Understanding Gauge

James Owen Weatherall*†

I consider two usages of the expression “gauge theory.” On one, a gauge theory is a theory with excess structure; on the other, a gauge theory is any theory appropriately related to classical electromagnetism. I make precise one sense in which one formulation of electromagnetism, the paradigmatic gauge theory on both usages, may be understood to have excess structure and then argue that gauge theories on the second usage, including Yang-Mills theory and general relativity, do not generally have excess structure in this sense.

---

1. Introduction. The word “gauge” is ubiquitous in modern physics. Our best physical theories are described, in various contexts, as “gauge theories.” The “gauge argument” allegedly reveals the underlying “logic of nature” (Martin 2002). Our theories regularly exhibit “gauge freedom,” “gauge structure,” and “gauge dependence.” Unfortunately, however, it is far from clear that the term has some univocal meaning across the many contexts in which it appears. It is a bit like “liberal” in American political discourse: it shows up everywhere, and no one knows what it means.

Here I focus on two strands of usage.[1] On the first strand, a “gauge theory” is a theory that exhibits excess structure or, in Earman’s words, “de-

---

*To contact the author, please write to: Department of Logic and Philosophy of Science, University of California, Irvine, CA 92697; e-mail: weatherj@uci.edu.

†This material is based on work supported by the National Science Foundation under grant 1331126. Thank you to Thomas Barrett, Gordon Belot, Ben Feintzeig, Richard Healey, David Malament, Sarita Rosenstock, and David Wallace for helpful discussions related to the material in this article; to Thomas Barrett, Ben Feintzeig, David Malament, J. B. Manchak, and Sarita Rosenstock for comments on an earlier draft; and to my fellow symposiasts—Thomas Barrett, Hans Halvorson, and Sahotra Sarkar—for a stimulating session at the 2014 PSA biennial meeting, at which this work was presented.

[1] There are still others—see, e.g., Weyl (1952). But I will not attempt a taxonomy.

Philosophy of Science, 83 (December 2016) pp. 1039–1049. 0031-8248/2016/8305-0033$10.00

scriptive fluff" (2004).[2] On this way of thinking about gauge, there is a mismatch between the mathematical structure used in the theory and the structure we take the world to have, in such a way that (perhaps) one could remove some structure from the theory without affecting its descriptive or representational power. Most famously, Earman and Norton (1987) argue that the so-called hole argument shows that general relativity is a gauge theory in just this sense; some have taken this as motivation for moving to a different, perhaps undiscovered, formalism for representing space-time (Earman 1989).[3] Similar considerations have motivated some views on Yang-Mills theory (Healey 2007; Rosenstock and Weatherall 2015).

The second strand of usage concerns a specific class of theories. Here one uses "gauge theory" to refer to various generalizations of classical electromagnetism that share a certain mathematical structure.[4] For instance, Trautman (1980) defines "gauge theory" as follows: "For me, a gauge theory is any physical theory of a dynamic variable which, at the classical level, may be identified with a connection on a principal bundle" (26). This turns out to be a large class containing most of our fundamental theories, including all Yang-Mills theories, general relativity, and Newton-Cartan theory. It is in this context that physicists seem to speak most often of gauge theories, usually as a synonym for "Yang-Mills theories."

It is easy to imagine that the two strands are closely related and, in particular, that all gauge theories in the second sense are also gauge theories in the first sense. But as I argue below, this is a mistake. In particular, I will articulate a precise sense in which electromagnetism—the paradigmatic example on both strands—may be understood to have excess structure, and thus to be a gauge theory in the first sense. I then consider whether other theories, such as Yang-Mills theory and general relativity, have excess structure in the same sense. I will argue that they do not. It follows that on at least one precise sense of what it means for a theory to have excess structure, the two strands of usage described above come apart.

2. Two Approaches to Electromagnetism. In what follows, we consider electromagnetism on the fixed background of Minkowski space-time,

---

[2] This way of speaking is common: see Ismael and van Fraassen (2003), Redhead (2003), Rovelli (2014), and many others.

[3] For an argument against this way of thinking about relativity theory, see Weatherall (2016b). For more on the relationship between the standard formalism of relativity theory and at least one widely discussed alternative, see Rosenstock, Barrett, and Weatherall (2015).

[4] Although I will say somewhat more about principal bundles and principal connections below, this will not be the occasion to review this formalism. For more on this topic, see Bleecker (1981), Palais (1981), and Weatherall (2016a).

(M, η_{ab}).[5] In this setting, there are two ways of characterizing models of ordinary electromagnetism.[6]

On one characterization, the principal dynamical variable is the electromagnetic field, represented by a two-form F_{ab} on space-time. The electromagnetic field is required to satisfy Maxwell's equations, which may be expressed as $\nabla_{[a} F_{bc]} = \mathbf{0}$ and $\nabla_a F^{ab} = J^b$, where ∇ is the Minkowski derivative operator and J^a is a smooth vector field representing the charge-current density on space-time. A model of the theory on this characterization might be written as a triple (M, η_{ab}, F_{ab}), where F_{ab} is any closed two form (i.e., any two form satisfying the first of Maxwell's equations).[7] Call this formulation of the theory EM_1.

On the second characterization, the dynamical field is the four-vector potential, represented by a one-form A_a on space-time. This field is required to satisfy a single differential equation: $\nabla_a \nabla^a A^b - \nabla^b \nabla_a A^a = J^b$, where again J^a is the charge-current density. A model of the theory may again be represented by a triple, (M, η_{ab}, A_a), where A_a is any one form.[8] I call this formulation of the theory EM_2.

These two formulations are systematically related. Given any model (M, η_{ab}, A_a) of EM_2, I can always define an electromagnetic field by $F_{ab} = \nabla_{[a}A_{b]}$. Since any F_{ab} thus defined is exact, it must also be closed, and thus the resulting triple (M, η_{ab}, F_{ab}) is a model of EM_1; moreover, this F_{ab} is associated with the same charge-current density as A_a. Conversely, given any model (M, η_{ab}, F_{ab}) of EM_1, since F_{ab} is closed, it must also be exact, and thus there exists a one-form A_a such that $F_{ab} = \nabla_{[a}A_{b]}$.[9] The triple (M, η_{ab}, A_b) is then a model of EM_2, again with the same charge-current density.

However, there is an important asymmetry in this relationship. Given any model of EM_2, there exists a unique corresponding model of EM_1, be-

---

[5] Minkowski space-time is a relativistic space-time where M is $\mathbb{R}^4$ and η_{ab} is flat and complete. We focus on this case for convenience; little of consequence turns on the limitation. Note that we work in the abstract index notation, following the sign conventions of Malament (2012).

[6] These two ways of thinking about electromagnetism are described in somewhat more detail in Weatherall (2015). Of course, there are other ways of characterizing the models of electromagnetism—including using the principal bundle formalism discussed below.

[7] One might also stipulate a source term, J^b, along with the model or perhaps limit attention to models for which the source term satisfies certain "physically reasonable" conditions. Here it suffices to permit arbitrary sources and to "read off" the charge-current density from the divergence of F_{ab}, using the second of Maxwell's equations.

[8] The same remarks apply here as in n. 7.

[9] This result holds globally because M is homeomorphic to $\mathbb{R}^4$. In the case of more general space-times, one would have to work locally; this difference raises interesting issues, but they are not relevant to the current discussion.

cause any smooth one form has a unique exterior derivative. But the converse is not true: given a model (M, η_{ab}, F_{ab}) of EM_1, there will generally be many corresponding models of EM_2, since if A_a is such that $F_{ab} = \nabla_{[a}A_{b]}$, then $A'_a = A_a + \nabla_a\chi$, for any smooth scalar field χ, also satisfies $F_{ab} = \nabla_{[a}A'_{b]} = \nabla_{[a}A_{b]} + \nabla_{[a}\nabla_{b]}\chi$, because for any smooth scalar field, $\nabla_{[a}\nabla_{b]}\chi = \mathbf{0}$. Transformations $A_a \mapsto A'_a$ of this form are sometimes known as *gauge transformations*.

It is this asymmetry that, I claim, supports the common view that electromagnetism has excess structure. The idea is that EM_1 and EM_2 both have all of the resources necessary to represent classical electromagnetic phenomena. Indeed, in both cases, one may take the empirical content of electromagnetism to be fully exhausted by the electromagnetic field associated with a given model—either directly in the case of EM_1 or as derived above in the case of EM_2. But there are prima facie distinct models of EM_2 associated with the same electromagnetic field. Thus, it would seem that these models of EM_2, although they differ in their mathematical properties, should be taken to have the same representational capacities. Intuitively, then, whatever structure distinguishes models of EM_2 related by gauge transformations must be irrelevant to the representational capacities of the models, at least as long as the empirical content is exhausted by the associated electromagnetic field. It is in this sense that electromagnetism—or really, EM_2—has excess structure.

3. Comparing Structure.

In the next section, I make the intuitive argument just given precise. First, however, I take a detour through pure mathematics. Mathematical objects often differ in how much structure they have. For instance, topological spaces have more structure than sets: a topological space (X, τ) consists in a set X, along with something more, namely, a collection τ of open subsets of X satisfying certain properties. Similarly, an inner product space has more structure than a vector space, and a Lie group has more structure than a smooth manifold. In this section, I use some basic category theory to capture these judgments as mathematical relationships between the theories.[10]

To begin, recall that various mathematical theories may be associated with categories. For instance, there is a category **Set**, whose objects are sets and whose arrows are functions. There is a category **Top**, whose objects are topological spaces and whose arrows are continuous functions. There are also functors between such categories. For instance, there is a functor $F : \mathbf{Top} \rightarrow \mathbf{Set}$ that takes every topological space (X, τ) to the set X and takes every continuous map $f : (X, \tau) \rightarrow (X', \tau')$ to the function $f : X \rightarrow X'$.

---

[10] For background on basic category theory, see Mac Lane (1998) or Leinster (2014); because of space constraints, I take notions like "category" and "functor" for granted.

Functors of this sort are often called "forgetful," because, intuitively speaking, they take objects of one category and forget something about them: in this case, they take topological spaces (X, τ) and forget about τ.

How can one tell whether a given functor is forgetful? There is a simple but insightful theory available, due to Baez, Bartel, and Dolan (2004; see also Barrett 2013). It requires a few further definitions, concerning properties that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ may have. First, we will say that F is *full* if for all objects A, B of $\mathcal{C}$, the map $(f : A \rightarrow B) \mapsto (F(f) : F(A) \rightarrow F(B))$ induced by F is surjective. Similarly, F is *faithful* if for all pairs of objects in $\mathcal{C}$, the induced map on arrows is injective. And F is *essentially surjective* if for every object X of $\mathcal{D}$, there is some object A of $\mathcal{C}$ and arrows $f : F(A) \rightarrow X$ and $f^{-1} : X \rightarrow F(A)$ such that $f^{-1} \circ f = 1_{F(A)}$ and $f \circ f^{-1} = 1_X$. (Such an arrow f is an *isomorphism*, so essentially surjective functors are surjective on objects "up to isomorphism.")

If a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is full, faithful, and essentially surjective, then the functor is said to realize an *equivalence* of categories. In such cases, F forgets nothing. Otherwise a functor is forgetful. In particular, a functor forgets (only) structure if it is faithful and essentially surjective but not full. A functor forgets (only) properties if it is full and faithful but not essentially surjective. And a functor forgets (only) stuff if it is full and essentially surjective but not faithful. In general, a given functor may be forgetful in more than one of these ways, but not in any other ways: any functor may be written as the composition of three functors that forget (no more than) structure, properties, and stuff, respectively.

The best way to make this categorization plausible is by considering examples. For instance, the functor we have already considered, $F : \mathbf{Top} \rightarrow \mathbf{Set}$, forgets only structure. This is because every set corresponds to some topological space (or other), which means that F is essentially surjective. Similarly, any distinct continuous functions $f, f' : (X, \tau) \rightarrow (X', \tau')$ must be distinct as functions, so F is faithful. But F is not full, because not every function $f : X \rightarrow X'$ is continuous, given topologies on X and X'. So in this case, the classification captures the pretheoretic intuition with which we began.

Similarly, we can define categories **Grp** and **AbGrp**, whose objects are groups and Abelian groups, respectively, and whose arrows are group homomorphisms; then, there is a functor $G : \mathbf{AbGrp} \rightarrow \mathbf{Grp}$ that takes Abelian groups and group homomorphisms to themselves. This functor is full and faithful, since it just acts as the identity on group homomorphisms between Abelian groups. But it is not essentially surjective because not every group is Abelian. So this functor forgets only properties—namely, the property of being Abelian. And finally, we can define a functor $H : \mathbf{Set} \rightarrow 1$, where 1 is the category with one object and one arrow (the identity on the one object). This functor takes every set to the unique object of 1 and every arrow to the unique arrow of 1. It is clearly full and essentially surjective, but not faithful, so it forgets only stuff. To see how, note that we may think of 1 as the category with the empty set as its only object; thus, H forgets all of the elements of the sets.

This classification of functors gives us a criterion for when a mathematical theory T_1 may be said to have more structure than another theory T_2, namely, when there exists a functor from the category associated with T_1 to the category associated with T_2 that forgets structure. Given two categories, there may be multiple functors between them, and it may be that not all such functors forget structure, even if there exists one that does. This means that comparative judgments of "amount of structure" between theories should be understood as relative to a choice of functor. This flexibility is a virtue: it allows us to explore various ways in which theories may be related.

4. A Diagnostic Tool. I now return to the question of interest. To begin, I use the criterion just developed to make the intuitive argument at the end of section 2 precise. I define two categories, corresponding to the two formulations of electromagnetism already discussed, and then define a functor between them that captures the relationship already discussed between EM$_1$ and EM$_2$.

The first category, $\mathbf{EM}_1$, has models (M, η_{ab}, F_{ab}) of EM$_1$ as objects and as arrows has maps that suitably preserve this structure. For present purposes, we take these to be isometries of Minkowski space-time that preserve the electromagnetic field, so that given two models, (M, η_{ab}, F_{ab}) and (M, η_{ab}, F'_{ab}), an arrow $\chi:(M, \eta_{ab}, F_{ab})\rightarrow(M, \eta_{ab}, F'_{ab})$ will be an isometry of (M, η_{ab}) such that $\chi^*(F'_{ab}) = F_{ab}$. Likewise, we may define a category $\mathbf{EM}_2$ whose objects are models (M, η_{ab}, A_a) of EM$_2$ and whose arrows are isometries of Minkowski space-time that preserve the four-vector potential. Given these categories, the map defined above, taking models (M, η_{ab}, A_a) of EM$_2$ to models $(M, \eta_{ab}, \nabla_{[a}A_{b]})$ of EM$_1$, becomes a functor $F:\mathbf{EM}_2\rightarrow\mathbf{EM}_1$ that take arrows of $\mathbf{EM}_2$—which, recall, are isometries of Minkowski space-time with an additional property—to the arrow of $\mathbf{EM}_1$ corresponding to the same isometry. (This action on arrows is well defined because, given any arrow $\chi:(M, \eta_{ab}, A_a)\rightarrow(M, \eta_{ab}, A'_a)$ of $\mathbf{EM}_2$, $\nabla_{[a}A_{b]} = \nabla_{[a}(\chi^*(A'_{b]})) = \chi^*(\nabla_{[a}A'_{b]})$.)[11] We then have the following result.

Proposition 1. F forgets only structure.

Proof. F is clearly faithful and essentially surjective. To see that it is not full, consider the object $A = (M, \eta_{ab}, \mathbf{0})$ of $\mathbf{EM}_1$. The object $X = (M, \eta_{ab}, \mathbf{0})$ of

---

[11] See Weatherall (2015) for further details on this functor.

$\mathbf{EM}_2$ maps to A. But now consider any nonconstant scalar field ψ. Then $X' = (M, \eta_{ab}, \nabla_a\psi)$ is a model of $\mathbf{EM}_2$, and $F(X) = F(X') = A$. If F were full, then, there would have to be an arrow $f : X \rightarrow X'$ such that $F(f) = 1_A$, but this is impossible, since by construction, there are no arrows between X and X'. QED

Proposition 1 provides a precise sense in which EM_2 has more structure than EM_1: the functor realizing the natural relationship between the theories forgets (only) structure. Recall that the intuitive argument was that there are distinct models of EM_2 corresponding to a single model of EM_1, and thus there must be features of the models of EM_2 that distinguish them, without making any difference to their empirical content. The present argument, meanwhile, is that there are models of EM_2 that fail to be isomorphic—by the standard of isomorphism used in defining EM_2—even though the corresponding models of EM_1 are isomorphic or even identical. This is captured in the formalism by the fact that there are arrows in $\mathbf{EM}_1$, which we may interpret as “structure-preserving maps” between models of EM_1, that are not structure-preserving maps between models of EM_2. The structure that these maps do not preserve is the structure that, on the intuitive argument, distinguished models of EM_2.

I take this to be strong evidence that the formal criterion given by forgetful functors captures the sense in which electromagnetism has excess structure. And since electromagnetism is the paradigmatic example of a gauge theory, I take this to be the sense of “excess structure” associated with the first strand of usage.[12] Of course, there may be other senses in which a theory might be thought to have excess structure, but I will not consider that question further here. Rather, I stipulate that the criterion developed here is salient and turn to a different question. Do gauge theories in the second sense—that is, the theories Trautman identifies—have excess structure?

No. First, consider electromagnetism, formulated now as a theory whose dynamical variable is a connection on a principal bundle over Minkowski space-time—that is, electromagnetism formulated as a gauge theory in Trautman’s sense. Call this theory EM_3. Models of EM_3 may be written (P, ω_α), where P is the total space of the (unique, trivial) principal bundle $U(1) \rightarrow P \xrightarrow{\pi} M$ over Minkowski space-time and ω_α is a principal connec-

---

[12] There are other examples of theories with excess structure in this sense, too—e.g., Newtonian gravitation, which is also sometimes described as a gauge theory (Malament 2012, 248), may be understood to have excess structure in just this sense. To see this, consider the discussion in Weatherall (2015, sec. 6) and observe that the functor from $\mathbf{NG}_1$ to $\mathbf{GNG}$ given by the Trautman geometrization theorem is faithful and essentially surjective but not full.

tion.[13] This theory is closely related to both EM_1 and EM_2 as already discussed: given any (global) section $\sigma : M \rightarrow P$, we may define a four-vector potential A_a as the pullback along σ of ω_α: $A_a = \sigma^*(\omega_\alpha)$. Similarly, we may define an electromagnetic field tensor F_{ab} as the pullback along σ of the *curvature* of the connection, defined by $\Omega_{\alpha\beta} = d_\alpha \omega_\beta$, where d is the exterior derivative on P: $F_{ab} = \sigma^*(\Omega_{\alpha\beta})$. Thus, A_a and F_{ab} may be thought of as representatives on M of the connection and curvature on P. In general, A_a will depend on the choice of section σ, whereas F_{ab} will not depend on that choice because $U(1)$ is an Abelian group.

Given this characterization of the theory, we can define yet another category, $\mathbf{EM}_3$, as follows: the objects of $\mathbf{EM}_3$ are models of EM_3, and the arrows are principal bundle isomorphisms (Ψ, ψ) that preserve both the connection on P and the metric on M: that is, pairs of diffeomorphisms $\Psi : P \rightarrow P$ and $\psi : M \rightarrow M$ such that $\psi^*(\eta_{ab}) = \eta_{ab}$, $\Psi^*(\omega_\alpha) = \omega_\alpha$, $\pi \circ \Psi = \psi \circ \pi$, and $\Psi(xg) = \Psi(x)g$ for any $x \in P$ and any $g \in U(1)$. Then we may define a functor $\tilde{F} : \mathbf{EM}_3 \rightarrow \mathbf{EM}_1$ as follows: $\tilde{F}$ acts on objects as $(P, \omega_\alpha) \mapsto (M, \eta_{ab}, \sigma^*(\Omega_{\alpha\beta}))$, for any global section $\sigma : M \rightarrow P$, and $\tilde{F}$ acts on arrows as $(\Psi, \psi) \mapsto \psi$. (Again, this action on arrows is well defined. Choose any section $\sigma : M \rightarrow P$. Then $\Psi^{-1} \circ \sigma \circ \psi$ is also a section of P. But since $F_{ab} = \sigma^*(\Omega_{\alpha\beta})$ and $F'_{ab} = \sigma^*(\Omega'_{\alpha\beta})$ are independent of the choice of section, $F_{ab} = (\Psi^{-1} \circ \sigma \circ \psi)^*(\Omega_{\alpha\beta}) = \psi^* \circ \sigma^* \circ \Psi^{-1*}(\Omega_{\alpha\beta}) = \psi^* \circ \sigma^*(\Omega'_{\alpha\beta}) = \psi^*(F'_{ab})$.)

Proposition 2. $\tilde{F}$ forgets nothing.

Proof. First, consider an arbitrary object (M, η_{ab}, F_{ab}) of $\mathbf{EM}_1$ and pick some A_a such that $F : \mathbf{EM}_2 \rightarrow \mathbf{EM}_1$. Then given any section $\sigma : M \rightarrow P$ and any one-form A_a, we can define a connection ω_α on P by assigning to each point $x \in \sigma[M]$ the one-form $\pi^*(A_a)$ and then extending this field on $\sigma[M]$ to all of P by requiring it to be equivariant. It follows that $\tilde{F}$ is essentially surjective. Now consider any two objects (P, ω_α) and (P, ω'_α) of $\mathbf{EM}_3$ and suppose there exist arrows $(\Psi, \psi), (\Psi', \psi') : (P, \omega_\alpha) \rightarrow (P, \omega'_\alpha)$ such that $\tilde{F}((\Psi, \psi)) = \psi = \psi' = \tilde{F}((\Psi', \psi'))$. It follows that $(\Psi' \circ \Psi^{-1}, 1_M)$ is a vertical bundle automorphism such that $(\Psi' \circ \Psi^{-1})^*(\omega'_\alpha) = \omega'_\alpha$. But by the equivariance of ω'_α, this is only possible if $\Psi' \circ \Psi^{-1} = 1_P$. Since Ψ and Ψ^{-1} are bijective, it follows that $\Psi = \Psi'$. Thus, $\tilde{F}$ is faithful. Finally, suppose there is an arrow $\psi : \tilde{F}((P, \omega_\alpha)) \rightarrow \tilde{F}((P, \omega'_\alpha))$. Then ψ is an isometry of Minkowski space-time such that, for any section $\sigma : M \rightarrow P$, $\psi^*(\sigma^*(d_\alpha \omega'_\beta)) = \sigma^*(d_\alpha \omega_\beta)$. Fix $\sigma : M \rightarrow P$. It follows that there exists a section $\tilde{\sigma} : M \rightarrow P$ such that $\tilde{\sigma}^*(\omega_\alpha) = \psi^*(\sigma^*(\omega'_\alpha))$. We may then define a diffeomorphism $\Psi : P \rightarrow P$ as follows. For any $x \in P$, there exists some $g \in U(1)$ such that

---

[13] Again, see Bleecker (1981), Palais (1981), or Weatherall (2016a) for more details on this theory.

$x = \tilde{\sigma} \circ \pi(x)g$. This relationship determines a smooth map $g : P \rightarrow U(1)$ satisfying $g(xh) = g(x)h$ for any $h \in U(1)$. We then define Ψ by $x \mapsto \sigma \circ \psi \circ \pi(x)g(x)$ for all $x \in P$. This map $\Psi : P \rightarrow P$ is a diffeomorphism such that $\Psi(xh) = \sigma \circ \psi \circ \pi(xh)g(xh) = \sigma \circ \psi \circ \pi(x)g(x)h = \Psi(x)h$ and $\pi \circ \Psi = \pi \circ \sigma \circ \psi \circ \pi = \psi \circ \pi$. Thus, $(\Psi, \psi) : P \rightarrow P$ is a principal bundle isomorphism. Moreover, since for any $x \in \tilde{\sigma}[M]$, $(\Psi_x)^*(\omega'_\alpha) = (\pi_x)^* \circ (\psi_{\pi(x)})^* \circ \overset{*}{\sigma}_{\psi \circ \pi(x)}(\omega'_\alpha) = (\pi_x)^* \circ (\tilde{\sigma}_{\pi(x)})^*(\omega_\alpha) = (\omega_\alpha)_{|x}$, it follows by the equivariance of ω_α and ω'_α that $\Psi^*(\omega'_\alpha) = \omega_\alpha$. So $\tilde{F}$ is full. QED

This result shows that EM_3 does not have excess structure in the sense that EM_2 does. To extend this to other gauge theories in the second sense, however, requires more work. The reason is that the criterion we have been using requires us to have two formulations, both of which are taken to be descriptively adequate and empirically equivalent, which we then compare. In other cases of interest, though, such as non-Abelian Yang-Mills theory or general relativity, it is not clear that we have a plausible second theory to consider.

Still, there is something one can say. It concerns the role of "gauge transformations" between models of EM_2, as described at the end of section 2. These are maps that relate models of EM_2 that have the same representational capacities even though they are not isomorphic. The criterion of excess structure described here, meanwhile, requires the existence of a functor between categories of models that fails to be full—or in other words, a standard of comparison between the theories relative to which one formulation has "more" arrows than the other or, alternatively, relative to which one of the formulations is "missing" arrows.[14] This suggests a rule of thumb for whether a theory, or a formulation of a theory, has excess structure in the sense described here, namely, the theory has models that are not isomorphic but that nevertheless we interpret as having precisely the same representational content. Indeed, whereas the criterion discussed above tells us when one theory or formulation has more structure than another, this second criterion evaluates whether any alternative formulation could have less structure and still do the same descriptive work—at least without equivocating between physical situations we now think are distinct.

How can we put this rule of thumb to work? Suppose you are given a theory and a collection of maps taking models to models with the same representational capacities—that is, one is presented with a candidate "gauge theory" and a class of "gauge transformations." One may then ask: are these gauge transformations naturally construed as isomorphisms of the models

---

[14] This way of speaking may be made somewhat more precise, by showing how adding arrows corresponding to gauge transformations to $\mathbf{EM}_2$ can lead to a new category that is equivalent to $\mathbf{EM}_1$. See Weatherall (2015).

of the theory, understood as mathematical objects? If the answer is yes, then it would seem that these maps do not signal excess structure, since these maps would not be "missing" from a natural category of models; conversely, if the answer is no, then there likely is excess structure in the formulation.

Applying this diagnostic to some examples of gauge theories in the second sense above, we immediately see that the moral concerning EM_3 generalizes to other Yang-Mills theories. For instance, models of an arbitrary Yang-Mills theory with structure group G may be written $(P, \omega^{\mathfrak{A}}{}_{\alpha})$, where P is a principal G-bundle over some space-time (M, g_{ab}) and $\omega^{\mathfrak{A}}{}_{\alpha}$ is a principal connection on P.[15] In this setting, a "gauge transformation" is often defined as a (vertical) principal bundle automorphism $(\Psi, 1_M)$ relating models $(P, \omega^{\mathfrak{A}}{}_{\alpha})$ and $(P, \Psi_*(\omega^{\mathfrak{A}}{}_{\alpha}))$ (see, e.g., Bleecker 1981, sec. 3.2). But these maps are just a special class of connection- and metric-preserving principal bundle isomorphisms, and so although they do map between models with the same representational resources, they are not "extra" maps, in the sense of the gauge transformations of EM_2. So Yang-Mills theory does not have excess structure in the sense discussed here.

Likewise, for general relativity, we characterize models of the theory as relativistic space-times, (M, g_{ab}). Here "gauge transformations" are often taken to be diffeomorphisms $\varphi : M \rightarrow M$ relating models (M, g_{ab}) and $(M, \varphi_*(g_{ab}))$ (see, e.g., Wald 1984; Earman and Norton 1987). But once again, these maps are just isometries (i.e., they are just isomorphisms of Lorentzian manifolds). So here, too, there is no excess structure.

5. Conclusion. I have isolated two strands of usage of the expression "gauge theory" in physics and philosophy of physics. According to one, a gauge theory is a theory that has excess structure; according to the other, a gauge theory is any theory whose dynamical variable is a connection on a principal bundle. I then endeavored to make precise the sense in which the paradigmatic example of a gauge theory (according to both strands)—classical electromagnetism—may be construed as having excess structure. From this discussion, I extracted a general criterion for when a theory has excess structure. From this criterion, I argued that gauge theories in the second sense need not have excess structure—and indeed, Yang-Mills theory and general relativity should not be construed as having excess structure in the sense that one formulation of electromagnetism does.

REFERENCES

Baez, John, Toby Bartel, and Jim Dolan. 2004. "Property, Structure, and Stuff." Quantum Gravity Seminar, University of California, Riverside. http://math.ucr.edu/home/baez/qg-spring2004/discussion.html.

---

[15] Again, for more on the notation used here, see Weatherall (2016a).

Barrett, Thomas. 2013. “How to Count Structure.” Philosophy of Physics Workshop, Carnegie Mellon University, September.

Bleecker, David. 1981. *Gauge Theory and Variational Principles*. Reading, MA: Addison-Wesley. Repr., Dover 2005.

Earman, John. 1989. *World Enough and Space-Time*. Cambridge, MA: MIT Press.

———. 2004. “Laws, Symmetry, and Symmetry Breaking: Invariance, Conservation Principles, and Objectivity.” *Philosophy of Science* 71 (5): 1227–41.

Earman, John, and John Norton. 1987. “What Price Spacetime Substantivalism? The Hole Story.” *British Journal for the Philosophy of Science* 38 (4): 515–25.

Healey, Richard. 2007. *Gauging What’s Real: The Conceptual Foundations of Contemporary Gauge Theories*. New York: Oxford University Press.

Ismael, Jenann, and Bas van Fraassen. 2003. “Symmetry as a Guide to Superfluous Structure.” In *Symmetries in Physics: Philosophical Reflections*, ed. Katherine Brading and Elena Castellani, 371–92. Cambridge: Cambridge University Press.

Leinster, Tom. 2014. *Basic Category Theory*. Cambridge: Cambridge University Press.

Mac Lane, Saunders. 1998. *Categories for the Working Mathematician*. 2nd ed. New York: Springer.

Malament, David. 2012. *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*. Chicago: University of Chicago Press.

Martin, Christopher A. 2002. “Gauge Principles, Gauge Arguments, and the Logic of Nature.” *Philosophy of Science* 69 (Proceedings): S221–S234.

Palais, Richard S. 1981. *The Geometrization of Physics*. Hsinchu: Institute of Mathematics, National Tsing Hua University.

Redhead, Michael. 2003. “The Interpretation of Gauge Symmetry.” In *Symmetries in Physics: Philosophical Reflections*, ed. Katherine Brading and Elena Castellani, 124–39. Cambridge: Cambridge University Press.

Rosenstock, Sarita, Thomas Barrett, and James Owen Weatherall. 2015. “On Einstein Algebras and Relativistic Spacetimes.” *Studies in History and Philosophy of Modern Physics* B 52:309–16.

Rosenstock, Sarita, and James Owen Weatherall. 2015. “A Categorical Equivalence between Generalized Holonomy Maps on a Connected Manifold and Principal Connections on Bundles over That Manifold.” Unpublished manuscript, arXiv. http://arxiv.org/abs/1504.02401.

Rovelli, Carlo. 2013. “Why Gauge?” *Foundation of Physics* 44 (1): 91–104.

Trautman, Andrzej. 1980. “Fiber Bundles, Gauge Fields, and Gravitation.” In *General Relativity and Gravitation*, ed. A. Held, 287–308. New York: Plenum.

Wald, Robert. 1984. *General Relativity*. Chicago: University of Chicago Press.

Weatherall, James Owen. 2015. “Are Newtonian Gravitation and Geometrized Newtonian Gravitation Theoretically Equivalent?” *Erkenntnis*, forthcoming. doi:10.1007/s10670-015-9783-5.

———. 2016a. “Fiber Bundles, Yang-Mills Theory, and General Relativity.” *Synthese* 193 (8): 2389–2424.

———. 2016b. “Regarding the ‘Hole Argument.’” *British Journal for Philosophy of Science*, forthcoming. doi:10.1093/bjps/axw012.

Weyl, Hermann. 1952. *Space-Time-Matter*. Mineola, NY: Dover.