\theta}), \quad \text{on } \{g > \gamma\}, \tag{2.1}$$

where

$$\widetilde{\theta} \stackrel{\text{def}}{=} \inf\{v \le 1 : \widetilde{m}_v = \widetilde{m}_1\},$$

and $\widetilde{m}$ is a Brownian meander, independent of g and m.

It remains to express the set $\{g > \gamma\}$ in terms of g, m and b (or $\widetilde{m}$). Note that

$$\begin{aligned} g > \gamma &\Longleftrightarrow |B_u| \ne L_u, \quad \text{for all } u \ge g \\ &\Longleftrightarrow |B_u| < L_u = L_g, \quad \text{for all } u \ge g \\ &\Longleftrightarrow \sup_{u \ge g} |B_u| < L_g \\ &\Longleftrightarrow S_m \stackrel{\text{def}}{=} \sup_{s \le 1} m_s < \sqrt{\frac{g}{1-g}}\ell_1, \end{aligned}$$

which means that

$$\{g > \gamma\} = \left\{ S_m < \sqrt{\frac{g}{1-g}}\widetilde{m}_1 \right\}. \tag{2.2}$$

In view of (2.1), we have

$$\mathbb{E}[f(g,\gamma)\mathbb{1}_{\{g>\gamma\}}] = \mathbb{E}[f(g, g(1-\widetilde{\theta}))\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}\widetilde{m}_1\}}], \tag{2.3}$$

where g, S_m and $(\widetilde{\theta}, \widetilde{m}_1)$ are independent. Note that the law of S_m is known ([**3**, p. 69]):

$$S_m^2 \stackrel{\text{law}}{=} T_\pi^{(3)},$$

where $T_a^{(3)}$ is as before the first hitting time at a of a three-dimensional Bessel process starting from zero.

2.2. *Second situation: $g < \gamma$.* Recall Lévy's identity in law:

$$(|B_t|, L_t, t \ge 0) \stackrel{\text{law}}{=} (S_t - B_t, S_t, t \ge 0),$$

where $S_t \stackrel{\text{def}}{=} \sup_{0 \le s \le t} B_s$. It immediately follows that

$$\begin{aligned} (g,\gamma) &\stackrel{\text{law}}{=} (\sup\{t \le 1 : S_t = B_t\}, \sup\{t \le 1 : B_t = 0\}) \\ &= (\rho, g), \end{aligned}$$

where $\rho \stackrel{\text{def}}{=} \sup\{t \le 1 : S_t = B_t\}$. We thus have to determine the law of (ρ, g) on the event $\{\rho < g\}$. Observe that on $\{\rho < g\}$,

$$\begin{aligned} \rho &= \sup\{t \le g : S_t = B_t\} \\ &= g \sup\left\{u \le 1 : \sup_{s \le u} b(s) = b(u)\right\} \\ &= g U_b, \end{aligned}$$

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where $U_b \stackrel{\text{def}}{=} \sup\{t \le 1 : S_b = b(t)\}$ and $S_b \stackrel{\text{def}}{=} \sup_{s \le 1} b(s)$.

It is well known that U_b is uniformly distributed in $(0, 1)$. It remains therefore to study the set $\{\rho < g\}$.

If $B_1 < 0$, then $\rho \le g$ almost surely, i.e. $\{B_1 < 0\} \subset \{\rho < g\}$.

If $B_1 > 0$, we have

$$\begin{aligned} \rho < g &\iff B_u < S_u = S_g, \quad \text{for all } u \ge g \\ &\iff \sup_{u \ge g} |B_u| < S_g \\ &\iff S_m < \sqrt{\frac{g}{1-g}} S_b. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{E}[f(g,\gamma)\mathbb{1}_{\{g>\gamma\}}] &= \mathbb{E}[f(g,\gamma)\mathbb{1}_{\{g>\gamma\}}\mathbb{1}_{\{B_1<0\}}] + \mathbb{E}[f(g,\gamma)\mathbb{1}_{\{g>\gamma\}}\mathbb{1}_{\{B_1>0\}}] \\ &= \tfrac{1}{2}\mathbb{E}[f(gU_b, g)] + \tfrac{1}{2}\mathbb{E}[f(gU_b, g)\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}S_b\}}]. \end{aligned}$$

This, combined with (2.3), completes the proof of Theorem 1.1. □

Remark. According to Theorem 1.1, to study the joint distribution of (g, γ) we need the law of $(\widetilde{\theta}, \widetilde{m}_1)$, which is characterized in the next section. We also need the law of (U_b, S_b), which can be described as follows. Let $(r_t, 0 \le t \le 1)$ denote a standard three-dimensional Bessel bridge, independent of the random variable U which is uniformly distributed in $(0, 1)$. According to Williams' identification, $(r_t, 0 \le t \le 1)$ is also a normalized Brownian excursion process. Using Vervaat's transformation relating the Brownian bridge with the normalized Brownian excursion [**1**], it is easily seen that

$$(U_b, S_b) \stackrel{\text{law}}{=} (1 - U, r_U). \tag{2.4}$$

The joint law of U and r_U is studied in the forthcoming Lemma 5.1 (see §5).

3. *Characterization of the joint law of* $(\widetilde{\theta}, \widetilde{m}_1)$

Let $(\beta_t, t \ge 0)$ be a one-dimensional Brownian motion starting from zero, and let $T_1 \stackrel{\text{def}}{=} \inf\{t > 0 : \beta_t = 1\}$. Define

$$g_{T_1} \stackrel{\text{def}}{=} \sup\{t < T_1 : \beta_t = 0\} \quad \text{and} \quad T^{(3)} = T_1 - g_{T_1}.$$

Recall that $(\beta_{g_{T_1}+t}, t \le T^{(3)})$ is a three-dimensional Bessel process starting from zero, considered until its first hitting time at one (this justifies the notation $T^{(3)}$). Moreover, $(\beta_u, u \le g_{T_1})$ and $(\beta_{g_{T_1}+t}, t \le T^{(3)})$ are independent.

PROPOSITION 3.1. *The joint law of $(\widetilde{\theta}, \widetilde{m}_1)$ is characterized by the following: for any Borel function $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$,*

$$\mathbb{E}(f(\widetilde{\theta}, \widetilde{m}_1)) = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[\frac{1}{\sqrt{T_1}} f\left(\frac{T^{(3)}}{T_1}, \frac{1}{\sqrt{T_1}}\right)\right]. \tag{3.1}$$

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Remark 3.1.1. According to Williams' path decomposition theorem for $(\beta_t, t \leq T_1)$,

$$(T_1, T^{(3)}) \stackrel{\text{law}}{=} (4U^2\widehat{T}_1 + T^{(3)}, T^{(3)}), \tag{3.2}$$

where, on the right-hand side, $\widehat{T}_1$, U and $T^{(3)}$ are independent variables, U being uniformly distributed in $(0, 1)$ and $\widehat{T}_1 \stackrel{\text{law}}{=} T_1$.

Proof of Proposition 3.1. We recall the following results (see also [**2**]).

(a) Imhof's relation:

$$\mathbb{E}(F(\widetilde{m}_u, u \leq 1)) = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[\frac{1}{R_1}F(R_u, u \leq 1)\right];$$

(b) absolute continuity between $(R_u, u \leq 1)$ and $(Z_u \stackrel{\text{def}}{=} R_{u\mathbb{L}_1}/\sqrt{\mathbb{L}_1}, u \leq 1)$, where $\mathbb{L}_1$ is the last exit time from one of R (three-dimensional Bessel process starting from zero):

$$\mathbb{E}(F(Z_u, u \leq 1)) = \mathbb{E}\left[\frac{1}{R_1^2}F(R_u, u \leq 1)\right];$$

(c) Williams' time reversal for R:

$$(R_u, u \leq \mathbb{L}_1) \stackrel{\text{law}}{=} (1 - \beta_{T_1 - u}, u \leq T_1).$$

Combining (a)–(c) yields

$$\mathbb{E}(F(\widetilde{m}_u, u \leq 1)) = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[\frac{1}{\sqrt{T_1}}F\left(\frac{1 - \beta_{T_1(1-u)}}{\sqrt{T_1}}, u \leq 1\right)\right]$$

or, equivalently,

$$\mathbb{E}(F(\widetilde{m}_1 - \widetilde{m}_{1-u}, u \leq 1)) = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[\frac{1}{\sqrt{T_1}}F\left(\frac{\beta_{T_1(1-u)}}{\sqrt{T_1}}, u \leq 1\right)\right].$$

This immediately yields the proposition. □

4. *Probabilistic discussions on analytical formulae*

4.1. We first discuss formula (1.2). To begin with, we transform the left-hand side of (1.2): as is well known,

$$\frac{x}{\sinh x} = \mathbb{E}\left[\exp\left(-\frac{x^2}{2}T_1^{(3)}\right)\right],$$

where $T_1^{(3)} = \inf\{t > 0 : R_t^{(3)} = 1\}$ and $R^{(3)}$ is a three-dimensional Bessel process, starting from zero. Hence, we have

$$\mathbb{E}\left[\frac{C_a}{\sinh(C_a)}\right] = \mathbb{E}\left[\exp\left(-\frac{a^2}{2}(C_1^2 T_1^{(3)})\right)\right].$$

Now, we use

$$C_1^2 \stackrel{\text{law}}{=} \frac{N^2}{(N')^2} \stackrel{\text{law}}{=} N^2 T_1$$

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where N and N' are two standard independent Gaussian variables, and $T_1 \overset{\text{def}}{=} \inf\{t > 0 : B_t = 1\}$ is assumed to be independent of N. Thus, we obtain

$$\begin{aligned}
\mathbb{E}\left[\frac{C_a}{\sinh(C_a)}\right] &= \mathbb{E}\left[\exp\left(-a|N|\sqrt{T_1^{(3)}}\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\frac{a|N|}{S_1^{(3)}}\right)\right] \quad \left(S_1^{(3)} \overset{\text{def}}{=} \sup_{s\le 1} R_s^{(3)}\right) \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathbb{E}\left[S_1^{(3)} \exp\left(-\frac{y^2 (S_1^{(3)})^2}{2}\right)\right] \exp(-ay)\,dy,
\end{aligned}$$

after some elementary change of variables.

Inspection of the right-hand side of (1.2) shows that it is the Laplace transform, with respect to the argument a, of the function

$$\frac{\pi/2}{(\cosh(\pi y/2))^2}.$$

Consequently, from the injectivity of the Laplace transform, we deduce that (1.2) is equivalent to

$$\sqrt{\frac{2}{\pi}}\mathbb{E}\left[S_1^{(3)} \exp\left(-\frac{y^2 (S_1^{(3)})^2}{2}\right)\right] = \frac{\pi/2}{(\cosh(\pi y/2))^2}. \tag{4.1}$$

Multiplying on both sides of (4.1) by y^a and integrating with respect to y over $\mathbb{R}_+$, we can determine all the positive moments of $T_1^{(3)}$: for $a > 0$,

$$\mathbb{E}[(T_1^{(3)})^a] = \frac{2^a \pi^{-2a+1/2}}{\Gamma(a+1/2)} \int_0^\infty \frac{x^{2a}}{(\cosh x)^2}\,dx. \tag{4.1$'$}$$

(See also Pitman and Yor [**13**].) We note that the negative moments of $T_1^{(3)}$, which are calculated in Yor [**18**, Ch. XI], are in formal agreement with (4.1′); see also, for similar computations related to the Riemann zeta function in terms of the sum $T_1^{(3)} + \widehat{T}_1^{(3)}$ of two independent copies of $T_1^{(3)}$, Williams [**16**, p. 369]. The integral on the right-hand side of (4.1′) may be explicitly computed in terms of the gamma and Riemann zeta functions (see Gradshteyn and Ryzhik [**7**, p. 352]); we obtain: for $a > 0$,

$$\mathbb{E}[(T_1^{(3)})^a] = \begin{cases} 2^{a+1}\pi^{-2a}(1-2^{1-2a})\Gamma(a+1)\zeta(2a), & \text{if } a \neq 1/2, \\ \sqrt{2/\pi}\ln 2, & \text{if } a = 1/2. \end{cases}$$

We now give two confirmations of (4.1).

(i) Recall that the Fourier transform (on $\mathbb{R}$) of the function

$$y \mapsto \frac{\pi/4}{(\cosh(\pi y/2))^2}$$

is $\xi/\sinh\xi$, see e.g. the table in Biane and Yor [**3**]. Thus, (4.1) is equivalent to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty dy \exp(i\xi y)\mathbb{E}\left[S_1^{(3)} \exp\left(-\frac{y^2 (S_1^{(3)})^2}{2}\right)\right] = \frac{\xi}{\sinh\xi},$$

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and it is now easily shown that the left-hand side is equal to

$$\mathbb{E}\left[\exp\left(-\frac{\xi^2}{2}T_1^{(3)}\right)\right],$$

which as we have already recalled is equal to $\xi/\sinh\xi$.

(ii) Our second confirmation of (4.1) consists in showing that it is equivalent to the well-known Kolmogorov–Smirnov law:

$$\sup_{u\leq 1}|b(u)|^2 \stackrel{\text{law}}{=} T_{\pi/2}^{(3)}. \tag{4.2}$$

To show this, we use a particular case of the following.

THEOREM A. (Agreement formula; Biane and Yor [**3**], Pitman and Yor [**12**]) *Let R and $\widehat{R}$ be two independent Bessel processes of dimension δ, starting from zero, T and $\widehat{T}$ their first hitting times of one. Define $\widetilde{R}$ by connecting the paths of R on $[0, T]$ and $\widehat{R}$ on $[0, \widehat{T}]$ back to back:*

$$\widetilde{R}_t \stackrel{\text{def}}{=} \begin{cases} R_t, & \text{if } t \leq T, \\ \widehat{R}_{T+\widehat{T}-t}, & \text{if } T \leq t \leq T+\widehat{T}, \end{cases}$$

and let $\widetilde{R}^{\text{br}}$ be obtained by Brownian scaling of $\widetilde{R}$ onto the time scale $[0, 1]$:

$$\widetilde{R}_u^{\text{br}} \stackrel{\text{def}}{=} \frac{\widetilde{R}_{u(T+\widehat{T})}}{\sqrt{T+\widehat{T}}}, \quad 0 \leq u \leq 1.$$

Let R^{br} be a standard Bessel bridge of dimension δ. Then for all positive or bounded measurable functions $F : C[0, 1] \to \mathbb{R}$,

$$\mathbb{E}[F(R^{\text{br}})] = c_\delta \mathbb{E}[F(\widetilde{R}^{\text{br}})(\widetilde{M}^{\text{br}})^{2-\delta}],$$

where

$$\widetilde{M}^{\text{br}} \stackrel{\text{def}}{=} \sup_{0\leq u\leq 1} \widetilde{R}_u^{\text{br}} = (T+\widehat{T})^{-1/2},$$

$$c_\delta \stackrel{\text{def}}{=} 2^{(\delta-2)/2}\Gamma(\delta/2).$$

Remark. A detailed study of the law of $\sup_{0\leq u\leq 1} R_u^{\text{br}}$ is made in Pitman and Yor [**13**] with the help of the agreement formula.

We now consider the particular case $\delta = 1$; we obtain the following relationship between $\sup_{u\leq 1}|b(u)|$ and $T+\widehat{T}$:

$$\sqrt{\frac{2}{\pi}}\mathbb{E}\left[f\left(\sup_{u\leq 1}|b(u)|\right)\right] = \mathbb{E}[(T+\widehat{T})^{-1/2} f((T+\widehat{T})^{-1/2})]. \tag{4.3}$$

Now, on the one hand, we have

$$\mathbb{E}\left[\exp\left(-\frac{y^2}{2}(T+\widehat{T})\right)\right] = \frac{1}{(\cosh y)^2}$$

and, on the other hand, $S_1^{(3)} \stackrel{\text{law}}{=} 1/\sqrt{T_1^{(3)}}$; together with these remarks, the comparison of (4.1) and (4.3) yields the well-known Kolmogorov–Smirnov identity (4.2).

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4.2. For the reader's convenience, we now present a few formulae and identities about the laws of the suprema of the Brownian bridge and meander processes (see Biane and Yor [**3**], Chung [**4**], Kennedy [**10**]):

$$S_m^2 \overset{\text{law}}{=} 4 \sup_{u \le 1} |b(u)|^2 \overset{\text{law}}{=} T_\pi^{(3)}, \tag{4.4}$$

$$\mathbb{P}(S_m < x) = \mathbb{P}\left(\sup_{u \le 1} |b(u)| < \frac{x}{2} \right) = \sum_{n=-\infty}^{\infty} (-1)^n \exp\left(-\frac{n^2 x^2}{2} \right) \tag{4.5}$$

$$= \frac{\sqrt{8\pi}}{x} \sum_{k=1}^{\infty} \exp\left(-\frac{(2k-1)^2 \pi^2}{2x^2} \right), \quad x > 0. \tag{4.6}$$

The next item is a collection of a few analytical formulae. From a probabilistic point of view, they ensue from (1.3). Note that (4.7) is a particular case of (1.4)†:

$$\int_{-\infty}^{\infty} \frac{b}{b^2 + x^2} \frac{x}{\sinh(\pi x)} \, dx = 1 + 2b \sum_{k=1}^{\infty} \frac{(-1)^k}{k+b}, \tag{4.7}$$

$$\int_{-\infty}^{\infty} \frac{b}{b^2 + x^2} \frac{1}{\cosh(\pi x)} \, dx = 4 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1) + 2b}, \tag{4.8}$$

$$\int_{-\infty}^{\infty} \frac{b}{b^2 + x^2} \frac{1}{(\cosh(\pi x))^2} \, dx = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{((2k-1) + 2b)^2}, \tag{4.9}$$

$$\int_{-\infty}^{\infty} \frac{b}{b^2 + x^2} \frac{\tanh(\pi x)}{x} \, dx = 8 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)((2k-1) + 2b)}. \tag{4.10}$$

Again, formulae (4.7)–(4.10) can be interpreted in a probabilistic way. They respectively lead to the following identities, where (4.11) is a rewriting of (4.1). We write $S_t^{(1)} \overset{\text{def}}{=} \sup_{0 \le s \le t} |B_s|$, $d_1 \overset{\text{def}}{=} \inf\{t > 1 : B_t = 0\}$ and $S_1^{(\mathrm{R})} \overset{\text{def}}{=} \sup_{0 \le s \le 1} B_s - \inf_{0 \le s \le 1} B_s$ ('R' for 'Range'). For $y \in \mathbb{R}$,

$$\sqrt{\frac{2}{\pi}} \mathbb{E}\left[S_1^{(3)} \exp\left(-\frac{y^2 (S_1^{(3)})^2}{2} \right) \right] = \frac{\pi/2}{(\cosh(\pi y/2))^2}, \tag{4.11}$$

$$\sqrt{\frac{2}{\pi}} \mathbb{E}\left[S_1^{(1)} \exp\left(-\frac{y^2 (S_1^{(1)})^2}{2} \right) \right] = \frac{1}{\cosh(\pi y/2)}, \tag{4.12}$$

$$\sqrt{\frac{2}{\pi}} \mathbb{E}\left[S_1^{(\mathrm{R})} \exp\left(-\frac{y^2 (S_1^{(\mathrm{R})})^2}{2} \right) \right] = \frac{\pi y/2}{\sinh(\pi y/2)}, \tag{4.13}$$

$$\sqrt{\frac{2}{\pi}} \mathbb{E}\left[S_{d_1}^{(1)} \exp\left(-\frac{y^2 (S_{d_1}^{(1)})^2}{2} \right) \right] = \frac{1}{\pi} \log\left(\coth\left(\frac{\pi |y|}{4} \right) \right). \tag{4.14}$$

Observe that by the above agreement formula (Theorem A, with $\delta = 1$), (4.13) can be rewritten as

$$\mathbb{E}\left[\exp\left(-\frac{y^2}{2} \sup_{u \le 1} b^2(u) \right) \right] = \frac{\pi y/2}{\sinh(\pi y/2)},$$

which is a re-confirmation of (4.2).

† For some enjoyable discussion about (4.7)–(4.9) of a purely analytical nature, see [**19**].

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Remark. The reader will find in Jurek [**9**] a useful discussion of the right-hand sides of (4.11)–(4.14), from the point of view of infinite divisibility.

5. *Proof of Theorem 1.2*

Applying Theorem 1.1 to the function $f(u, v) = uv$ yields

$$\begin{aligned}\mathbb{E}(g\gamma) &= \tfrac{1}{2}\mathbb{E}(g^2) + \tfrac{1}{2}\mathbb{E}[g^2 U_b \mathbb{1}_{\{S_m < \sqrt{g/(1-g)}S_b\}}] + \mathbb{E}[g^2(1-\widetilde{\theta})\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}\widetilde{m}_1\}}] \\ &\stackrel{\text{def}}{=} I_1 + I_2 + I_3,\end{aligned}$$

with obvious notation. Since g has the arcsine law, it immediately follows that

$$I_1 = \tfrac{3}{32}. \tag{5.1}$$

The computation of I_2 is based on the following lemma. In the rest of the section, $(r_t, 0 \le t \le 1)$ denotes a standard three-dimensional Bessel bridge, independent of the random variable U which is uniformly distributed in $(0, 1)$.

LEMMA 5.1. *For any $a > 0$, both quantities $\mathbb{E}[(1-U)\mathbb{1}_{\{r_U > a\}}]$ and $\mathbb{E}[U\mathbb{1}_{\{r_U > a\}}]$ are equal to $e^{-2a^2}/2$. Consequently, $\mathbb{P}(r_U > a) = e^{-2a^2}$.*

Proof. Recall that $(r_t, t < 1)$ can be realized as $((1-t)R_{t/(1-t)}, t < 1)$, where R is a three-dimensional Bessel process starting from zero. Accordingly,

$$(U, r_U^2) \stackrel{\text{law}}{=} (U, U(1-U)R_1^2).$$

A few lines of elementary computations, together with the fact that R_1^2 is a chi-square variable, yield $\mathbb{E}[(1-U)\mathbb{1}_{\{r_U > a\}}] = e^{-2a^2}/2$.

On the other hand, by the symmetry of the Bessel bridge, $(1-U, r_U) \stackrel{\text{law}}{=} (U, r_U)$, which completes the proof of the lemma. □

We now evaluate I_2. By (2.4) and Lemma 5.1,

$$\begin{aligned} I_2 &= \frac{1}{2}\mathbb{E}[g^2(1-U)\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}r_U\}}] \\ &= \frac{1}{4}\mathbb{E}\left[g^2 \exp\left(-\frac{2(1-g)}{g}S_m^2\right)\right].\end{aligned}$$

Using the exact Laplace transform of S_m^2 (taking into account the identity in law (4.5)), we obtain

$$\begin{aligned} I_2 &= \frac{1}{4}\mathbb{E}\left[g^2 \frac{2\sqrt{(1-g)/g}\,\pi}{\sinh(2\sqrt{(1-g)/g}\,\pi)}\right] \\ &= \int_0^\infty \frac{dx}{(1+x^2)^3}\frac{x}{\sinh(2\pi x)},\end{aligned}$$

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the second equality following from (1.3). On the other hand, differentiating on both sides of equation (1.4) gives, for $a > 0$ and $b > 0$,

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{(b^2+x^2)^2}\frac{x}{\sinh(ax)} = \frac{1}{4ab^3} + \frac{a}{2b}\sum_{k=1}^\infty \frac{(-1)^k}{(ab+k\pi)^2}, \tag{5.2}$$

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{(b^2+x^2)^3}\frac{x}{\sinh(ax)} = \frac{3}{16ab^5} + \frac{a}{8b^3}\sum_{k=1}^\infty \frac{(-1)^k}{(ab+k\pi)^2} + \frac{a^2}{4b^2}\sum_{k=1}^\infty \frac{(-1)^k}{(ab+k\pi)^3}. \tag{5.3}$$

Applying (5.3) to $a = 2\pi$ and $b = 1$ yields

$$I_2 = \tfrac{37}{32} - \tfrac{1}{48}\pi^2 - \tfrac{3}{4}\zeta(3). \tag{5.4}$$

It remains to compute I_3. Recall the distribution function of S_m from (4.6). By conditioning on $(g, \widetilde{\theta}, \widetilde{m}_1)$, we obtain

$$I_3 = \sqrt{8\pi}\,\mathbb{E}\left[g^2\sum_{k=1}^\infty I_4\left(\frac{(2k-1)^2\pi^2}{2}\frac{1-g}{g}\right)\right], \tag{5.5}$$

where

$$I_4(\lambda) \stackrel{\text{def}}{=} \mathbb{E}\left[\frac{1-\widetilde{\theta}}{\widetilde{m}_1}\exp\left(-\frac{\lambda}{\widetilde{m}_1^2}\right)\right], \quad \lambda > 0.$$

Note that, by means of Proposition 3.1, the derivative of $\lambda \mapsto I_4(\lambda)$ is

$$\begin{aligned} I_4'(\lambda) &= -\mathbb{E}\left[\frac{1-\widetilde{\theta}}{\widetilde{m}_1^3}\exp\left(-\frac{\lambda}{\widetilde{m}_1^2}\right)\right] \\ &= -\sqrt{\frac{\pi}{2}}\,\mathbb{E}[(4U^2\widehat{T}_1)\exp(-\lambda(4U^2\widehat{T}_1 + T^{(3)}))]. \end{aligned}$$

Since $\mathbb{E}e^{-a\widehat{T}_1} = \exp(-\sqrt{2a})$ and $\mathbb{E}e^{-aT^{(3)}} = \sqrt{2a}/\sinh\sqrt{2a}$ for all $a > 0$, this yields

$$I_4'(\lambda) = -\sqrt{\frac{\pi}{2}}\left(\frac{e^{-\sqrt{2\lambda}}}{2\lambda} - \frac{e^{-2\sqrt{2\lambda}}}{\sqrt{2\lambda}\sinh\sqrt{2\lambda}}\right).$$

Going back to (5.5), we have

$$I_3 = -2\pi\,\mathbb{E}\left[g^2\int_{(2k-1)\pi\sqrt{(1-g)/g}}^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-2x}}{\sinh x}\right)dx\right].$$

From here, a few lines of elementary computations based on (5.2) and (5.3), together with the fact that g has the arcsine law, readily yield the following:

$$I_3 = -\tfrac{9}{8} + \tfrac{1}{16}\pi^2 + \tfrac{17}{32}\zeta(3). \tag{5.6}$$

Since $\mathbb{E}(g\gamma) = I_1 + I_2 + I_3$, assembling (5.1), (5.4) and (5.6) completes the proof of Theorem 1.2. $\square$

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6. *Martingale representations*

The covariance between g and γ may be expressed as the expectation of the covariation between the martingales with final values g and γ. Thus, it is natural to look for the explicit Itô–Clark representation of g (hence, of γ) as a stochastic integral.

Let, for $0 < t \leq 1$,

$$g_t \stackrel{\text{def}}{=} \sup\{s \leq t : B_s = 0\},$$
$$\gamma_t \stackrel{\text{def}}{=} \sup\{s \leq t : |B_s| = L_s\},$$

(thus $g = g_1$ and $\gamma = \gamma_1$). Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of B. For any bounded Borel function φ, and $t \in (0, 1)$,

$$\mathbb{E}(\varphi(g) \mid \mathcal{F}_t) = \varphi(g_t)\mathbb{P}(g \leq t \mid \mathcal{F}_t) + \mathbb{E}(\varphi(g)\mathbb{1}_{\{g>t\}} \mid \mathcal{F}_t). \tag{6.1}$$

According to Jeulin [**8**] (see also Yor [**18**, p. 42]),

$$\mathbb{P}(g \leq t \mid \mathcal{F}_t) = \Psi\left(\frac{|B_t|}{\sqrt{1-t}}\right), \tag{6.2}$$

where

$$\Psi(x) \stackrel{\text{def}}{=} \sqrt{\frac{2}{\pi}} \int_0^x e^{-y^2/2}\, dy. \tag{6.3}$$

On the other hand, $\{g > t\} = \{\inf_{t\leq u\leq 1} |B_u| = 0\}$. On $\{g > t\}$, g is identical to $\sup\{u \in (t, 1) : B_u = 0\}$. By means of the (strong) Markov property, it is easily checked that

$$\mathbb{E}(\varphi(g)\mathbb{1}_{\{g>t\}} \mid \mathcal{F}_t) = \sqrt{\frac{2}{\pi}} \int_t^1 \frac{\varphi(u)}{\sqrt{2\pi(u-t)(1-u)}} \exp\left(-\frac{B_t^2}{2(u-t)}\right) du. \tag{6.4}$$

We mention that the constant $\sqrt{2/\pi}$ on the right-hand side of (6.4) comes from $1/\mathbb{E}(m_1)$.

In the particular case $\varphi(x) = x$, we have the following.

PROPOSITION 6.1. *Define the martingales* $M_t^{(g)} \stackrel{\text{def}}{=} \mathbb{E}(g \mid \mathcal{F}_t)$, $M_t^{(\gamma)} \stackrel{\text{def}}{=} \mathbb{E}(\gamma \mid \mathcal{F}_t)$, *and the Brownian motion* $\beta_t \stackrel{\text{def}}{=} \int_0^t \operatorname{sgn}(B_s)\, dB_s$. *Then*

$$M_t^{(g)} = \mathbb{E}(g) + \int_0^t \mu_s^{(g)}\, dB_s, \tag{6.5}$$

$$M_t^{(\gamma)} = \mathbb{E}(\gamma) + \int_0^t \mu_s^{(\gamma)} \operatorname{sgn}(B_s)\, dB_s. \tag{6.6}$$

Here, $\mathbb{E}(g) = \mathbb{E}(\gamma) = 1/2$, *and*

$$\mu_s^{(g)} \stackrel{\text{def}}{=} -\sqrt{\frac{2}{\pi}} \frac{(s-g_s)\operatorname{sgn}(B_s)}{\sqrt{1-s}} \exp\left(-\frac{B_s^2}{2(1-s)}\right) - \left(1 - \Psi\left(\frac{|B_s|}{\sqrt{1-s}}\right)\right) B_s, \tag{6.7}$$

$$\mu_s^{(\gamma)} \stackrel{\text{def}}{=} -\sqrt{\frac{2}{\pi}} \frac{(s-\gamma_s)\operatorname{sgn}(\beta_s)}{\sqrt{1-s}} \exp\left(-\frac{\beta_s^2}{2(1-s)}\right) - \left(1 - \Psi\left(\frac{|\beta_s|}{\sqrt{1-s}}\right)\right) \beta_s, \tag{6.8}$$

where Ψ *is the function defined in (6.3). Consequently, we obtain*

$$\operatorname{cov}(g, \gamma) = \mathbb{E}\left(\int_0^1 \mu_s^{(g)} \mu_s^{(\gamma)} \operatorname{sgn}(B_s)\, ds\right). \tag{6.9}$$

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Proof. By (6.1),

$$dM_t^{(g)} = g_t\, dZ_t + Z_t\, dg_t + dX_t,$$

where

$$Z_t \stackrel{\text{def}}{=} \Psi\left(\frac{|B_t|}{\sqrt{1-t}}\right),$$
$$X_t \stackrel{\text{def}}{=} \mathbb{E}(g\mathbb{1}_{\{g>t\}} \mid \mathcal{F}_t),$$

and Ψ is the function defined in (6.3). In view of (6.4), we have

$$X_t = H(t, B_t),$$

where

$$H(t,x) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_t^1 \frac{u}{\sqrt{(u-t)(1-u)}} \exp\left(-\frac{x^2}{2(u-t)}\right) du.$$

From the formula

$$\frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{v(1-v)}} \exp\left(-\frac{x^2}{2v}\right) dv = \sqrt{\frac{2}{\pi}}\int_x^\infty e^{-y^2/2}\, dy, \tag{6.10}$$

we easily deduce that

$$H(t,x) = t\left(1 - \Psi\left(\frac{|x|}{\sqrt{1-t}}\right)\right) + (1-t)h\left(\frac{|x|}{\sqrt{1-t}}\right), \tag{6.11}$$

with

$$h(y) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^1 \frac{\sqrt{v}}{\sqrt{1-v}} \exp\left(-\frac{y^2}{2v}\right) dv.$$

Applying Itô's formula to the semimartingales Z_t and $X_t = H(t, B_t)$ (with H in the form of (6.11)) readily yields (6.7).

To check (6.8), we rewrite (6.7) as $\mu_s^{(g)} \stackrel{\text{def}}{=} F(B_u, u \le s)$. We can apply the same argument to the Brownian motion $\beta_t = \int_0^t \operatorname{sgn}(B_s)\, dB_s = |B_t| - L_t$, to see that

$$M_t^{(\gamma)} = \mathbb{E}(\gamma) + \int_0^t \mu_s^{(\gamma)}\, d\beta_s = \frac{1}{2} + \int_0^t \mu_s^{(\gamma)} \operatorname{sgn}(B_s)\, dB_s,$$

where $\mu_s^{(\gamma)} = F(\beta_u, u \le s)$. Using the form of F in (6.7) yields (6.11). This completes the proof of the proposition. □

Remark 6.1.1. In general, for any bounded Borel function φ, we can evaluate the martingale representation of $M_t^{(\varphi,g)} \stackrel{\text{def}}{=} \mathbb{E}(\varphi(g) \mid \mathcal{F}_t)$ by means of (6.1), (6.2) and (6.4). Indeed, instead of (6.5), we have

$$M_t^{(\varphi,g)} = \mathbb{E}(\varphi(g)) + \int_0^t \mu_s^{(\varphi,g)}\, dB_s,$$

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where

$$\mu_s^{(\varphi,g)} \stackrel{\text{def}}{=} -\sqrt{\frac{2}{\pi}}\frac{(\varphi(s)-\varphi(g_s))\,\mathrm{sgn}(B_s)}{\sqrt{1-s}}\exp\left(-\frac{B_s^2}{2(1-s)}\right) - \frac{B_s}{\pi(1-s)}\int_0^1 \frac{\varphi(s+(1-s)u)-\varphi(s)}{u^{3/2}(1-u)^{1/2}}\exp\left(-\frac{B_s^2}{2(1-s)u}\right)du.$$

The martingale representation of $\mathbb{E}(\varphi(\gamma) \mid \mathcal{F}_t)$ is derived directly from the preceding formulae. Consequently, we obtain an analytical formula for $\mathrm{cov}(\varphi(g), \varphi(\gamma))$ which generalizes (6.9) above.

Remark 6.1.2. Formula (6.10) may be seen as an analytical application of the identity in law,

$$N^2 \stackrel{\text{law}}{=} 2\mathcal{E}g,$$

where $\mathcal{E}$ denotes an exponential variable with mean one, independent of g, and N is as before a Gaussian $\mathcal{N}(0,1)$ variable. For further discussions on (6.10) and related formulae, see Yor [**17**].

7. *Moments of* (θ, m_1)

We determine the joint moments of θ and m_1.

PROPOSITION 7.1. *For* $(p,q) \in \mathbb{N}^2$,

$$\mathbb{E}(\theta^p m_1^q) = \frac{\pi^{1/2}}{\Gamma((2p+q+1)/2)2^{(2p+q+2)/2}}\int_0^\infty \lambda^{2p+q-1}(1-e^{-2\lambda})f_p(\lambda)\,d\lambda, \quad (7.1)$$

where

$$f_p(\lambda) = \mathbb{E}\left[(T^{(3)})^p \exp\left(-\frac{\lambda^2}{2}T^{(3)}\right)\right].$$

Moreover, f_p *satisfies the recurrence relation:*

$$\begin{cases} -\lambda f_{p+1}(\lambda) = f_p'(\lambda), \\ f_0(\lambda) = \lambda/\sinh(\lambda). \end{cases}$$

When $p=1$, *(7.1) becomes*

$$\mathbb{E}(\theta m_1^q) = \frac{\pi^{1/2}\Gamma(q+1)}{\Gamma((q+3)/2)2^{(q+2)/2}}((q+1)(2-2^{-(q+1)})\zeta(q+2)-(q+2)), \quad (7.2)$$

where $\zeta(\cdot)$ *is as before the Riemann zeta function. In particular,*

$$\mathbb{E}(\theta) = \tfrac{1}{4}\pi^2 - 2.$$

Proof. It is an immediate consequence of Proposition 3.1 that

$$\mathbb{E}(\theta^p m_1^q) = \sqrt{\frac{\pi}{2}}\mathbb{E}((T^{(3)})^p T_1^{-(2p+q+1)/2}).$$

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Since

$$x^{-(2p+q+1)/2} = \frac{1}{\Gamma((2p+q+1)/2)} \int_0^\infty e^{-tx} t^{(2p+q-1)/2}\, dt$$
$$= \frac{1}{\Gamma((2p+q+1)/2)2^{(2p+q-1)/2}} \int_0^\infty e^{-\lambda^2 x/2} \lambda^{2p+q}\, d\lambda,$$

it follows from (3.2) that (writing $a(p,q) \stackrel{\text{def}}{=} \pi^{1/2}/\,\Gamma((2p+q+1)/2)2^{(2p+q)/2}$)

$$\mathbb{E}(\theta^p m_1^q) = a(p,q) \int_0^\infty \lambda^{2p+q} \mathbb{E}\left((T^{(3)})^p \exp\left(-\frac{\lambda^2}{2} T^{(3)}\right)\right) \mathbb{E}\left(\exp\left(-\frac{\lambda^2}{2} 4U^2 T_1\right)\right) d\lambda$$
$$= \frac{a(p,q)}{2} \int_0^\infty \lambda^{2p+q-1} \mathbb{E}\left((T^{(3)})^p \exp\left(-\frac{\lambda^2}{2} T^{(3)}\right)\right) (1-e^{-2\lambda})\, d\lambda,$$

proving (7.1).

To check (7.2), note that by the relation between f_0 and f_1,

$$\mathbb{E}(\theta m_1^q) = \frac{a(1,q)}{2} \int_0^\infty \lambda^q (1-e^{-2\lambda}) f_0'(\lambda)\, d\lambda$$
$$= -\frac{a(1,q)}{2} \int_0^\infty \frac{\lambda}{\sinh \lambda} (q\lambda^{q-1}(1-e^{-2\lambda}) + 2\lambda^q e^{-2\lambda})\, d\lambda$$
$$= a(1,q) \int_0^\infty \frac{e^{-\lambda}}{1-e^{-2\lambda}} (q\lambda^q (1-e^{-2\lambda}) + 2\lambda^{q+1} e^{-2\lambda})\, d\lambda$$
$$= a(1,q)\left(q\Gamma(q+1) + 2\sum_{n=0}^\infty \int_0^\infty \lambda^{q+1} e^{-(3+2n)\lambda}\, d\lambda\right)$$
$$= a(1,q)\left(q\Gamma(q+1) + 2\Gamma(q+2) \sum_{n=0}^\infty \frac{1}{(3+2n)^{q+2}}\right).$$

Since

$$\sum_{n=0}^\infty \frac{1}{(3+2n)^{q+2}} = \zeta(q+2) - 1 - 2^{-(q+2)}\zeta(q+2)$$
$$= (1-2^{-(q+2)})\zeta(q+2) - 1,$$

this yields (7.2), and thus completes the proof of the proposition. □

8. *Some related computations*

8.1. *The probability of $g > \gamma$.* Recall that m_1 is Rayleigh distributed, i.e. $\mathbb{P}(m_1 \in dx) = xe^{-x^2/2} \mathbb{1}_{\{x>0\}}\, dx$. By (2.2),

$$\mathbb{P}(g > \gamma) = \mathbb{P}\left(S_m < \frac{\sqrt{g}}{\sqrt{1-g}} \tilde{m}_1\right)$$
$$= \mathbb{E}\left[\exp\left(-\frac{1}{2}\frac{g}{1-g} S_m^2\right)\right]$$
$$= \mathbb{E}\left(\frac{\pi\sqrt{g/(1-g)}}{\sinh(\pi\sqrt{g/(1-g)})}\right).$$

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In view of (1.3), this implies

$$\mathbb{P}(g > \gamma) = 2\int_0^\infty \frac{y}{\sinh(\pi y)} \frac{dy}{1+y^2} = 2\ln 2 - 1.$$

8.2. *The expectation of* $\max(g, \gamma)$. According to Theorem 1.1,

$$\mathbb{E}(\max(g,\gamma)) = \tfrac{1}{2}\mathbb{E}(g) + \tfrac{1}{2}\mathbb{E}(g\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}S_b\}}) + \mathbb{E}(g\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}\widetilde{m}_1\}}).$$

Since $S_b \overset{\text{law}}{=} r_U \overset{\text{law}}{=} \frac{1}{2}\widetilde{m}_1$, and since $\widetilde{m}_1$ is Rayleigh distributed, we obtain

$$\mathbb{E}(g\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}S_b\}}) = \mathbb{E}\left(g\frac{2\pi\sqrt{(1-g)/g}}{\sinh(2\pi\sqrt{(1-g)/g})}\right),$$

$$\mathbb{E}(g\mathbb{1}_{\{S_m < \sqrt{g/(1-g)}\widetilde{m}_1\}}) = \mathbb{E}\left(g\frac{\pi\sqrt{(1-g)/g}}{\sinh(\pi\sqrt{(1-g)/g})}\right).$$

Again, by making use of (1.3), we arrive easily at

$$\mathbb{E}(\max(g,\gamma)) = \tfrac{3}{2} - \tfrac{1}{12}\pi^2.$$

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