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Fast and slow points of Birkhoff sums

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Abstract. We investigate the growth rate of the Birkhoff sums $S_{n,\alpha} f(x) = \sum_{k=0}^{n-1} f(x + k\alpha)$, where f is a continuous function with zero mean defined on the unit circle $\mathbb{T}$ and (α, x) is a 'typical' element of $\mathbb{T}^2$. The answer depends on the meaning given to the word 'typical'. Part of the work will be done in a more general context.

Key words: Birkhoff sum, typical/generic properties, group rotation, coboundary, law of iterated logarithm

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1. *Introduction*

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the unit circle and let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be irrational. Denote by $\mathcal{C}_0(\mathbb{T})$ the set of continuous functions on $\mathbb{T}$ with zero mean, and by $S_{n,\alpha} f(x)$ the nth Birkhoff sum, $S_{n,\alpha} f(x) = \sum_{k=0}^{n-1} f(x + k\alpha)$. The rotation $R_\alpha : x \mapsto x + \alpha$ defines a uniquely ergodic transformation on $\mathbb{T}$ with respect to the (normalized) Lebesgue measure λ. Hence for all $f \in \mathcal{C}_0(\mathbb{T})$ we know that $S_{n,\alpha} f(x) = o(n)$ for all $x \in \mathbb{T}$. There are many possible ways to make this result more precise. We can, for example, fix x and ask for the size of $S_{n,\alpha} f(x)$ for a large proportion of $n \leq N$. We can also fix n and ask for the size of $S_{n,\alpha} f(x)$ for a large set of initial conditions x (see, for example, [**2**, **3**, **5**, **11**, **13**, **14**] for results in this direction).

Our purpose in this paper is quite different. We want to investigate the typical growth of $S_{n,\alpha} f(x)$. There are several ways to understand this problem. We can fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ (respectively, $x \in \mathbb{T}$) and ask for the behaviour of $S_{n,\alpha} f(x)$ for f in a generic subset of $\mathcal{C}_0(\mathbb{T})$ and for a typical $x \in \mathbb{T}$ (respectively, for a typical $\alpha \in \mathbb{T}$). We can also consider it as a problem in two variables and ask for the behaviour of $S_{n,\alpha} f(x)$ for f in a generic subset of $\mathcal{C}_0(\mathbb{T})$ and for a typical $(\alpha, x) \in \mathbb{T}^2$. There are also several ways to understand the word 'typical'. We can look for a residual set of the parameter space or for a set of full Lebesgue measure.

We shall try to put this in a general context. If we fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then we consider the Birkhoff sums associated to a uniquely ergodic transformation on the compact metric space $\mathbb{T}$. Hence, let us fix Ω an infinite compact metric space and $T : \Omega \to \Omega$ an invertible continuous map such that T is uniquely ergodic. Let μ be the ergodic measure, which is regular and continuous. We will also assume that it has full support (equivalently, that all orbits of T are dense). For $x \in \Omega$ and $f \in \mathcal{C}_0(\Omega)$, the Birkhoff sum $S_{n,T} f(x)$ is now defined by $\sum_{k=0}^{n-1} f(T^k x)$. Using $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n) = o(n)$ for $f \in \mathcal{C}_0(\Omega)$, let us define

$$\mathcal{E}_\psi(f) = \left\{x \in \Omega;\ \limsup_n \frac{|S_{n,T} f(x)|}{\psi(n)} = +\infty\right\}.$$

The set $\mathcal{E}_\psi(f)$ has already been studied by several authors. In particular, it was shown by Krengel [**15**] (when $\Omega = [0, 1]$) and later by Liardet and Volný [**17**] that, for all functions f in a residual subset of $\mathcal{C}_0(\Omega)$, $\mu(\mathcal{E}_\psi(f)) = 1$. We complete this result by showing that $\mathcal{E}_\psi(f)$ is also residual.

THEOREM 1.1. *Suppose that $\psi : \mathbb{N} \to \mathbb{N}$ satisfies $\psi(n) = o(n)$. There exists a residual set $\mathcal{R} \subset \mathcal{C}_0(\Omega)$ such that, for any $f \in \mathcal{R}$, $\mathcal{E}_\psi(f)$ is residual and of full μ-measure in Ω.*

Remark 1.2. It is important to remark that the conclusion of Theorem 1.1 deeply depends on the space of functions we consider and on its topology. For example, in the context of irrational rotations, Herman showed in [**10**] that $C^{1+\varepsilon}$ zero-mean functions are coboundaries, and their Birkhoff sums are bounded. On the other hand, for functions of bounded variations, the growth of Birkhoff sums can be estimated very precisely using Koksma's inequality (see [**16**, Ch. 2, Theorem 5.1]).

If we allow α to vary in our initial problem, then the natural framework is that of topological groups. Hence, we fix a compact and connected metric abelian group $(G, +)$. By [**16**, Ch. 4, Corollary 4.4], G is a monothetic group, that is, it possesses a dense cyclic subgroup. Let μ be the Haar measure on G. It is invariant under each translation or group rotation $T_u(x) = x + u$. We define G_0 as the set of $u \in G$ such that T_u is ergodic. By well-known results of ergodic theory, u belongs to G_0 if and only if $\{nu;\ n \in \mathbb{Z}\}$ is dense in G; in this case T_u is uniquely ergodic, only the Haar measure is invariant with respect to T_u. Moreover, G_0 is always non-empty, it is dense and its Haar measure is equal to 1 (see [**16**, Ch. 4, Theorem 4.5]).

Contrary to what happens in Theorem 1.1, the growth of $S_{n,u} f(x)$ for a typical $(u, x) \in G^2$ is not the same from the topological and from the probabilistic points of view. For the latter , the typical growth of $S_{n,u} f(x)$ has order $n^{1/2}$.

THEOREM 1.3.

(i) *For all $\nu > 1/2$ and all $f \in L^2_0(G)$,*

$$\mu \otimes \mu\left(\left\{(u, x) \in G^2;\ \limsup_n \frac{|S_{n,u} f(x)|}{n^\nu} \geq 1\right\}\right) = 0.$$

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(ii) *There exists a residual subset $\mathcal{R} \subset \mathcal{C}_0(G)$ such that, for all $f \in \mathcal{R}$,*

$$\mu \otimes \mu\left(\left\{(u, x) \in G^2;\ \limsup_n \frac{|S_{n,u} f(x)|}{n^{1/2}} = +\infty\right\}\right) = 1.$$

From a topological point of view, the typical growth of $S_{n,u} f(x)$ has order n. Indeed, for $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n) = o(n)$, let us introduce

$$\mathfrak{E}_\psi(f) = \left\{(u, x) \in G^2;\ \limsup_n \frac{|S_{n,u} f(x)|}{\psi(n)} = +\infty\right\}.$$

THEOREM 1.4. *Suppose that $\psi : \mathbb{N} \to \mathbb{N}$ satisfies $\psi(n) = o(n)$. There exists a residual set $\mathcal{R}^* \subset \mathcal{C}_0(G) \times G^2$ such that, for any $(f, u, x) \in \mathcal{R}^*$, we have $(u, x) \in \mathfrak{E}_\psi(f)$.*

We remark that, by the Kuratowski–Ulam theorem, Theorem 1.4 implies that there exists a residual set $\mathcal{R} \subset \mathcal{C}_0(G)$ such that, for every $f \in \mathcal{R}$, the set $\mathfrak{E}_\psi(f)$ is residual in G^2.

The last possibility is to fix $x \in G$ and allow u to vary. Without loss of generality, we may assume that $x = 0$. Again, topologically speaking, the typical growth of $S_{n,u} f(0)$ is not better than $o(n)$.

COROLLARY 1.5. *Suppose that $\psi : \mathbb{N} \to \mathbb{N}$ satisfies $\psi(n) = o(n)$. There exists a residual set $\mathcal{R} \subset \mathcal{C}_0(G)$ such that, for any $f \in \mathcal{R}$, the set $\{u \in G;\ (u, 0) \in \mathfrak{E}_\psi(f)\}$ is residual in G.*

Finally, we return to irrational rotations where we would like to get more precise statements. Let us fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and set

$$\mathcal{F}_\psi(f) = \left\{x \in \mathbb{T};\ \limsup_n \frac{|S_{n,\alpha} f(x)|}{\psi(n)} < +\infty\right\}.$$

When $\psi(n) = n^\nu$, $\nu \in (0, 1)$, we simply denote by $\mathcal{F}_\nu(f)$ the set $\mathcal{F}_\psi(f)$. We already know by the results mentioned before Theorem 1.1 that $\lambda(\mathcal{F}_\psi(f)) = 0$ for f in a residual subset of $\mathcal{C}_0(\mathbb{T})$, where λ is the Lebesgue measure on $\mathbb{T}$. It turns out that a much stronger result is true: generically, these sets have zero Hausdorff dimension!

THEOREM 1.6. *For any $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n) = o(n)$, there exists a residual subset $\mathcal{R}$ of $\mathcal{C}_0(\mathbb{T})$ such that, for any $f \in \mathcal{R}$, $\dim_{\mathcal{H}}(\mathcal{F}_\psi(f)) = 0$.*

We then do a similar study for Hölder functions $f \in \mathcal{C}_0^\xi(\mathbb{T})$, $\xi \in (0, 1)$. Recall that a function f belongs to $\mathcal{C}_0^\xi(\mathbb{T})$ if it has zero mean and if there exists a constant $C > 0$ such that, for all $x, y \in \mathbb{T}$,

$$|f(x) - f(y)| \le C|x - y|^\xi.$$

The infimum of such constants C is denoted by $\mathrm{Lip}_\xi(f)$.

For a function $f \in \mathcal{C}_0^\xi(\mathbb{T})$, we have better bounds on $S_{n,\alpha} f(x)$ depending on ξ and on the arithmetical properties of α. Indeed, it is known (see [**16**, Ch. 2, Theorem 5.4]) that $|S_{n,\alpha} f(x)| \le n \cdot \mathrm{Lip}_\xi(f)(D_n^*(\alpha))^\xi$, where $D_n^*(\alpha)$ is the discrepancy of the sequence $(\alpha, 2\alpha, \ldots, n\alpha)$ defined by

$$|D_n^*(\alpha)| = \sup_{I \subset \mathbb{T}} \left|\frac{\mathrm{card}\{1 \le i \le n;\ i\alpha \in I\}}{n} - |I|\right|.$$

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For instance, if α has type 1 (for example, if α is an irrational algebraic number), using the well-known estimates of the discrepancy, we get that $|S_{n,\alpha} f(x)| = O(n^{1-\xi+\varepsilon})$ for all $\varepsilon > 0$. In other words, for all $\nu > 1 - \xi$, $\mathcal{F}_\nu(f) = \mathbb{T}$. We investigate the case $\nu \le 1 - \xi$ and show that the Hausdorff dimension of $\mathcal{F}_\nu(f)$ cannot always be large.

THEOREM 1.7. *Let $\xi \in (0, 1)$. There exists $f \in \mathcal{C}_0^\xi(\mathbb{T})$ such that, for all $\nu \in (0, 1 - \xi)$,*

$$\dim_{\mathcal{H}}(\mathcal{F}_\nu(f)) \le \sqrt{\frac{\xi}{1-\nu}}.$$

This theorem is in stark contrast to [**8**, Theorem 4.1]. In the latter paper, a similar study of fast Birkhoff averages of subshifts is done. In this case, the sets which correspond to $\mathcal{F}_\nu(f)$ always have maximal dimension.

2. *Useful lemmas*

In this section we provide lemmas which will be used several times in the proof of our main theorems. The first one allows us to approximate step functions by continuous functions. In the statement of the theorem we use the standard notation $\mathbf{1}_B(x)$ for the function which equals 1 if $x \in B$ and equals 0 if not.

LEMMA 2.1. *Let Ω be a metric space, let μ be a continuous Borel probability measure on Ω. Let g be a step function such that $\int_\Omega g(x)\,d\mu(x) = 0$ and $\delta > 0$. Then there exists $f \in \mathcal{C}_0(\Omega)$ such that $\|f\|_\infty \le 2\|g\|_\infty$ and $f = g$ except on a set of measure at most δ.*

Proof. Let $\varepsilon > 0$ be very small and $\{a_1, \ldots, a_n\}$ be the finite set $g(\Omega)$. We can write $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$, where $A_i = \{x \in \Omega;\ g(x) = a_i\}$. Since the measure μ is regular, we can find compact sets $K_1, \ldots, K_n$ and open sets $U_1, \ldots, U_n$ such that

$$K_i \subset A_i \subset U_i,$$
$$\mu(U_i) - \varepsilon \le \mu(A_i) \le \mu(K_i) + \varepsilon.$$

By Urysohn's lemma, one may find functions $\varphi_i \in \mathcal{C}(\Omega)$ such that

$$0 \le \varphi_i \le 1, \quad \varphi_i = 1 \text{ on } K_i, \quad \varphi_i = 0 \text{ outside } U_i.$$

We then set $h = \sum_{i=1}^n a_i \varphi_i$. It is clear that

$$\mu(\{a_i \mathbf{1}_{A_i} \ne a_i \varphi_i\}) \le \mu(U_i \backslash K_i).$$

Therefore,

$$\mu(\{h \ne g\}) \le \sum_{i=1}^n \mu(U_i \backslash K_i) \le 2n\varepsilon.$$

If $k = \max(-\|g\|_\infty, \min(h, \|g\|_\infty))$, we now have $\|k\|_\infty \le \|g\|_\infty$ and

$$\mu(\{k \ne g\}) \le \mu(\{h \ne g\}) \le 2n\varepsilon.$$

The function k is continuous but is not necessarily in $\mathcal{C}_0(\Omega)$. Nevertheless, we observe that

$$\left|\int_\Omega k(x)\,d\mu(x)\right| = \left|\int_\Omega (k(x) - g(x))\,d\mu(x)\right| \le \|k - g\|_\infty \mu(\{k \ne g\}) \le 4n\varepsilon\|g\|_\infty$$

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and we can modify k to obtain a zero mean. Let $a \in \Omega$ and $r > 0$ be such that $0 < \mu(B(a, r)) \leq \mu(B(a, 2r)) < \delta/2$ and let $\varphi_0 \in \mathcal{C}(\Omega)$ with $\varphi_0 = 1$ on the closed ball $\bar{B}(a, r)$, $\varphi_0 = 0$ outside $B(a, 2r)$ and $0 \leq \varphi_0 \leq 1$. We set

$$f = k - \frac{\int_\Omega k\, d\mu}{\int_\Omega \varphi_0\, d\mu}\varphi_0.$$

Then $f \in \mathcal{C}_0(\Omega)$, $f = g$ except on a set of measure at most $2n\varepsilon + \delta/2$ and

$$\|f\|_\infty \leq \|g\|_\infty + \frac{|\int_\Omega k\, d\mu|}{\int_\Omega \varphi_0\, d\mu} \leq \|g\|_\infty + \frac{4n\varepsilon\|g\|_\infty}{\mu(B(a, r))}.$$

Choosing $\varepsilon > 0$ sufficiently small then gives the result. □

Our second lemma is a way to construct continuous functions in $\mathcal{C}_0(\Omega)$ with large Birkhoff sums on large subsets. As usual, $\psi : \mathbb{N} \to \mathbb{N}$ satisfies $\psi(n) = o(n)$. We denote by E^c the complement of the set E.

LEMMA 2.2. *Let Ω be a metric space and T be a uniquely ergodic transformation on Ω with a non-atomic ergodic probability measure μ. Let $J, M \in \mathbb{N}$, $C > 0$, $\varepsilon > 0$. Then there exist $f \in \mathcal{C}_0(\Omega)$, $m \geq M$ and a compact set $E \subset \Omega$ such that $\|f\|_\infty \leq \varepsilon$, $\mu(E) > 1 - \varepsilon$ and*

$$\textit{for all } x \in E,\ \textit{for all } j \in \{1, \ldots, J\}, \quad |S_{m,T^j} f(x)| \geq C\psi(m).$$

Proof. Set $\overline{\varepsilon} = \varepsilon/3$. We begin by fixing $m \in \mathbb{N}$, any integer greater than M, and such that $m\overline{\varepsilon} \geq C\psi(m)$. Let $n \gg m$, to be fixed later. We then consider a Rokhlin tower associated to T, $2n$ and $\overline{\varepsilon}$ (see, for instance, [**7**]). Namely, we consider $A \subset \Omega$ such that the sets $T^k(A)$, $0 \leq k \leq 2n - 1$, are pairwise disjoint and $\mu(\bigcup_{k=0}^{2n-1} T^k(A)) > 1 - \overline{\varepsilon}$. We then consider a function g equal to $\overline{\varepsilon}$ on $\bigcup_{k=0}^{n-1} T^k(A)$, equal to $-\overline{\varepsilon}$ on $\bigcup_{k=n}^{2n-1} T^k(A)$ and equal to zero elsewhere.

We set

$$F = \left(\bigcup_{k=0}^{n-1-mJ} T^k(A)\right) \cup \left(\bigcup_{k=n}^{2n-1-mJ} T^k(A)\right) := F_1 \cup F_2.$$

Then, for any $x \in F_1$, for any $\ell \leq m - 1$, for any $j \in \{1, \ldots, J\}$,

$$T^{\ell j}(x) \in \bigcup_{k=0}^{n-1} T^k(A).$$

It follows that $S_{m,T^j} g(x) = m\overline{\varepsilon}$. In the same way, for any $x \in F_2$, for any $j \in \{1, \ldots, J\}$, $S_{m,T^j} g(x) = -m\overline{\varepsilon}$.

Finally, for any $x \in F$, for any $j \in \{1, \ldots, J\}$,

$$|S_{m,T^j} g(x)| = m\overline{\varepsilon} \geq C\psi(m).$$

Moreover,

$$\mu(F) = 2(n - mJ)\mu(A) \geq 2(n - mJ) \cdot \frac{1 - \overline{\varepsilon}}{2n} \geq 1 - 2\overline{\varepsilon}$$

provided n is large enough.

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Thanks to Lemma 2.1, we can approximate g by a continuous function $f \in \mathcal{C}_0(\Omega)$ with $\|f\|_\infty \le 2\bar{\varepsilon}$ and $f = g$ except on a set $\mathcal{N}$ of measure $\eta > 0$, with $mJ\eta < \bar{\varepsilon}$. Fix $j \in \{1, \ldots, J\}$. Then $S_{m,T^j} f(x) = S_{m,T^j} g(x)$ except if $x \in \bigcup_{k=0}^{m-1} T^{-kj}(\mathcal{N})$. Let $\mathcal{N}' = \bigcup_{k=0}^{m-1} \bigcup_{j=1}^{J} T^{-kj}(\mathcal{N})$. Then $\mu(\mathcal{N}') \le mJ\eta < \bar{\varepsilon}$. Moreover, $|S_{m,T^j} f(x)| \ge C\psi(m)$ for all $j \in \{1, \ldots, J\}$ and all $x \in F \cap \mathcal{N}'^c =: E_0$. Clearly, $\mu(E_0) > 1 - 3\bar{\varepsilon} = 1 - \varepsilon$. We conclude by taking for E the closure of E_0. □

3. *Fast and slow points of Birkhoff sums—I*

In this section we prove Theorems 1.1 and 1.4. Their proofs share many similarities and depend heavily on Lemma 2.2 applied in suitable situations. We will also need that if T is a uniquely ergodic transformation on Ω, then the set of $\mathcal{C}_0(\Omega)$-coboundaries for T, namely the set of functions $g - g \circ T$ for some $g \in \mathcal{C}_0(\Omega)$, is dense in $\mathcal{C}_0(\Omega)$ (see, for instance, [**17**, Lemma 1]). It is convenient to work with a coboundary since its Birkhoff sums are uniformly bounded.

Proof of Theorem 1.1. Let (h_l) be a dense sequence of coboundaries in $\mathcal{C}_0(\Omega)$ and let $C_l > 0$ be such that $\sup_n \|S_{n,T} h_l\|_\infty \le C_l$. Let f_l, E_l and m_l be given by Lemma 2.2 for $C = l + C_l + 1$, $M = l$, $J = 1$, $\varepsilon = 1/l$. We set $g_l = h_l + f_l$ and we observe that, for $x \in E_l$,

$$|S_{m_l,T}\, g_l(x)| \ge (l + C_l + 1)\psi(m_l) - C_l \ge (l + 1)\psi(m_l).$$

Since E_l is compact and g_l is continuous, we can choose $\delta_l > 0$ and an open set $F_l \subset \Omega$ containing E_l such that, for any $f \in B(g_l, \delta_l)$, for any $x \in F_l$,

$$|S_{m_l,T}\, f(x)| \ge l\psi(m_l). \tag{1}$$

Let $\mathcal{R} = \bigcap_{L\ge1} \bigcup_{l\ge L} B(g_l, \delta_l)$, which is a residual set in $\mathcal{C}_0(\Omega)$, and pick $f \in \mathcal{R}$. There exists an increasing sequence (l_k) going to $+\infty$ such that $f \in B(g_{l_k}, \delta_{l_k})$ for all k. We set $F = \limsup F_{l_k} = \bigcap_{K\ge1} \bigcup_{k\ge K} F_{l_k}$. Since $\mu(F_{l_k}) \ge \mu(E_{l_k}) \ge 1 - (1/l_k)$ the set F has full measure. Moreover, since μ has full support and $\mu(\bigcup_{k\ge K} F_{l_k}) = 1$ for all K, F is also residual in Ω. Finally, if x belongs to F, then (1) is true for infinitely many l, which shows Theorem 1.1. □

In the next proof Ω is replaced by the compact connected metric abelian group G and we consider uniquely ergodic translations T_v. We recall that for these translations, all non-constant characters γ are $\mathcal{C}_0$-coboundaries: they can be written as $\gamma = \gamma_0 \circ T_v - \gamma_0$, where $\gamma_0 = \gamma/(\gamma(v) - 1)$.

Proof of Theorem 1.4. Let us denote by $\hat{G}$ the set of characters of G, that is, the set of continuous homeomorphisms from G to the unit circle $S_1 = \{z \in \mathbb{C};\, |z| = 1\}$. A trigonometric polynomial is then a finite linear combination of characters. Since G is compact we can choose a sequence (h_l) of trigonometric polynomials which is dense in $\mathcal{C}_0(G)$ (see [**18**, §1.5.2]). Let $v \in G_0$, that is, T_v is ergodic. Since h_l is a $\mathcal{C}_0$-coboundary for all T_{jv}, $j = 1, \ldots, l$, there exists $C_l > 0$ such that

$$\sup_n \sup_{j\in\{1,\ldots,l\}} \|S_{n,jv} h_l\|_\infty \le C_l.$$

Let f_l, E_l and m_l be given by Lemma 2.2 for $T = T_v$, $C = C_l + l + 1$, $M = l$, $J = l$, $\varepsilon = 1/l$. Set $g_l = h_l + f_l$ and observe that, for $x \in E_l$, $j \in \{1, \ldots, l\}$,

$$|S_{m_l, jv} g_l(x)| \geq (l + C_l + 1)\psi(m_l) - C_l \geq (l + 1)\psi(m_l).$$

Since $\{jv;\ j = 1, \ldots, l\} \times E_l$ is compact in $G \times G$ and g_l is continuous, we can choose $\delta_l > 0$ and an open set $H_l \subset G \times G$ such that $\{jv;\ j = 1, \ldots, l\} \times E_l \subset H_l$ and, for any $(f, u, x) \in B(g_l, \delta_l) \times H_l$,

$$|S_{m_l, u} f(x)| > l\psi(m_l). \tag{2}$$

We now observe that $\bigcup_{l \geq L} \{g_l\} \times \{jv;\ j = 1, \ldots, l\} \times E_l$ is dense in $\mathcal{C}_0(G) \times G \times G$ for any $L \geq 1$. Hence, $\mathcal{R}^* = \bigcap_{L \geq 1} \bigcup_{l \geq L} B(g_l, \delta_l) \times H_l$ is a residual subset of $\mathcal{C}_0(G) \times G^2$ and any $(f, u, x) \in \mathcal{R}^*$ satisfies that (u, x) belongs to $\mathfrak{E}_\psi(f)$ since (2) is true for infinitely many integers l. □

Proof of Corollary 1.5. This corollary follows easily from Theorem 1.4 and from the Kuratowski–Ulam theorem. Indeed, we know that there exist a residual set $\mathcal{R} \subset \mathcal{C}_0(G)$ and $x \in G$ such that, for all $f \in \mathcal{R}$, $\{u \in G;\ (u, x) \in \mathfrak{E}_\psi(f)\}$ is residual. Now, setting $\mathcal{R}' = \{f(\cdot - x);\ f \in \mathcal{R}\}$, for any $f \in \mathcal{R}'$, $\{u \in G;\ (u, 0) \in \mathfrak{E}_\psi(f)\}$ is residual. □

4. *Fast and slow points of Birkhoff sums—II*

We turn to the proof of Theorem 1.3. Its first part heavily depends on the following Menshov–Rademacher inequality (see, for instance, [**6**, Ch. 4]).

LEMMA 4.1. *Let $X_1, \ldots, X_N$ be a sequence of orthonormal random variables and $c_1, \ldots, c_N$ be a sequence of real numbers. Then*

$$\mathbb{E}\bigg(\max_{1 \leq n \leq N} \bigg(\sum_{j=1}^{n} c_j X_j\bigg)^2\bigg) \leq \log_2^2(4N) \sum_{n=1}^{N} c_n^2.$$

Proof of Theorem 1.3 part (i). Recall that $\int_G f(x)\, d\mu(x) = 0$. Without loss of generality, we suppose $\|f\|_2 = 1$ and we consider $X_k(u, x) = f(x + ku)$ as a random variable on the probability space $(G^2, \mu \otimes \mu)$. Next we show that $(X_k)_{k \geq 1}$ is an orthonormal sequence. Indeed, let $\sum_{\gamma \in \hat{G}} \hat{f}(\gamma)\gamma$ be the Fourier expansion of f. Then, for k, $j \geq 1$,

$$\int_{G^2} X_k \overline{X_j}\, d\mu \otimes d\mu = \sum_{\gamma, \gamma' \in \hat{G}} \hat{f}(\gamma)\overline{\hat{f}(\gamma')} \int_G \gamma(x)\overline{\gamma'(x)}\, d\mu(x) \int_G \gamma(ku)\overline{\gamma'}(ju)\, d\mu(u).$$

Now, $\int_G \gamma(x)\overline{\gamma'(x)}\, d\mu(x)$ is zero provided $\gamma \neq \gamma'$ and is equal to 1 otherwise. Moreover, let us fix $\gamma \in \hat{G}$ and set $\gamma_k(u) = \gamma(ku)$, $\gamma_j(u) = \gamma(ju)$. Then $\int_G \gamma_k \overline{\gamma_j}\, d\mu = 0$ except if $\gamma_k = \gamma_j$, that is, except if $\gamma^{k-j} = 1$. If $k \neq j$, using that $\hat{G}$ is torsion-free since G is compact and connected, this can only happen if $\gamma = 1$. Therefore, we have shown that

$$\int_{G^2} X_k \overline{X_j}\, d\mu \otimes d\mu = \begin{cases} \displaystyle\sum_{\gamma} |\hat{f}(\gamma)|^2 = 1 & \text{if } k = j, \\ |\hat{f}(1)|^2 = 0 & \text{otherwise.} \end{cases}$$

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Applying Lemma 4.1 with $c_j = 1$ yields

$$\int_{G^2} \max_{1\leq n\leq N} |S_{n,u} f(x)|^2 \, d\mu(u) \otimes d\mu(x) \leq \log_2^2(4N)N. \tag{3}$$

Let $\nu > 1/2$ and, for $k \geq 1$,

$$E_k = \{(u, x) \in G^2; \exists n \in \{2^k, \ldots, 2^{k+1} - 1\}, |S_{n,u} f(x)| \geq n^\nu\}.$$

Using Markov's inequality and (3), we get

$$\begin{aligned}\mu \otimes \mu(E_k) &\leq \mu \otimes \mu\Big(\max_{1\leq n\leq 2^{k+1}} |S_{n,u} f(x)| \geq 2^{\nu k}\Big)\\ &\leq \frac{1}{2^{2k\nu}} \log_2^2 (4 \cdot 2^{k+1}) \cdot 2^{k+1} \leq Ck^2 2^{k(1-2\nu)}.\end{aligned}$$

Since $\sum_k \mu \otimes \mu(E_k) < \infty$, the Borel–Cantelli lemma implies that $\mu \otimes \mu(\limsup_k E_k) = 0$ and the conclusion follows. □

Remark 4.2. In fact, the same proof shows that, for any $\varepsilon > 0$,

$$\mu \otimes \mu\bigg(\bigg\{(u, x) \in G^2; \limsup_n \frac{|S_{n,u} f(x)|}{n^{1/2} \log^{3/2+\varepsilon}(n)} \geq 1\bigg\}\bigg) = 0.$$

To prove the second part of Theorem 1.3, we shall use both a Baire category and a probabilistic argument. The probabilistic part is based on the following lemma, which is a consequence of the proof of the law of the iterated logarithm given in [**4**].

We recall that a random variable $X : (\Omega, \mathcal{A}, P) \to \mathbb{R}$ has a Rademacher distribution if $P(X = 1) = P(X = -1) = 1/2$.

LEMMA 4.3. *Let $\varepsilon > 0$ and $M \in \mathbb{N}$. There exists $N \geq M$ such that, for any sequence (Y_k) of independent Rademacher variables defined on the same probability space $(\Omega, \mathcal{A}, P)$,*

$$P\bigg(\sup_{M\leq n\leq N} \frac{|\sum_{k=1}^n Y_k(\omega)|}{\sqrt{n \log \log n}} > \frac{1}{2}\bigg) > 1 - \varepsilon.$$

Observe that the value of N only depends on the joint distribution of $\{Y_k\}$ and not on the Y_k themselves. This will be important in the following proof.

The following lemma is the key point of our proof.

LEMMA 4.4. *Let $\varepsilon \in (0, 1)$, $C > 0$ and $M \in \mathbb{N}$. There exist $f \in \mathcal{C}_0(G)$, $N > M$ and $F \subset G^2$ with $\|f\|_\infty \leq \varepsilon$, $\mu \otimes \mu(F) > 1 - \varepsilon$ and*

$$(u, x) \in F \implies \sup_{M\leq n\leq N} \frac{|S_{n,u} f(x)|}{n^{1/2}} \geq C.$$

Proof. Without loss of generality, we may assume that $\sqrt{\log \log M} > 2C/\varepsilon$. Lemma 4.3 gives us a value of N associated to ε and M. We then consider a sequence (X_k) of independent Rademacher variables defined on the same probability space $(\Omega, \mathcal{A}, P)$. We select a symmetric neighbourhood $\mathcal{O}$ of $0 \in G$ so that, setting

$$E_{\mathcal{O}} = \{u \in G; (j' - j)u \notin \mathcal{O} + \mathcal{O} \text{ for all } 0 \leq j, j' \leq N, \ j \neq j'\},$$

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we have $\mu(E_{\mathcal{O}}) > 1-\varepsilon$. This is possible since, denoting by $(\mathcal{O}_l)$ a basis of symmetric neighbourhoods of 0 in G, the full measure set $G_0 = \{u \in G;\ T_u \text{ is ergodic}\}$ satisfies

$$G_0 \subset \{u \in G;\ ku \neq 0 \text{ for all } k \in \mathbb{Z}\backslash\{0\}\} \subset \bigcup_l E_{\mathcal{O}_l}.$$

By compactness of G, G is contained in a finite union $(x_1+\mathcal{O}) \cup \cdots \cup (x_K+\mathcal{O})$. We set $A_1 = x_1 + \mathcal{O}$ and, for $2 \leq k \leq K$, $A_k = (x_k+\mathcal{O})\backslash(A_1 \cup \cdots \cup A_{k-1})$. The sets $A_1, \ldots, A_k$ provide a Borelian partition of G.

We then split each A_k into a disjoint sum $A_k = B_k \cup B'_k$ with $\mu(B_k) = \mu(B'_k) = \mu(A_k)/2$. For $1 \leq k \leq K$, define φ_k by $\varphi_k = (\mathbf{1}_{B_k} - \mathbf{1}_{B'_k})$, where $\mathbf{1}_A$ denotes the characteristic function of the set A. Finally, we put

$$g(x,\omega) = \sum_{k=1}^{K} \varepsilon X_k(\omega)\varphi_k(x),$$

so that

$$S_{n,u}g(x,\omega) = \varepsilon \sum_{j=0}^{n-1}\sum_{k=1}^{K} X_k(\omega)\varphi_k(x+ju).$$

Let us fix $u \in E_{\mathcal{O}}$. For all $x \in G$ and all $j \in \{0,\ldots,N-1\}$, there exists exactly one integer $k \in \{1,\ldots,K\}$, which we will denote by $k(j,u,x)$, such that $\varphi_k(x+ju) \neq 0$. Hence, for $(u,x) \in E_{\mathcal{O}} \times G$ and $n \leq N$,

$$S_{n,u}g(x,\omega) = \varepsilon \sum_{j=0}^{n-1} X_{k(j,u,x)}(\omega)\varphi_{k(j,u,x)}(x+ju).$$

Moreover, for $j \neq j'$, the integers $k(j,u,x)$ and $k(j',u,x)$ are different: otherwise, $(j-j')u$ would belong to $\mathcal{O}+\mathcal{O}$.

Recall that x and u are respectively fixed in G and $E_{\mathcal{O}}$. Applying Lemma 4.3 to the sequence $(X_{k(j,u,x)}\varphi_{k(j,u,x)}(x+ju))_{0\leq j\leq N-1}$ which is a sequence of independent Rademacher variables, we get the existence of $\Omega_{u,x} \subset \Omega$ such that $P(\Omega_{u,x}) > 1-\varepsilon$ and

$$(u,x,\omega) \in E_{\mathcal{O}} \times G \times \Omega_{u,x} \implies \sup_{M\leq n\leq N} \frac{|S_{n,u}g(x,\omega)|}{\sqrt{n\log\log n}} \geq \frac{\varepsilon}{2}.$$

Hence

$$(u,x,\omega) \in E_{\mathcal{O}} \times G \times \Omega_{u,x} \implies \sup_{M\leq n\leq N} \frac{|S_{n,u}g(x,\omega)|}{\sqrt{n}} \geq \frac{\varepsilon}{2}\sqrt{\log\log M} > C.$$

Keeping in mind that $\mu(E_{\mathcal{O}}) > 1-\varepsilon$ holds as well, by Fubini's theorem we can select and fix $\omega \in \Omega$ such that

$$\mu \otimes \mu\left(\left\{(u,x) \in G^2;\ \sup_{M\leq n\leq N} \frac{|S_{n,u}g(x,\omega)|}{\sqrt{n}} > C\right\}\right) > (1-\varepsilon)^2 > 1-2\varepsilon. \tag{4}$$

Given $\delta > 0$, according to Lemma 2.1, the function $g = g(\cdot,\omega)$ can be approximated by a continuous function $f \in \mathcal{C}_0(G)$ such that $\|f\|_\infty \leq 2\varepsilon$ and which coincides with g except in a set of measure less than δ/N. It follows that for every $u \in G$ and for any $n \in \{M,\ldots,N\}$, $S_{n,u}f(x) = S_{n,u}g(x)$ except in a set of measure less than δ. Finally, if δ is sufficiently small, inequality (4) is still satisfied if we replace g by f. □

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Proof of Theorem 1.3, part (ii). Let (h_l) be a sequence of trigonometric polynomials dense in $\mathcal{C}_0(G)$. For all $l \geq 1$ and all $u \in G_0$, since h_l is a $\mathcal{C}_0$-coboundary for T_u, we know that $\sup_n \|S_{n,u}h_l\|_\infty < +\infty$. We then find $G_l \subset G_0$ with $\mu(G_l) > 1 - 1/l$ and $C_l > 0$ such that, for all $u \in G_l$, $\sup_n \|S_{n,u}h_l\|_\infty \leq C_l$. We apply Lemma 4.4 with $\varepsilon = 1/l$, $C = l + C_l + 1$ and $M_l = l$. We get a function $f_l \in \mathcal{C}_0(G)$, an integer $N_l \geq M_l$ and a set $F_l \subset G^2$. We define $g_l = h_l + f_l$ and $E_l = F_l \cap (G_l \times G)$ so that $\mu \otimes \mu(E_l) \geq 1 - 2/l$. The way we constructed all these objects ensures that, for any $(u, x) \in E_l$,

$$\sup_{M_l \leq n \leq N_l} \frac{|S_{n,u}g_l(x)|}{n^{1/2}} \geq l + 1.$$

This yields the existence of a $\delta_l > 0$ such that, for any $f \in B(g_l, \delta_l)$ and any $(u, x) \in E_l$,

$$\sup_{M_l \leq n \leq N_l} \frac{|S_{n,u}f(x)|}{n^{1/2}} \geq l.$$

Finally, we consider the residual set $\mathcal{R} = \bigcap_{L \geq 1} \bigcup_{l \geq L} B(g_l, \delta_l)$, and we pick $f \in \mathcal{R}$. There exists an increasing sequence (l_k) such that $f \in B(g_{l_k}, \delta_{l_k})$. Let $E = \limsup_k E_{l_k}$, which has full measure, and pick $(u, x) \in E$. There exists a subsequence (l'_k) of (l_k) such that $(u, x) \in E_{l'_k}$ for all k. We then have

$$\sup_{M_{l'_k} \leq n \leq N_{l'_k}} \frac{|S_{n,u}f(x)|}{n^{1/2}} \geq l'_k$$

which allows us to conclude. □

Remark 4.5. The proof gives slightly more than announced: there exists a residual set $\mathcal{R} \subset \mathcal{C}_0(G)$ such that, for all $\varepsilon \in (0, 1/2)$ and all $f \in \mathcal{R}$,

$$\mu \otimes \mu\left(\left\{(u, x) \in G^2;\ \limsup_n \frac{|S_{n,u}f(x)|}{n^{1/2}(\log\log n)^{1/2-\varepsilon}} = +\infty\right\}\right) = 1.$$

5. *Fast and slow points for irrational rotations on the circle*

Throughout this section, we fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

5.1. *A partition of* $\mathbb{T}$.

To get an estimate of the Hausdorff dimension of $\mathcal{F}_\psi(f)$, which is more precise than the result already obtained on its measure, we will need a refinement of Rokhlin towers specific to the irrational rotation R_α defined by $R_\alpha(x) = x + \alpha$ (mod 1) for all $x \in \mathbb{T}$. We shall use the following system of partitions of $\mathbb{T}$ associated to R_α, as described, for instance, in [**19**, Lecture 9, Theorem 1]. Let (p_n/q_n) be the nth convergent of α in its continued fraction expansion. Define

$$\Delta_0^{(n)} = \begin{cases} [0, \{q_n\alpha\}) & \text{if } n \text{ is even,} \\ [\{q_n\alpha\}, 1) & \text{if } n \text{ is odd.} \end{cases}$$

Denote also $\Delta_j^{(n)} = R_\alpha^j(\Delta_0^{(n)})$. For any $n \geq 1$, the intervals $\Delta_j^{(n)}$, $0 \leq j < q_{n+1}$, and $\Delta_j^{(n+1)}$, $0 \leq j < q_n$, are pairwise disjoint and their union is the whole of $\mathbb{T}$. We shall denote by d_n the length of $\Delta_0^{(n)}$. It is well known that

$$\frac{1}{2q_{n+1}} \leq d_n \leq \frac{1}{q_{n+1}}.$$

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In order to prove Theorems 1.6 and 1.7, we will need to use the outer Hausdorff measures $\mathcal{H}^s$. Let us recall the definition. Let $E \subset \mathbb{T}$ and $s > 0$. The s-Hausdorff measure of E is defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E),$$

where

$$\mathcal{H}^s_\delta(E) = \inf_{r \in \mathcal{R}_\delta(E)} \sum_{I \in r} |I|^s$$

and $\mathcal{R}_\delta(E)$ is the set of (countable) coverings of E with intervals I of length $|I| \leq \delta$.

5.2. *Continuous functions.* The main step towards the proof of Theorem 1.6 is the following lemma which partially improves Lemma 2.2.

LEMMA 5.1. *Let $M \in \mathbb{N}$, $C > 0$, $s \in (0, 1)$, $\delta > 0$ and $\varepsilon > 0$. Then there exist $f \in \mathcal{C}_0(\mathbb{T})$ with $\|f\|_\infty \leq \varepsilon$, a compact set $E \subset \mathbb{T}$, and an integer $m \geq M$ such that*

$$\text{for all } x \in E, \quad |S_{m,\alpha} f(x)| \geq C\psi(m); \tag{5}$$

$$\mathcal{H}^s_\delta(E^c) < \varepsilon. \tag{6}$$

Proof. Let $m \geq M$ be such that $m\varepsilon > C\psi(m)$. Let n be a large integer and consider the partition of $\mathbb{T}$ described in §5.1,

$$\mathbb{T} = \bigcup_{0 \leq j < q_{n+1}} \Delta_j^{(n)} \cup \bigcup_{0 \leq j < q_n} \Delta_j^{(n+1)},$$

where the convergents of α are p_n/q_n. Since it will be easier to deal with even numbers we put $\widetilde{q}_n = 2\lfloor q_n/2 \rfloor$, $n \in \mathbb{N}$, which is the greatest even integer less than q_n. Hence $\widetilde{q}_n$ and $\widetilde{q}_{n+1}$ are even. We define a continuous function f with zero mean such that:

- on $\Delta_j^{(n)}$, $0 \leq j < (\widetilde{q}_{n+1})/2$, and on $\Delta_j^{(n+1)}$, $0 \leq j < \widetilde{q}_n/2$, $f = \varepsilon$ except on two very small intervals of size $\eta > 0$ where f is affine to ensure that f vanishes at the boundary of $\Delta_j^{(n)}$ and $\Delta_j^{(n+1)}$;
- on $\Delta_j^{(n)}$, $(\widetilde{q}_{n+1})/2 \leq j < \widetilde{q}_{n+1}$, and on $\Delta_j^{(n+1)}$, $\widetilde{q}_n/2 \leq j < \widetilde{q}_n$, $f = -\varepsilon$ except on two very small intervals of size $\eta > 0$ where f is affine to ensure that f vanishes at the boundary of $\Delta_j^{(n)}$ and $\Delta_j^{(n+1)}$;
- if $x \notin \bigcup_{0 \leq j < \widetilde{q}_{n+1}} \Delta_j^{(n)} \cup \bigcup_{0 \leq j < \widetilde{q}_n} \Delta_j^{(n+1)}$ we set $f(x) = 0$.

We set $\Gamma_j^{(n)}$ (respectively, $\Gamma_j^{(n+1)}$) the (largest) subinterval of $\Delta_j^{(n)}$ (respectively, $\Delta_j^{(n+1)}$) such that $|f| = \varepsilon$ and we let

$$E = \bigcup_{0 \leq j < (\widetilde{q}_{n+1})/2 - m} \Gamma_j^{(n)} \cup \bigcup_{((\widetilde{q}_{n+1})/2) \leq j < \widetilde{q}_{n+1} - m} \Gamma_j^{(n)}$$
$$\cup \bigcup_{0 \leq j < (\widetilde{q}_n/2) - m} \Gamma_j^{(n+1)} \cup \bigcup_{(\widetilde{q}_n/2) \leq j < \widetilde{q}_n - m} \Gamma_j^{(n+1)}.$$

If x belongs to E, then $f(x + j\alpha) = f(x)$ for all $j = 0, \ldots, m-1$ and $|f(x)| = \varepsilon$. Therefore, we have $|S_{m,\alpha} f(x)| = m\varepsilon > C\psi(m)$. On the other hand, E^c is the union of at most:

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- $(2m+2)$ intervals of size d_n;
- $(2m+2)$ intervals of size d_{n+1};
- $2(\widetilde{q}_{n+1}+\widetilde{q}_n)$ intervals of size η.

Hence, we have

$$\mathcal{H}^s_\delta(E^c) \le (2m+2)d_n^s + (2m+2)d_{n+1}^s + 2(\widetilde{q}_{n+1}+\widetilde{q}_n)\eta^s < \varepsilon$$

if we choose n sufficiently large and then η sufficiently small. $\square$

Proof of Theorem 1.6. We mimic the proof of Theorem 1.1. Recall that

$$\mathcal{F}_\psi(f) = \left\{x \in \mathbb{T};\ \limsup_n \frac{|S_{n,\alpha} f(x)|}{\psi(n)} < +\infty\right\}.$$

Let (h_l) be a sequence of coboundaries which is dense in $\mathcal{C}_0(\mathbb{T})$. Then, for any $l \ge 1$, there exists $C_l > 0$ such that $\sup_n \|S_{n,\alpha} h_l\|_\infty \le C_l$. Let f_l, E_l and m_l be given by Lemma 5.1 for $C = l + C_l + 1$, $M = l$ and $\varepsilon = s = \delta = 1/l$. We set $g_l = h_l + f_l$ and observe that, for $x \in E_l$,

$$|S_{m_l,\alpha} g_l(x)| \ge (l + C_l + 1)\psi(m_l) - C_l \ge (l+1)\psi(m_l).$$

There exists $\delta_l > 0$ such that, for any $f \in B(g_l, \delta_l)$ and any $x \in E_l$,

$$|S_{m_l,\alpha} f(x)| \ge l\psi(m_l).$$

Since the sequence (g_l) is dense in $\mathcal{C}_0(\mathbb{T})$, $\mathcal{R} = \bigcap_{L\ge 1}\bigcup_{l\ge L} B(g_l, \delta_l)$ is a residual subset of $\mathcal{C}_0(\mathbb{T})$. Pick $f \in \mathcal{R}$. There exists an increasing sequence (l_k) such that $f \in B(g_{l_k}, \delta_{l_k})$. We set

$$E = \limsup_k E_{l_k} := \bigcap_{K\ge 1}\bigcup_{k\ge K} E_{l_k}$$

and observe that, for any $x \in E$,

$$\limsup_n \frac{|S_{n,\alpha} f(x)|}{\psi(n)} = +\infty.$$

Moreover, by taking the complement of the limsup set E defined above, $E^c = \bigcup_{K\ge 1}\bigcap_{k\ge K} E_{l_k}^c$. For any $s \in (0,1)$, the properties of the sets E_l ensure that $\mathcal{H}^s(\bigcap_{k\ge K} E_{l_k}^c) = 0$. Since $\mathcal{F}_\psi \subset E^c$, we conclude that $\dim_{\mathcal{H}}(\mathcal{F}_\psi) \le s$ and therefore $\dim_{\mathcal{H}}(\mathcal{F}_\psi) = 0$. $\square$

5.3. *Hölder functions.* We now modify the previous construction to adapt it to Hölder continuous functions.

LEMMA 5.2. *Let $M \in \mathbb{N}$, $\nu \in (0,1)$, $\xi \in (0,1)$ with $\nu + \xi < 1$, $A > 0$, $\sqrt{\xi/(1-\nu)} < s \le 1$, $\delta > 0$, $\varepsilon > 0$. There exist a continuous function $f \in \mathcal{C}_0(\mathbb{T})$ with $\|f\|_\infty \le 1$, $\mathrm{Lip}_\xi(f) \le 1$, an integer $N \ge M$, and a compact set $E \subset \mathbb{T}$ such that*

$$\text{for all } x \in E \quad \text{there exists } m \in \{M, \ldots, N\},\ |S_{m,\alpha} f(x)| \ge A m^\nu, \tag{7}$$

$$\mathcal{H}^s_\delta(E^c) < \varepsilon. \tag{8}$$

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Proof. The construction of f will be more or less difficult depending on the arithmetical properties of α. Let (p_n/q_n) be the nth convergent of α in its continued fraction expansion. For each $n \geq 0$, there exists $\tau_n \geq 1$ such that $q_{n+2} = q_n^{\tau_n}$. We define

$$\tau := \liminf_n \tau_n \in [1, +\infty].$$

We then fix $\nu' \in (0, 1)$ such that $\nu' > \nu$, $\xi + \nu' < 1$ and

$$\sqrt{\frac{\xi}{1-\nu'}} < s. \tag{9}$$

If, moreover, $\tau < \sqrt{(1-\nu)/\xi}$, we also require that $\tau < \sqrt{(1-\nu')/\xi}$.

Let n be a large integer and consider the partition of $\mathbb{T}$ described in §5.1:

$$\mathbb{T} = \bigcup_{0 \leq j < q_{n+1}} \Delta_j^{(n)} \cup \bigcup_{0 \leq j < q_n} \Delta_j^{(n+1)}.$$

Again for ease of notation we suppose that q_n and q_{n+1} are even; if not, a modification similar to the one used in the proof of Lemma 5.1 can be used.

First case: $\tau \geq \sqrt{(1-\nu)/\xi}$. Then, for n large enough, $\tau_n s > 1 + \eta$ for some fixed $\eta > 0$. We fix such an n and we then define f as follows:

- on $\Delta_j^{(n)} = (a_j, b_j)$, $0 \leq j < (q_{n+1})/2$, f is equal to $(x - a_j)^\xi$ on $[a_j, (a_j + b_j)/2]$, equal to $(b_j - x)^\xi$ on $[(a_j + b_j)/2, b_j]$;
- on $\Delta_j^{(n)} = (a_j, b_j)$, $(q_{n+1})/2 \leq j < q_{n+1}$, f is equal to $-(x - a_j)^\xi$ on $[a_j, (a_j + b_j)/2]$, equal to $-(b_j - x)^\xi$ on $[(a_j + b_j)/2, b_j]$;
- f is equal to 0 otherwise.

It is then clear that $\|f\|_\infty \leq 1$, $\mathrm{Lip}_\xi(f) \leq 1$ and $\int_{\mathbb{T}} f\, d\lambda = 0$. Recalling that $d_n = b_j - a_j$ for $0 \leq j < q_{n+1}$, we then set

$$\delta_0 = \sqrt{\frac{\xi}{1-\nu'}} \in (0, 1),$$

$$\gamma_0 = \frac{1}{\delta_0} = \sqrt{\frac{1-\nu'}{\xi}} > 1,$$

$$\Gamma_j = (a_j + d_n^{\gamma_0}, b_j - d_n^{\gamma_0}), \quad 0 \leq j < q_{n+1},$$

$$E_0 = \bigcup_{j=0}^{((q_{n+1})/2)-1-\lfloor q_{n+1}^{\delta_0}\rfloor} \Gamma_j \cup \bigcup_{j=(q_{n+1})/2}^{q_{n+1}-1-\lfloor q_{n+1}^{\delta_0}\rfloor} \Gamma_j.$$

Observe that if $y \in \Gamma_j$, then $|f(y)| \geq d_n^{\gamma_0 \xi}$ and that $R_\alpha(\Gamma_j) \subset \Gamma_{j+1}$, $0 \leq j < q_{n+1} - 1$. It follows that, for $x \in E_0$, with constants C which do not depend on n and may change from line to line,

$$\begin{aligned} |S_{\lfloor q_{n+1}^{\delta_0}\rfloor, \alpha} f(x)| &\geq C \lfloor q_{n+1}^{\delta_0} \rfloor d_n^{\gamma_0 \xi} \\ &\geq C q_{n+1}^{\delta_0} q_{n+1}^{-\gamma_0 \xi} \\ &\geq C q_{n+1}^{\delta_0 (1 - (\gamma_0/\delta_0)\xi)} \\ &\geq C q_{n+1}^{\delta_0 \nu'} \\ &\geq A \lfloor q_{n+1}^{\delta_0} \rfloor^\nu \end{aligned} \tag{10}$$

provided n is large enough. Thus (7) is satisfied with $m = \lfloor q_{n+1}^{\delta_0} \rfloor$ and $E = E_0$ for large values of n. Moreover, E_0^c is contained in the union of:

- $2\lfloor q_{n+1}^{\delta_0} \rfloor + 2$ intervals of size d_n (the intervals $\Delta_j^{(n)}$ which are not considered);
- $2q_{n+1}$ intervals of size $d_n^{\gamma_0}$ (the extreme parts of the intervals $\Delta_j^{(n)}$);
- q_n intervals of size d_{n+1} (the intervals of the following generation $\Delta_j^{(n+1)}$).

Hence, for n large enough,

$$\mathcal{H}_\delta^s(E^c) \leq C(q_{n+1}^{\delta_0} q_{n+1}^{-s} + q_{n+1} q_{n+1}^{-\gamma_0 s} + q_n q_n^{-\tau_n s}).$$

Since $\delta_0 - s < 0$, $1 - \gamma_0 s < 0$ and $1 - \tau_n s < -\eta$, (8) is also satisfied provided n is large enough.

Second case: $\tau < \sqrt{(1-\nu)/\xi}$. This time, the intervals coming from $\bigcup_j \Delta_j^{(n+1)}$ are too long to be neglected with respect to the $\mathcal{H}^s$-measure. By the choice of ν', we know that there exist integers n as large as we want such that

$$1 \leq \sqrt{\tau_n} \leq \tau_n < \sqrt{\frac{1-\nu'}{\xi}}; \tag{11}$$

we will fix such an n later. We keep the same values for δ_0, γ_0, Γ_j and E_0 and the same definition for f on $\bigcup_{0 \leq j < q_{n+1}} \Delta_j^{(n)}$ as in the first case. On the other hand, we define f on $\Delta_j^{(n+1)} = (u_j, v_j)$ by imposing $f(x) = (x - u_j)^\xi$ on $[u_j, (u_j + v_j)/2]$, $f(x) = (v_j - x)^\xi$ on $[(u_j + v_j)/2, v_j]$ if $0 \leq j < q_n/2$ and $f(x) = -(x - u_j)^\xi$ on $[u_j, (u_j + v_j)/2]$, $f(x) = -(v_j - x)^\xi$ on $[(u_j + v_j)/2, v_j]$ if $q_n/2 \leq j < q_n$. We then set

$$\delta_{1,n} = \delta_1 = \sqrt{\tau_n}\sqrt{\frac{\xi}{1-\nu'}} \in (0, 1),$$

$$\gamma_{1,n} = \gamma_1 = \frac{1}{\delta_1} = \frac{1}{\sqrt{\tau_n}}\sqrt{\frac{1-\nu'}{\xi}} > 1,$$

$$\Theta_j = (u_j + d_{n+1}^{\gamma_1}, v_j - d_{n+1}^{\gamma_1}),$$

$$E_1 = \bigcup_{j=0}^{(q_n/2)-1-\lfloor q_n^{\delta_1} \rfloor} \Theta_j \cup \bigcup_{j=q_n/2}^{q_n-1-\lfloor q_n^{\delta_1} \rfloor} \Theta_j,$$

and $E = E_0 \cup E_1$. Remember that $d_{n+1} \approx q_n^{-\tau_n}$. We can still use (10) and can deduce analogously for any $x \in E_1$,

$$\begin{aligned} |S_{\lfloor q_n^{\delta_1} \rfloor, \alpha} f(x)| &\geq C q_n^{\delta_1} d_{n+1}^{\gamma_1 \xi} \\ &\geq C q_n^{\delta_1 - \tau_n \gamma_1 \xi} \\ &\geq C \lfloor q_n^{\delta_1} \rfloor^{\nu'} \\ &\geq A \lfloor q_n^{\delta_1} \rfloor^{\nu} \end{aligned} \tag{12}$$

provided n is large enough. From now on we can fix a sufficiently large n. The set E^c consists of at most:

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- $2\lfloor q_{n+1}^{\delta_0}\rfloor+2$ intervals of size d_n;
- $2q_{n+1}$ intervals of size $d_n^{\gamma_0}$;
- $2\lfloor q_n^{\delta_1}\rfloor+2$ intervals of size d_{n+1};
- $2q_n$ intervals of size $d_{n+1}^{\gamma_1}$.

Thus,

$$\mathcal{H}_\delta^s(E^c)\le C(q_{n+1}^{\delta_0-s}+q_{n+1}^{1-\gamma_0 s}+q_n^{\delta_1-\tau_n s}+q_n^{1-\tau_n\gamma_1 s}).$$

By using (9) and (11) we conclude exactly as before since

$$\delta_1-\tau_n s\le\tau_n\bigg(\sqrt{\frac{\xi}{1-\nu'}}-s\bigg)\le\sqrt{\frac{1-\nu'}{\xi}}\bigg(\sqrt{\frac{\xi}{1-\nu'}}-s\bigg)<0$$

and

$$1-\tau_n\gamma_1 s\le 1-\sqrt{\frac{1-\nu'}{\xi}}s<0. \qquad \square$$

Proof of Theorem 1.7. We will prove slightly more than announced. Let $\mathcal{E}^\xi$ be the closed subspace of $\mathcal{C}_0^\xi(\mathbb{T})$ defined by

$$\mathcal{E}^\xi=\{f\in\mathcal{C}_0(\mathbb{T});\ \forall x,y\in\mathbb{T},\ |f(x)-f(y)|\le|x-y|^\xi\}=\{f\in\mathcal{C}_0(\mathbb{T});\ \mathrm{Lip}_\xi(f)\le 1\}.$$

The space $\mathcal{E}^\xi$, equipped with the norm of the uniform convergence is now again a separable complete metric space. We will prove that, for all functions f in a residual subset of $\mathcal{E}^\xi$, for all $\nu\in(0,1-\xi)$, $\dim_{\mathcal{H}}(\mathcal{F}_\nu(f))\le\sqrt{\xi/(1-\nu)}$. Since $\mathcal{F}_\nu(f)\subset\mathcal{F}_{\widetilde{\nu}}(f)$ provided $\nu\le\widetilde{\nu}$, it is sufficient to prove this inequality for ν belonging to a sequence (ν_k) which is dense in $(0,1-\xi)$. Now, the countable intersection of residual sets remaining residual, we just have to prove that, for a fixed $\nu\in(0,1-\xi)$, all functions f in a residual subset of $\mathcal{E}^\xi$ satisfy $\dim_{\mathcal{H}}(\mathcal{F}_\nu)\le\sqrt{\xi/(1-\nu)}$.

Let (h_l) be a sequence of $\mathcal{C}_0$-coboundaries which is dense in $\mathcal{E}^\xi$ and with $\mathrm{Lip}_\xi(h_l)\le 1-1/l$. For any $l\ge 1$, there exists $C_l>0$ such that $\sup_n\|S_{n,\alpha}h_l\|_\infty\le C_l$. Let f_l, N_l and E_l be given by Lemma 5.2 with $s=\sqrt{\xi/(1-\nu)}+1/l$, $\delta=\varepsilon=1/l$, $M=l$ and $A=l(C_l+l+1)$. We set $g_l=h_l+(1/l)f_l$ so that $g_l\in\mathcal{E}^\xi$, (g_l) is dense in $\mathcal{E}^\xi$ and, for any x in the compact set E_l, there exists $m\in\{l,\dots,N_l\}$ with

$$|S_{m,\alpha}g_l(x)|\ge(l+C_l+1)m^\nu-C_l\ge(l+1)m^\nu.$$

We can then find $\delta_l>0$ such that, for all $f\in B(g_l,\delta_l)$ and all $x\in E_l$, there exists $m\in\{l,\dots,N_l\}$ with

$$|S_{m,\alpha}f(x)|\ge lm^\nu \quad\text{and}\quad \mathcal{H}_{1/l}^s(E_l^c)<\frac{1}{l}.$$

We set $\mathcal{R}=\bigcap_{L\ge1}\bigcup_{l\ge L}B(g_l,\delta_l)\cap\mathcal{E}^\xi$, which is a residual subset of $\mathcal{E}^\xi$. Pick $f\in\mathcal{R}$. There exists an increasing sequence (l_k) such that $f\in B(g_{l_k},\delta_{l_k})$. We set $E=\limsup_k E_{l_k}$ and observe that, for any $x\in E$,

$$\limsup_n\frac{|S_{n,\alpha}f(x)|}{n^\nu}=+\infty$$

so that $\mathcal{F}_\nu\subset E^c$. Now the construction of the sets E_l ensures that

$$\dim_{\mathcal{H}}(E^c)\le\sqrt{\frac{\xi}{1-\nu}}. \qquad \square$$

Question 5.3. Is the value $\sqrt{(1-\xi)/\nu}$ optimal? In particular, it does not depend on the type of α, which may appear surprising.

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6. *Miscellaneous remarks*

6.1. *Open questions.* Our study suggests further questions. The first is related to Corollary 1.5.

Question 6.1. Does there exist $\nu \in [1/2, 1]$ such that:
(i) for all $\gamma > \nu$, for all $f \in \mathcal{C}_0(G)$,

$$\mu\left(\left\{u \in G;\ \limsup_n \frac{S_{n,u} f(0)}{n^\gamma} \geq 1\right\}\right) = 0;$$

(ii) for all $\gamma < \nu$, there exists a residual subset $\mathcal{R}$ of $\mathcal{C}_0(G)$ such that, for all $f \in \mathcal{R}$,

$$\mu\left(\left\{u \in G;\ \limsup_n \frac{S_{n,u} f(0)}{n^\gamma} = +\infty\right\}\right) = 1?$$

It can be shown that $\nu = 1/2$ works for (ii). Indeed, Lemma 4.4 and Fubini's theorem imply that, for all $\varepsilon \in (0, 1)$, all $C > 0$ and all $M \in \mathbb{N}$, there exist $x \in G$, $f \in \mathcal{C}_0(G)$, $N > M$ and $E \subset G$ with $\|f\|_\infty < \varepsilon$, $\mu(E) > 1 - \varepsilon$ and $\sup_{M \leq n \leq N} (S_{n,u} f(x))/n^{1/2} \geq C$ for $u \in E$. Translating f if necessary, we may assume that $x = 0$. We then conclude exactly as in the proof of Theorem 1.3.

Second, Theorem 1.6 improves Theorem 1.1 for rotations of the circle by replacing nowhere dense sets with the more precise notion of sets with zero Hausdorff dimension. There are also enhancements of meager sets, for instance σ-porous sets (see [**20**]).

Question 6.2. Does there exist a residual subset $\mathcal{R}$ of $\mathcal{C}_0(\mathbb{T})$ such that, for any $f \in \mathcal{R}$, $\mathfrak{E}_\psi(f)$ is σ-porous?

In the spirit of Theorem 1.3, the next step would be to perform a multifractal analysis of the exceptional sets. To be precise, let $f \in \mathcal{C}_0(\mathbb{T})$ and $\nu \in (1/2, 1)$. Let us set

$$\mathcal{E}^-(\nu, f) = \left\{(\alpha, x) \in \mathbb{T}^2;\ \limsup_n \frac{\log |S_{n,\alpha} f(x)|}{\log n} \geq \nu\right\}.$$

These sets have Lebesgue measure zero.

Question 6.3. Can we majorize the Hausdorff dimension of $\mathcal{E}^-(\nu, f)$?

We could also replace everywhere the lim sup by lim inf.

Question 6.4. Let $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n) = o(n)$ and $\lim_{n\to+\infty} \psi(n) = +\infty$. Does there exist $f \in \mathcal{C}_0(\Omega)$ such that $\{x \in \Omega;\ \liminf_n |S_{n,T} f(x)|/|\psi(n)| = +\infty\}$ is residual?

We would like to thank the referee who suggested to us that if we replace 'residual' by 'non-negligible set' in this question, the answer is negative. More precisely, we can prove that, for any $f \in \mathcal{C}_0(\Omega)$, the set

$$A = \left\{x \in \Omega;\ \liminf_{n\to+\infty} |S_{n,T} f(x)| = +\infty\right\}$$

is μ-negligible. This is a consequence of a result due to Atkinson [**1**]. Proceeding towards a contradiction, suppose that $\mu(A) > 0$. Then we can select an integer N such that

$$A_N = \left\{x \in \Omega;\ \inf_{n \geq N} |S_{n,T} f(x)| > 100\right\}$$

is of positive measure. Let $4\varepsilon = \mu(A_N)$ and let K be a sufficiently large integer such that $(K-2)/K > 1-\varepsilon$. We then consider a Rokhlin tower associated to T, KN and ε (see, for instance, [7]). Namely, we consider $B \subset \Omega$ such that the sets $T^k(B)$, $0 \le k < KN$, are pairwise disjoint and $\mu(\bigcup_{0\le k<KN} T^k(B)) > 1-\varepsilon$. The choice of K ensures that

$$\mu\bigg(\bigcup_{N\le k<(K-1)N} T^k(B)\bigg) > (1-\varepsilon)^2 > 1-2\varepsilon.$$

We can then find k_0 such that $N \le k_0 < (K-1)N$ and $\mu(T^{k_0}(B) \cap A_N) > 0$. Let $A'_N = T^{k_0}(B) \cap A_N \subset A_N$. The sets $T^n(A'_N)$ are disjoint for $-N \le n \le N$.

Define $S_{n,T} f(x) = -S_{-n,T} f(T^n x)$ when $n < 0$. According to Atkinson's paper, we can find $n \in \mathbb{Z}$, $n \ne 0$, such that if we put

$$A^*_N = A'_N \cap T^{-n}(A'_N) \cap \{x : |S_{n,T} f(x)| < 1\}$$

then $\mu(A^*_N) > 0$. By the Rokhlin lemma assumption, $|n| > N$. If $n > 0$ then this contradicts the definition of $A_N \supset A'_N \supset A^*_N$. If $n < 0$ then we can take $T^n(A^*_N) \subset A'_N \subset A_N$. If $y \in T^n(A^*_N)$ then $x = T^{-n}(y) \in A^*_N$ and $1 > |S_{n,T} f(x)| = |S_{-n,T} f(T^n x)| = |S_{-n,T} f(y)| > 100$ is again a contradiction.

6.2. *Other sums.* The study of $S_{n,\alpha} f(x)$ is a particular case of the series $\sum_{n\ge1} a_n f(x+n\alpha)$. In the particular case $a_n = 1/n$ this series, also called the one-sided ergodic Hilbert transform, was thoroughly investigated in [9].

In [9], the authors show that, for any non-polynomial function $f \in \mathcal{C}^2_0(\mathbb{T})$ with values in $\mathbb{R}$, there exists a residual set $\mathcal{R}_f$ of irrational numbers depending on f such that, for every $\alpha \in \mathcal{R}_f$,

$$\limsup_N \sum_{n=1}^{N} \frac{f(x+n\alpha)}{n} = +\infty$$

for almost every $x \in \mathbb{T}$ and they ask if this holds for every $x \in \mathbb{T}$ (they show that this is the case if $\hat{f}(n) = 0$ when $n \le 0$). We provide a counterexample.

Example 6.5. Let $a \in (0, 1)$ and $f \in \mathcal{C}^2_0(\mathbb{T})$ be defined by its Fourier coefficients $\hat{f}(0) = 0$, $\hat{f}(n) = ia^n$ for $n > 0$, $\hat{f}(n) = -ia^{-n}$ for $n < 0$. A small computation shows that

$$f(x) = \frac{iae^{2\pi ix}}{1-ae^{2\pi ix}} - \frac{iae^{-2\pi ix}}{1-ae^{-2\pi ix}} = \frac{-2a\sin(2\pi x)}{1-2a\cos(2\pi x)+a^2}.$$

We shall prove that the one-sided ergodic Hilbert transform of f is bounded at $x = 0$. Indeed, setting

$$G_N(t) = \sum_{n=1}^{N} \frac{e^{2\pi int}}{n},$$

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it is easy to show that

$$\sum_{n=1}^{N} \frac{f(n\alpha)}{n} = \sum_{k>0} ia^k G_N(k\alpha) - \sum_{k>0} ia^k G_N(-k\alpha)$$
$$= i \sum_{k>0} a^k (G_N(k\alpha) - \overline{G_N(k\alpha)})$$
$$= -2 \sum_{k>0} a^k \Im m(G_N(k\alpha)).$$

Now, it is well known that the imaginary part of $G_N(t)$, namely $\sum_{n=1}^{N} (\sin(2\pi nt)/n)$, is uniformly bounded in N and t (see, for instance, [**12**, p. 4]).

Question 6.6. Can we investigate, in the spirit of this paper and of [**9**], the case $a_n = n^{-a}$, with $0 < a < 1$?

6.3. *Coboundaries in $\mathcal{C}_0^\xi(\mathbb{T})$.* The natural norm in $\mathcal{C}_0^\xi(\mathbb{T})$ is given by

$$\|f\|_\xi = \sup_{x\in\mathbb{T}} |f(x)| + \sup_{\substack{x,y\in\mathbb{T}\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\xi}. \tag{13}$$

One may wonder whether, in Theorem 1.7, we have residuality in $(\mathcal{C}_0^\xi(\mathbb{T}), \|\ \|_\xi)$ instead of in $(\mathcal{E}^\xi, \|\ \|_\infty)$. A natural way to find out would be to prove that the coboundaries are dense in $\mathcal{C}_0^\xi(\mathbb{T})$. This is not the case, which shows again that $\mathcal{C}_0^\xi(\mathbb{T})$ is a weird space.

In $\mathcal{C}_0^\xi(\mathbb{T})$ we denote the ball of radius r centred at $f \in \mathcal{C}_0^\xi(\mathbb{T})$ by $B_0^\xi(f, r)$, that is, $g \in B_0^\xi(f, r)$ if and only if $\|g - f\|_\xi < r$. We shall prove the following precise statement.

THEOREM 6.7. *For any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, for any $\xi \in (0, 1)$, there exists $f \in \mathcal{C}_0^\xi(\mathbb{T})$ such that, for any $g \in B_0^\xi(f, 0.1)$, the function g is not a $\mathcal{C}_0$- (and hence not a $\mathcal{C}_0^\xi$-)coboundary, that is, there is no $u \in \mathcal{C}_0(\mathbb{T})$ such that $g = u \circ R_\alpha - u$. Hence $\mathcal{C}_0$-coboundaries are not dense in $\mathcal{C}_0^\xi(\mathbb{T})$.*

Proof. By induction we select $n_1 = 1$, $n_k \in \mathbb{N}$, $J_k \subset [n_k, n_{k+1}) \cap \mathbb{Z}$ with the following properties. If we let $h_k = (k/n_{k+1})^{1/\xi}$ then the intervals

$$\{[j\alpha - h_k, j\alpha + 3h_k];\ j \in J_k, k \in \mathbb{N}\} \text{ are pairwise disjoint} \tag{14}$$

(all these intervals are considered mod 1 on $\mathbb{T}$),

$$\lambda\bigg(\bigcup_{j\in J_k} [j\alpha - h_k, j\alpha + 3h_k]\bigg) < \frac{1}{100^{k+2}}, \tag{15}$$

$$m_k \overset{\text{def}}{=} \#J_k > 0.99 \cdot n_{k+1}. \tag{16}$$

For this property we can use that $\bigcup_{k'<k} \bigcup_{j\in J_{k'}} [j\alpha - h_{k'}, j\alpha + 3h_{k'}]$ is a union of intervals, which by (15) are of total measure less than $1/200$ and the sequence $(j\alpha)$ is uniformly distributed on $\mathbb{T}$, especially if we suppose that the n_k are denominators of

suitable convergents of α and recall §5.1. We also suppose that J_k is maximal possible, by which we mean that if $j \in [n_k, n_{k+1}) \cap \mathbb{Z}$ and $j \notin J_k$ then

$$[j\alpha - h_k, j\alpha + 3h_k] \cap \bigcup_{k'<k} \bigcup_{j' \in J_{k'}} [j'\alpha - h_{k'}, j'\alpha + 3h_{k'}] \neq \emptyset. \tag{17}$$

By the definition of h_k and (16) we have

$$m_k \cdot h_k^{\xi} > 0.99 \cdot k. \tag{18}$$

Next we define f. On an interval $[j\alpha - h_k, j\alpha + 3h_k]$, $j \in J_k$, $k \in \mathbb{N}$, we define f by $f(j\alpha - h_k) = f(j\alpha + h_k) = f(j\alpha + 3h_k) = 0$ and

$$f(j\alpha) = h_k^{\xi}, \quad f(j\alpha + 2h_k) = -h_k^{\xi}; \tag{19}$$

otherwise f is linear on each $[j\alpha + nh_k, j\alpha + (n+1)h_k]$ with $n \in [-1, 0, 1, 2]$. If $x \notin \bigcup_{k\in\mathbb{N}} \bigcup_{j\in J_k} [j\alpha - h_k, j\alpha + 3h_k]$ then we set $f(x) = 0$.

It is obvious that $f \in \mathcal{C}_0^{\xi}(\mathbb{T})$ with $\mathrm{Lip}_{\xi}(f) \leq 1$.

Suppose that $g \in B_0^{\xi}(f, 0.1)$ and, proceeding towards a contradiction, suppose that $g = u \circ R_{\alpha} - u$ with a $u \in \mathcal{C}_0(\mathbb{T})$. Then there exists K_u such that $|u| \leq K_u$.

Clearly, for any $x \in \mathbb{T}$ and any $n \in \mathbb{N}$, we have

$$|S_{n,\alpha} g(x)| = \left| \sum_{j=0}^{n-1} g(x + j\alpha) \right| = |u(x + (n+1)\alpha) - u(x)| \leq 2K_u. \tag{20}$$

We will prove in (28) and (29) that for any function $g \in B_0^{\xi}(f, 0.1)$, its Birkhoff sums are not bounded and this will provide a contradiction.

Suppose k is fixed. Since $g \in B_0^{\xi}(f, 0.1)$ we have, for any $j \in J_k$,

$$\frac{|f(j\alpha) - g(j\alpha) - (f(j\alpha + 2h_k) - g(j\alpha + 2h_k))|}{|2h_k|^{\xi}} < 0.1. \tag{21}$$

This and (19) imply that, for $j \in J_k$,

$$g(j\alpha) - g(j\alpha + 2h_k) \geq 0.9 \cdot 2^{\xi} h_k^{\xi} > 0.9 h_k^{\xi} = 0.9 \frac{k}{n_{k+1}}. \tag{22}$$

Next we consider the cases where $j \notin J_k$, $j \in [n_k, n_{k+1})$. Then (17) applies. Suppose first that there exists $k' < k$, $j' \in J_{k'}$, such that $j\alpha, j\alpha + 2h_k \in [j'\alpha - h_{k'}, j'\alpha + 3h_{k'}]$. The construction of f on $[j'\alpha - h_{k'}, j'\alpha + 3h_{k'}]$ ensures that

$$|f(j\alpha) - f(j\alpha + 2h_k)| \leq 2h_k (h_{k'})^{\xi - 1} < 0.001 \cdot h_k^{\xi} \tag{23}$$

provided n_{k+1} was chosen sufficiently large.

If $j\alpha \notin \bigcup_{k'<k} \bigcup_{j'\in J_{k'}} [j'\alpha - h_{k'}, j'\alpha + 3h_{k'}]$ then either $f(j\alpha) = 0$ or $j\alpha \in \bigcup_{k'>k} \bigcup_{j'\in J_{k'}} [j'\alpha - h_{k'}, j'\alpha + 3h_{k'}]$. In the latter case $|f(j\alpha)| \leq h_{k'}^{\xi}$ with $k' > k$ and we can suppose by the inductive definition of the $h_{k'}$ that $h_{k'} < 0.0005^{1/\xi} \cdot h_k$. Thus

$$|f(j\alpha)| \leq 0.0005 \cdot h_k^{\xi}. \tag{24}$$

Similarly, if $j\alpha + 2h_k \notin \bigcup_{k'<k} \bigcup_{j'\in J_{k'}} [j'\alpha - h_{k'}, j'\alpha + 3h_{k'}]$ we can suppose that

$$|f(j\alpha + 2h_k)| \leq 0.0005 \cdot h_k^{\xi}. \tag{25}$$

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If one of $j\alpha$, $j\alpha + 2h_k$ belongs to a $[j'\alpha - h_{k'}, j'\alpha + 3h_{k'}]$, $k' < k$, $j' \in J_{k'}$, and the other is not an element of any such interval, then $f(x) = 0$ at some x in $[j\alpha, j\alpha + 2h_k]$ and a combination of (23) and (24), or (25) is applicable.

Summarizing, we have finally shown that, for all $j \in [n_k, n_{k+1})\backslash J_k$,

$$|f(j\alpha) - f(j\alpha + 2h_k)| < 0.002 \cdot h_k^{\xi}. \tag{26}$$

Since $g \in B_0^{\xi}(f, 0.1)$ by (21) and (26) we obtain

$$|g(j\alpha) - g(j\alpha + 2h_k)| < 0.102 \cdot h_k^{\xi} \cdot 2^{\xi}. \tag{27}$$

We claim that either

$$\sum_{j=n_k}^{n_{k+1}-1} g(j\alpha) \geq \frac{1}{4} n_{k+1} h_k^{\xi} > \frac{k}{4} \tag{28}$$

(see (18) as well) or

$$\sum_{j=n_k}^{n_{k+1}-1} g(j\alpha + 2h_k) \leq -\frac{1}{4} n_{k+1} h_k^{\xi} < -\frac{k}{4}. \tag{29}$$

It is clear that for large k this will contradict (20).

Next suppose that the negation of (28) and the negation of (29) hold. This implies

$$\sum_{j=n_k}^{n_{k+1}-1} (g(j\alpha) - g(j\alpha + 2h_k)) < 2 \cdot \frac{k}{4} = \frac{1}{2} n_{k+1} h_k^{\xi}. \tag{30}$$

By (22) and (16),

$$\sum_{j \in J_k} (g(j\alpha) - g(j\alpha + 2h_k)) \geq \# J_k \cdot 0.9 \frac{k}{n_{k+1}} > 0.99 \cdot n_{k+1} \cdot 0.9 h_k^{\xi}. \tag{31}$$

On the other hand, by (16) and (27),

$$\sum_{\substack{j=n_k \\ j \notin J_k}}^{n_{k+1}-1} |g(j\alpha) - g(j\alpha + 2h_k)| < 0.01 \cdot n_{k+1} \cdot 0.102 \cdot h_k^{\xi} \cdot 2^{\xi}. \tag{32}$$

Now (31) and (32) contradict (30). □

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References

[1] G. Atkinson. Recurrence of co-cycles and random walks. *J. Lond. Math. Soc. (2)* **13**(3) (1976), 486–488.

[2] J. Beck. Randomness of the square root of 2 and the giant leap, Part 1. *Period. Math. Hungar.* **60**(2) (2010), 137–242.

[3] J. Beck. Randomness of the square root of 2 and the giant leap, Part 2. *Period. Math. Hungar.* **62**(2) (2011), 127–246.

[**4**] P. Billingsley. *Probability and Measure (Wiley Series in Probability and Statistics)*. Wiley, New York, 1995.
[**5**] M. Bromberg and C. Ulcigrai. A temporal central limit theorem for real-valued cocycles over rotations. *Ann. Inst. Henri Poincaré Probab. Stat.* **54**(4) (2018), 2304–2334.
[**6**] J. L. Doob. *Stochastic Processes (Wiley Publications in Statistics)*. John Wiley & Sons Inc., New York; Chapman & Hall, Limited, London, 1953.
[**7**] T. Eisner, B. Farkas, M. Haase and R. Nagel. *Operator Theoretic Aspects of Ergodic Theory (Graduate Texts in Mathematics)*. Springer International Publishing, Cham, 2015.
[**8**] A. Fan and J. Schmeling. On fast Birkhoff averaging. *Math. Proc. Cambridge Philos. Soc.* **135** (2003), 443–467.
[**9**] A. Fan and J. Schmeling. Everywhere divergence of one-sided ergodic Hilbert transform. *Ann. Inst. Fourier (Grenoble)* **68**(6) (2018), 2477–2500.
[**10**] M. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publ. Math. Inst. Hautes Études Sci.* **49** (1979), 5–233.
[**11**] F. Huveneers. Subdiffusive behavior generated by irrational rotations. *Ergod. Th. & Dynam. Sys.* **29**(4) (2009), 1217–1233.
[**12**] J.-P. Kahane. *Séries de Fourier Absolument Convergentes (Ergebnisse der Mathematik und Ihrer Grenzgebiete)*. Springer, Berlin, 1970.
[**13**] H. Kesten. Uniform distribution mod 1. *Ann. of Math. (2)* **71** (1960), 445–471.
[**14**] H. Kesten. Uniform distribution mod 1. II. *Acta Arith.* **7** (1961/1962), 355–380.
[**15**] U. Krengel. On the speed of convergence in the ergodic theorem. *Monatsh. Math.* **86** (1978), 3–6.
[**16**] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences (Dover Books on Mathematics)*. Dover Publications, Mineola, NY, 2006.
[**17**] P. Liardet and D. Volný. Continuous and differentiable functions in dynamical systems. *Israel J. Math.* **98** (1997), 29–60.
[**18**] W. Rudin. *Fourier Analysis on Groups (Interscience Tracts in Pure and Applied Mathematics, 12)*. Interscience Publishers, New York, 1962.
[**19**] Y. Sinai. *Topics in Ergodic Theory (Princeton Mathematical Series, 44)*. Princeton University Press, Princeton, NY, 1994.
[**20**] L. Zajiček. Porosity and σ-porosity. *Real Anal. Exchange* **13** (1987–1988), 314–350.