

A SHARP LOWER BOUND FOR CHOOSING THE MAXIMUM OF AN INDEPENDENT SEQUENCE

PIETER C. ALLAART* ** AND

JOSÉ A. ISLAS,* *** *University of North Texas*

Abstract

In this paper we consider a variation of the full-information secretary problem where the random variables to be observed are independent but not necessarily identically distributed. The main result is a sharp lower bound for the optimal win probability. Precisely, if X_1, \dots, X_n are independent random variables with known continuous distributions and $V_n(X_1, \dots, X_n) := \sup_{\tau} \mathbb{P}(X_{\tau} = M_n)$, where $M_n := \max\{X_1, \dots, X_n\}$ and the supremum is over all stopping times adapted to X_1, \dots, X_n , then $V_n(X_1, \dots, X_n) \geq (1 - 1/n)^{n-1}$, and this bound is attained. The method of proof consists in reducing the problem to that of a sequence of random variables taking at most two possible values, and then applying Bruss' sum-the-odds theorem, Bruss (2000). In order to obtain a sharp bound for each n , we improve Bruss' lower bound, Bruss (2003), for the sum-the-odds problem.

Keywords: Choosing the maximum; sum-the-odds theorem; stopping time

2010 Mathematics Subject Classification: Primary 60G40

Secondary 62L15

1. Introduction

In the classical secretary problem or best-choice problem, a known number n of applicants for a single position are interviewed one-by-one in a random order, and after each interview a manager must decide whether or not to hire the applicant just interviewed, based solely on his or her relative rank among the applicants interviewed thus far. The objective is to hire the best applicant. It is well known that under the optimal strategy, the probability of success decreases monotonically to $1/e \approx 0.3679$ as $n \rightarrow \infty$. An entertaining account of the history of the secretary problem and some of its many variations can be found in [3]; see also [5].

One natural extension is the full-information best-choice problem, in which each applicant, independent of the others, can be assigned a numerical score whose distribution is known in advance, and the objective is to hire the applicant with the highest score. The mathematical framework for this problem is as follows. Here and throughout this paper, let X_1, \dots, X_n be independent random variables with known continuous distributions, let $M_n := \max\{X_1, \dots, X_n\}$, and define

$$V_n(X_1, \dots, X_n) := \sup_{\tau} \mathbb{P}(X_{\tau} = M_n), \quad (1.1)$$

where the supremum is over all stopping times adapted to the natural filtration of X_1, \dots, X_n . Gilbert and Mosteller [4] solved this optimal stopping problem in the case when X_1, \dots, X_n are

Received 12 November 2015; revision received 13 January 2016.

* Postal address: Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203-5017, USA.

** Email address: allaart@unt.edu

*** Email address: joseislas@my.unt.edu

independent and identically distributed (i.i.d.). They showed that the optimal win probability is independent of the distribution of the X_i , and decreases monotonically to 0.5802.

It is natural to ask how much lower the optimal win probability can be if we drop the assumption that the random variables are identically distributed. In this case, it should be fairly clear that the optimal win probability depends on the distributions of X_1, \dots, X_n , and we aim to find a sharp lower bound. Our main result is as follows.

Theorem 1.1. *For each $n \in \mathbb{N}$, we have*

$$V_n(X_1, \dots, X_n) \geq \left(1 - \frac{1}{n}\right)^{n-1}, \quad (1.2)$$

and this bound is attained.

Remark 1.1. The bound in the theorem decreases to $1/e \approx 0.3679$ as $n \rightarrow \infty$. This is quite a bit smaller than the limiting win probability 0.5802 in the i.i.d. case, which should not come as a surprise. (Intuitively, when the distributions become progressively more spread out—as they do in the extremal case; see Lemma 3.1 below—there is more uncertainty about the future than when all distributions are identical, resulting in a lower optimal win probability.) Comparing our bound with the classical secretary problem, we see, moreover, that, in the worst case and for large n , a gambler who has full information can do no better than a gambler who observes only relative ranks.

Remark 1.2. One might ask whether the continuity assumption about the X_i is needed in Theorem 1.1. The way our objective function is defined in (1.1) implies that we win if we stop with a value that is at least tied for the overall maximum. Thus, it seems that the possibility of ties should only make it easier to win and, consequently, the lower bound (1.2) should continue to hold when the random variables X_1, \dots, X_n are permitted to have atoms. However, it is not clear to us how to prove this formally.

Our method of proof consists of two steps. First, we construct a sequence Y_1, \dots, Y_n of simple random variables (each of which, in fact, takes on at most two possible values) such that $V_n(X_1, \dots, X_n) \geq V_n(Y_1, \dots, Y_n)$. Next, we use Bruss' sum-the-odds theorem [1] to obtain an explicit expression for $V_n(Y_1, \dots, Y_n)$, which we then minimize by establishing a lower bound for the sum-the-odds problem which improves on that given by Bruss [2]. This lower bound, though not difficult to obtain, is interesting in its own right.

We have not found a more direct way to prove the main result, though we do not rule out the possibility that one exists. However, we believe that the technique of proof used here is of independent interest and may be applicable to other optimal stopping problems of a similar nature.

2. Bruss' sum-the-odds theorem

Bruss [1] considered the problem of stopping at the last success in a sequence of independent Bernoulli trials. Specifically, let A_1, \dots, A_n be independent events with $\mathbb{P}(A_i) = p_i$, $i = 1, \dots, n$, and let $\mathbf{1}_i := \mathbf{1}_{A_i}$, the indicator random variable corresponding to A_i , for $i = 1, \dots, n$. If we think of the value 1 as representing a success and 0 as representing a failure, the problem of stopping at the last success comes down to finding a stopping time τ that maximizes $\mathbb{P}(\mathbf{1}_\tau = 1, \mathbf{1}_{\tau+1} = \dots = \mathbf{1}_n = 0)$.

Theorem 2.1. (Sum-the-odds theorem [1].) *Let $\mathbf{1}_1, \mathbf{1}_2, \dots, \mathbf{1}_n$ be independent indicator random variables. Let $p_j := \mathbb{E}(\mathbf{1}_j)$, $q_j := 1 - p_j$ and $r_j := p_j/q_j$, for $j = 1, \dots, n$. Consider the problem of stopping at the last success; that is, the optimal stopping problem*

$$v := v(p_1, \dots, p_n) := \sup_{\tau} \mathbb{P}(\mathbf{1}_{\tau} = 1, \mathbf{1}_{\tau+1} = \dots = \mathbf{1}_n = 0).$$

Then the optimal rule is to stop on the first index (if any) k with $\mathbf{1}_k = 1$ and $k \geq s$, where

$$s := \sup \left\{ 1, \sup \left\{ 1 \leq k \leq n : \sum_{j=k}^n r_j \geq 1 \right\} \right\},$$

with $\sup\{\emptyset\} := -\infty$. Moreover, the optimal win probability is given by

$$v = v(p_1, \dots, p_n) = \left(\prod_{j=s}^n q_j \right) \left(\sum_{k=s}^n r_k \right).$$

To see how this may be applied to the best-choice problem, let A_i be the event that the i th applicant is best so far. In the classical secretary problem, where only relative ranks are observed and all $n!$ orderings of the applicants are equally likely, the events A_1, \dots, A_n are independent, so Bruss' theorem applies to give the optimal stopping rule. However, in the full-information case we are considering here, we have $A_i = \{X_i \geq \max\{X_1, \dots, X_{i-1}\}\}$ for $i = 1, \dots, n$, and these events are generally *not* independent when X_1, \dots, X_n are not identically distributed. Consider, for instance, the case when $n = 3$, $X_1 \equiv 1$, $X_2 = 0$ or 3 , each with probability $\frac{1}{2}$, and $X_3 = 0, 2$ or 4 each with probability $\frac{1}{3}$; one checks easily that A_2 and A_3 are not independent.

To overcome this issue, we will reduce the problem to that of a sequence of random variables which is simple enough that the sum-the-odds theorem *does* apply to it; see the next section. But first we need a sharp lower bound for the optimal win probability $v(p_1, \dots, p_n)$.

Bruss [2] showed that if $\sum_{k=1}^n r_k \geq 1$ then $v(p_1, \dots, p_n) > e^{-1}$, and this is the best *uniform* lower bound. However, for fixed n we can do somewhat better. Let b_n denote the lower bound in Theorem 1.1, that is,

$$b_n := \left(1 - \frac{1}{n} \right)^{n-1}.$$

Theorem 2.2. *If $\sum_{k=1}^n r_k \geq 1$ then*

$$v = v(p_1, \dots, p_n) \geq b_{n+1} = \left(1 - \frac{1}{n+1} \right)^n.$$

The bound is attained when $p_1 = p_2 = \dots = p_n = 1/(n+1)$.

Lemma 2.1. *The sequence (b_n) is strictly decreasing, and $\lim_{n \rightarrow \infty} b_n = e^{-1}$.*

Proof. This follows since $b_{n+1}^{-1} = (1 + 1/n)^n$, which increases in n and has limit e . □

Proof of Theorem 2.2. We initially follow Bruss' proof [2] of the bound e^{-1} . For convenience, we reindex the p_i, q_i , and r_i by redefining $p_i := p_{n-i+1}, q_i := q_{n-i+1}$, and $r_i := r_{n-i+1}$ for $i = 1, 2, \dots, n$. In this new notation, let $R_k := r_1 + \dots + r_k$ and let $t := \inf\{k : R_k \geq 1\}$. Then we can write v as

$$v = R_t \prod_{k=1}^t q_k. \tag{2.1}$$

If $t = 1$ then $r_1 \geq 1$, and, hence, $v = r_1 q_1 = p_1 \geq \frac{1}{2}$. Thus, the statement holds for $t = 1$. Assume, therefore, that $t \geq 2$. Use $q_j = (1 + r_j)^{-1}$ to write v in the form

$$v = \frac{R_t}{(1 + r_t) \prod_{j=1}^{t-1} (1 + r_j)}. \tag{2.2}$$

Using the inequality of arithmetic and geometric means, we have

$$\left(\prod_{j=1}^{t-1} (1 + r_j) \right)^{1/(t-1)} \leq \frac{\sum_{j=1}^{t-1} (1 + r_j)}{t - 1} = 1 + \frac{R_{t-1}}{t - 1},$$

and substituting this into (2.2), we obtain

$$v \geq \frac{R_t}{(1 + r_t)(1 + R_{t-1}/(t - 1))^{t-1}}.$$

We now refine the analysis from the second part of Bruss' proof. It is simpler to consider v^{-1} ; that is,

$$v^{-1} \leq \frac{(1 + r_t)(1 + R_{t-1}/(t - 1))^{t-1}}{R_{t-1} + r_t} = g_t(R_{t-1}, r_t),$$

where

$$g_t(x, y) := \frac{1 + y}{x + y} \left(1 + \frac{x}{t - 1} \right)^{t-1}.$$

In view of

$$R_{t-1} = \sum_{k=1}^{t-1} r_k < 1 \leq R_t = R_{t-1} + r_t,$$

we need to maximize $g_t(x, y)$ over $0 \leq x < 1, 1 - x \leq y$. For fixed x in this range, $g_t(x, y)$ is clearly decreasing in y , so that

$$g_t(x, y) \leq g_t(x, 1 - x) = (2 - x) \left(1 + \frac{x}{t - 1} \right)^{t-1}.$$

Elementary calculus shows that this function is maximized at $x = x^* := (t - 1)/t$, and so

$$v^{-1} \leq g_t(x^*, 1 - x^*) = \left(\frac{t + 1}{t} \right)^t.$$

Hence,

$$v \geq \left(\frac{t}{t + 1} \right)^t = b_{t+1} \geq b_{n+1},$$

where the last inequality follows from Lemma 2.1. When $p_1 = p_2 = \dots = p_n = 1/(n + 1)$, we have $q_k = n/(n + 1)$ and $r_k = 1/n$ for each k , so that $t = n$ and $R_t = 1$, and (2.1) yields $v = b_{n+1}$. □

3. Proof of the main theorem

We first introduce some notation and terminology. For $j = 1, \dots, n$, let F_j denote the distribution function of X_j . By our assumption, each F_j is continuous. For $k = 1, \dots, n$, we will call X_k a *candidate* if $X_k = \max\{X_1, \dots, X_k\}$. If we observe a candidate $X_k = x$ and we stop, we win if and only if none of the future observations X_{k+1}, \dots, X_n exceeds x . This happens with probability

$$U_k(x) := \mathbb{P}(X_{k+1} \leq x, \dots, X_n \leq x) = \prod_{j=k+1}^n F_j(x). \tag{3.1}$$

On the other hand, if we have observed $X_1 = x_1, \dots, X_k = x_k$ and we continue, we win if and only if we stop at some time $k < \tau \leq n$ and $X_\tau = \max\{m, X_{k+1}, \dots, X_n\}$, where $m := \max\{x_1, \dots, x_k\}$. Thus, the optimal win probability if we continue is

$$\begin{aligned} W_k(m) &:= \sup_{k < \tau \leq n} \mathbb{P}(X_\tau = M_n \mid \max\{X_1, \dots, X_k\} = m) \\ &= \sup_{k < \tau \leq n} \mathbb{P}(X_\tau \geq \max\{m, X_{k+1}, \dots, X_n\}). \end{aligned}$$

It is straightforward to verify, by conditioning on the value of X_{k+1} , the recursion

$$W_k(m) = F_{k+1}(m)W_{k+1}(m) + \int_m^\infty \max\{U_{k+1}(x), W_{k+1}(x)\} dF_{k+1}(x) \tag{3.2}$$

for $k = 1, \dots, n - 1$, since an observation smaller than m at time $k + 1$ would force us to continue.

The following definition identifies a class of sequences of random variables which will turn out to be extremal for our problem.

Definition 3.1. A sequence Y_1, \dots, Y_n of independent random variables will be called a *V-sequence* if there exist constants a_1, a_2, \dots, a_n , and b_2, \dots, b_n with $a_1 < a_2 < \dots < a_n$ and $a_1 > b_2 > b_3 > \dots > b_n$, and

- (i) $\mathbb{P}(Y_1 = a_1) = 1$, and
- (ii) $\mathbb{P}(Y_i = a_i \text{ or } b_i) = 1$ for $i = 2, \dots, n$.

We will show first that, if Y_1, \dots, Y_n is any *V-sequence* then the optimal win probability $V_n(Y_1, \dots, Y_n)$ for this sequence is greater than or equal to the bound in Theorem 1.1. Then, for a given sequence X_1, \dots, X_n of independent continuous random variables, we will construct a specific *V-sequence* Y_1, \dots, Y_n (whose distributions depend on those of X_1, \dots, X_n , of course) such that $V_n(X_1, \dots, X_n) \geq V_n(Y_1, \dots, Y_n)$. This will clearly establish the lower bound of Theorem 1.1.

Lemma 3.1. *If Y_1, \dots, Y_n is a V-sequence then*

$$V_n(Y_1, \dots, Y_n) \geq \left(1 - \frac{1}{n}\right)^{n-1},$$

and this bound is attained.

Proof. For $k = 1, \dots, n - 1$, we define the indicator random variables

$$\mathbf{1}_k := \begin{cases} 1 & \text{if } Y_{k+1} > \max\{Y_1, \dots, Y_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $p_k := \mathbb{E}(\mathbf{1}_k) = \mathbb{P}(Y_{k+1} = a_{k+1})$, $q_k := 1 - p_k$, and $r_k := p_k/q_k$. We have

$$V_n(Y_1, \dots, Y_n) = \max\{U_1(a_1), W_1(a_1)\}. \tag{3.3}$$

Note that if we decide to continue after observing Y_1 , we win if and only if we choose the last success in the sequence $\mathbf{1}_1, \dots, \mathbf{1}_{n-1}$, which is an independent sequence since for each $k \geq 1$, $\mathbf{1}_k = 1$ if and only if $Y_{k+1} = a_{k+1}$. Thus, $W_1(a_1) = v(p_1, \dots, p_{n-1})$, so that if $\sum_{k=1}^{n-1} r_k \geq 1$, Theorem 2.2 gives

$$W_1(a_1) \geq \left(1 - \frac{1}{n}\right)^{n-1}.$$

Suppose now that $\sum_{k=1}^{n-1} r_k \leq 1$. Stopping at the first observation, we obtain the win probability

$$U_1(a_1) = q_1 \cdots q_{n-1} = \prod_{i=1}^{n-1} (1 + r_i)^{-1}.$$

Using the inequality of arithmetic and geometric means, we have

$$\left(\prod_{i=1}^{n-1} (1 + r_i)\right)^{1/(n-1)} \leq \frac{\sum_{k=1}^{n-1} (1 + r_k)}{n - 1} \leq \frac{n}{n - 1}.$$

Thus,

$$U_1(a_1) \geq \left(\frac{n}{n - 1}\right)^{-(n-1)} = \left(1 - \frac{1}{n}\right)^{n-1}.$$

In both cases, (3.3) yields the desired lower bound.

To see that the bound is attained, let $p_i := \mathbb{P}(Y_{i+1} = a_{i+1}) = 1/n$, $q_i := 1 - p_i$, and $r_i := p_i/q_i = 1/(n - 1)$, for $i = 1, \dots, n - 1$. If we skip the first observation and play optimally from then on, we win if and only if we stop at the last success of $\mathbf{1}_1, \dots, \mathbf{1}_{n-1}$, that is, $W_1(a_1) = v(p_1, \dots, p_{n-1})$. Note that $\inf\{s : \sum_{j=s}^n r_j \geq 1\} = 2$. From Theorem 2.1, we obtain

$$v(p_1, \dots, p_{n-1}) = \sum_{j=2}^n \frac{1}{n} \left(\frac{n - 1}{n}\right)^{n-2} = \left(1 - \frac{1}{n}\right)^{n-1}.$$

If, on the other hand, we choose to stop at the first observation, the win probability is $U_1(a_1) = q_1 \cdots q_{n-1} = (1 - 1/n)^{n-1}$ as well. Hence,

$$V_n(Y_1, \dots, Y_n) = \left(1 - \frac{1}{n}\right)^{n-1},$$

as required. □

Lemma 3.2. For the optimal stopping problem (1.1), there exists a sequence of critical values x_1^*, \dots, x_{n-1}^* such that at an observation $1 \leq k < n$ it is optimal to stop if and only if $X_k \geq x_k^*$ and X_k is a candidate. In other words, the optimal stopping rule is

$$\tau^* := \min\{k \leq n : X_k \geq \max\{X_1, \dots, X_{k-1}, x_k^*\}\},$$

or $\tau^* = n$ if no such k exists.

Proof. For each $i = 1, \dots, n-1$, $U_i(x)$ is continuous and increasing, since it is the product of continuous and increasing functions. Moreover, $\lim_{x \rightarrow -\infty} U_i(x) = 0$ and $\lim_{x \rightarrow \infty} U_i(x) = 1$. For each $i = 1, \dots, n-1$, we have

$$W_i(x) = \sup_{i < \tau \leq n} \mathbb{P}(X_\tau \geq \max\{x, X_{i+1}, \dots, X_n\}).$$

This shows that W_i is continuous and decreasing. We claim that

$$\lim_{x \rightarrow -\infty} W_i(x) > 0 \tag{3.4}$$

and

$$\lim_{x \rightarrow \infty} W_i(x) < 1. \tag{3.5}$$

It then follows that the graphs of U_i and W_i intersect each other at some (not necessarily unique) point x_i^* .

To see (3.4), choose $k \in \{i + 1, \dots, n\}$ such that $\mathbb{P}(X_k = \max\{X_{i+1}, \dots, X_n\}) > 0$. Then

$$\begin{aligned} L &:= \lim_{x \rightarrow -\infty} \mathbb{P}(X_k \geq x, X_k = \max\{X_{i+1}, \dots, X_n\}) \\ &= \mathbb{P}(X_k = \max\{X_{i+1}, \dots, X_n\}) \\ &> 0, \end{aligned}$$

so, by considering the stopping rule $\tau \equiv k$,

$$\begin{aligned} W_i(x) &\geq \mathbb{P}(X_k \geq \max\{x, X_{i+1}, \dots, X_n\}) \\ &= \mathbb{P}(X_k \geq x, X_k = \max\{X_{i+1}, \dots, X_n\}) \\ &\rightarrow L > 0 \quad \text{as } x \rightarrow -\infty. \end{aligned}$$

This gives (3.4). Next, choose x such that $\mathbb{P}(\max\{X_{i+1}, \dots, X_n\} \geq x) < 1$. Then

$$W_i(x) \leq \sup_{i < \tau \leq n} \mathbb{P}(X_\tau \geq x) \leq \mathbb{P}(\max\{X_{i+1}, \dots, X_n\} \geq x) < 1,$$

which implies (3.5). □

When X_1, \dots, X_n are i.i.d., the critical values x_1^*, \dots, x_{n-1}^* form a decreasing sequence; see [4]. This remains the case if X_1, \dots, X_n are merely assumed to be independent.

Lemma 3.3. We have $x_1^* \geq x_2^* \geq \dots \geq x_{n-1}^*$.

Proof. It suffices to show that

$$W_{k+1}(x) \geq U_{k+1}(x) \implies W_k(x) \geq U_k(x), \quad x \in \mathbb{R}, k = 1, \dots, n-2.$$

But this is almost obvious, since assuming the inequality on the left, from (3.2), we have

$$W_k(x) \geq F_{k+1}(x)W_{k+1}(x) \geq F_{k+1}(x)U_{k+1}(x) = U_k(x),$$

where the last equality is a consequence of (3.1). □

Lemma 3.4. *For each $i \in \{2, 3, \dots, n - 1\}$, there exists a constant $c_i > x_1^*$ such that*

$$\int_{x_1^*}^{\infty} U_i(x) \, dF_i(x) = U_i(c_i) \mathbb{P}(X_i > x_1^*). \tag{3.6}$$

Proof. If $\mathbb{P}(X_i > x_1^*) = 0$, we can choose $c_i > x_1^*$ arbitrarily, and both sides of (3.6) will be 0. So assume that $\mathbb{P}(X_i > x_1^*) > 0$. Since U_i is increasing and bounded by 1, we have

$$U_i(x_1^*) \mathbb{P}(X_i > x_1^*) \leq \int_{x_1^*}^{\infty} U_i(x) \, dF_i(x) \leq \mathbb{P}(X_i > x_1^*),$$

so

$$U_i(x_1^*) \leq \frac{1}{\mathbb{P}(X_i > x_1^*)} \int_{x_1^*}^{\infty} U_i(x) \, dF_i(x) \leq 1.$$

If the inequality on the left is strict, then, since $U_i(x)$ is continuous and tends to 1 as $x \rightarrow \infty$, we can apply the intermediate value theorem to obtain $c_i > x_1^*$ such that

$$U_i(c_i) = \frac{1}{\mathbb{P}(X_i > x_1^*)} \int_{x_1^*}^{\infty} U_i(x) \, dF_i(x).$$

Otherwise, it must be the case that U_i is constant (and, hence, equal to 1) on $[x_1^*, \infty)$, and we can choose c_i to be any number strictly greater than x_1^* . In either case, we obtain (3.6). \square

Lemma 3.5. (Reduction lemma.) *There exists a V-sequence X'_1, \dots, X'_n such that $V_n(X_1, \dots, X_n) \geq V_n(X'_1, \dots, X'_n)$.*

Proof. Let c_2, \dots, c_{n-1} be the constants from Lemma 3.4. Choose numbers a_2, \dots, a_n and b_2, \dots, b_n such that $x_1^* < a_2 < a_3 < \dots < a_n$ and $a_j \leq c_j$ for $j = 2, \dots, n - 1$, and $x_1^* > b_2 > \dots > b_n$. Set $X'_1 := x_1^*$ and, for $2 \leq i \leq n$, define

$$X'_i := \begin{cases} a_i & \text{if } X_i > x_1^*, \\ b_i & \text{if } X_i \leq x_1^*. \end{cases}$$

Note that the random variables X'_1, \dots, X'_n are independent. The idea of the proof is to replace the random variables X_1, \dots, X_n by their counterparts X'_1, \dots, X'_n one at a time, starting with X_1 , and to show that each such replacement does not increase the optimal win probability. In order to do so, we need to introduce analogs of the functions U_j and W_j for the sequence $X'_1, \dots, X'_k, X_{k+1}, \dots, X_n$, where $k = 0, 1, \dots, n$. First, introduce random variables

$$Y_j^{(k)} := \begin{cases} X'_j & \text{if } 1 \leq j \leq k, \\ X_j & \text{if } k < j \leq n, \end{cases} \quad k = 0, 1, \dots, n, \quad j = 1, 2, \dots, n.$$

Now define

$$U_j^{(k)}(x) := \mathbb{P}(Y_{j+1}^{(k)} \leq x, \dots, Y_n^{(k)} \leq x) = \mathbb{P}(Y_{j+1}^{(k)} \leq x) \dots \mathbb{P}(Y_n^{(k)} \leq x),$$

$$W_j^{(k)}(x) := \sup_{j < \tau \leq n} \mathbb{P}(Y_\tau^{(k)} = \max\{x, Y_{j+1}^{(k)}, \dots, Y_n^{(k)}\}),$$

for $k = 0, 1, \dots, n$ and $j = 1, 2, \dots, n$. We also write

$$V_n^{(k)} := V_n(Y_1^{(k)}, \dots, Y_n^{(k)}), \quad k = 0, 1, \dots, n,$$

so $V_n^{(0)} = V_n(X_1, \dots, X_n)$. Observe that

$$k \leq j \implies W_j^{(k)} = W_j \quad \text{and} \quad U_j^{(k)} = U_j. \tag{3.7}$$

Since

$$\mathbb{P}(X'_i \leq x_1^*) = \mathbb{P}(X'_i = b_i) = \mathbb{P}(X_i \leq x_1^*), \quad i = 2, 3, \dots, n,$$

we also have

$$U_j^{(k)}(x_1^*) = U_j(x_1^*), \quad j = 1, \dots, n - 1, k = 0, 1, \dots, n. \tag{3.8}$$

We will show that, for all $0 \leq k \leq n - 1$, $V_n^{(k)} \geq V_n^{(k+1)}$. First,

$$\begin{aligned} V_n^{(0)} &= V_n(X_1, X_2, \dots, X_n) \\ &= \mathbb{E}[\max\{U_1(X_1), W_1(X_1)\}] \\ &= \int_{-\infty}^{\infty} \max\{U_1(x), W_1(x)\} dF_1(x) \\ &= \int_{-\infty}^{x_1^*} W_1(x) dF_1(x) + \int_{x_1^*}^{\infty} U_1(x) dF_1(x) \\ &\geq \int_{-\infty}^{x_1^*} W_1(x_1^*) dF_1(x) + \int_{x_1^*}^{\infty} U_1(x_1^*) dF_1(x) \\ &= U_1(x_1^*) \\ &= W_1(x_1^*) \\ &= V_n(X'_1, X_2, \dots, X_n) \\ &= V_n^{(1)}, \end{aligned}$$

where the inequality follows since W_1 is decreasing and U_1 is increasing.

Next, we show, for every $1 \leq k \leq n - 1$, that $V_n^{(k)} \geq V_n^{(k+1)}$. In view of (3.8), this will follow if we prove that

$$W_j^{(k)}(x_1^*) \geq W_j^{(k+1)}(x_1^*), \quad j \leq k < n, \tag{3.9}$$

for $j = 1, \dots, n - 1$; we can then take $j = 1$. We prove (3.9) by backward induction on j . First, (3.9) holds for $j = n - 1$ since

$$W_{n-1}^{(n-1)}(x_1^*) = \mathbb{P}(X_n > x_1^*) = \mathbb{P}(X'_n > x_1^*) = W_{n-1}^{(n)}(x_1^*).$$

Suppose that (3.9) holds with j replaced by $j + 1$; that is,

$$W_{j+1}^{(k)}(x_1^*) \geq W_{j+1}^{(k+1)}(x_1^*), \quad j < k < n. \tag{3.10}$$

We first claim that

$$U_{j+1}^{(k)}(a_{j+1}) \geq U_{j+1}^{(k+1)}(a_{j+1}), \quad j < k < n. \tag{3.11}$$

This follows since $a_{j+1} > x_1^* \geq x_{j+1}^*$, $U_{j+1}^{(k)}$ is increasing and $W_{j+1}^{(k)}$ is decreasing, and so, for $j < k < n$,

$$\begin{aligned} U_{j+1}^{(k)}(a_{j+1}) &\geq U_{j+1}^{(k)}(x_1^*) \\ &= U_{j+1}(x_1^*) \\ &\geq W_{j+1}(x_1^*) \\ &= W_{j+1}^{(j+1)}(x_1^*) \\ &\geq W_{j+1}^{(k)}(x_1^*) \\ &\geq W_{j+1}^{(k)}(a_{j+1}), \end{aligned}$$

where we used (3.8) in the first equality, (3.7) in the second equality, and the induction hypothesis (3.10) in the third inequality.

We will also need

$$U_{j+1}^{(k)}(a_{j+1}) \geq U_{j+1}^{(k+1)}(a_{j+1}), \quad j < k < n. \tag{3.12}$$

To see this, note that

$$\frac{U_{j+1}^{(k)}(a_{j+1})}{U_{j+1}^{(k+1)}(a_{j+1})} = \frac{\mathbb{P}(X_{k+1} \leq a_{j+1})}{\mathbb{P}(X'_{k+1} \leq a_{j+1})},$$

and

$$\mathbb{P}(X'_{k+1} \leq a_{j+1}) = \mathbb{P}(X'_{k+1} = b_{k+1}) = \mathbb{P}(X_{k+1} \leq x_1^*) \leq \mathbb{P}(X_{k+1} \leq a_{j+1}),$$

since $k > j$ implies that $a_{k+1} > a_{j+1}$. Now using (3.11), the induction hypothesis (3.10), (3.12), and finally (3.11) again, we obtain, for $k > j$,

$$\begin{aligned} W_j^{(k)}(x_1^*) &= \mathbb{P}(X'_{j+1} = b_{j+1})W_{j+1}^{(k)}(x_1^*) + \mathbb{P}(X'_{j+1} = a_{j+1})U_{j+1}^{(k)}(a_{j+1}) \\ &\geq \mathbb{P}(X'_{j+1} = b_{j+1})W_{j+1}^{(k+1)}(x_1^*) + \mathbb{P}(X'_{j+1} = a_{j+1})U_{j+1}^{(k+1)}(a_{j+1}) \\ &= W_j^{(k+1)}(x_1^*). \end{aligned}$$

It remains to verify (3.9) for $k = j$. Here, we use (3.6) and the fact (by our choice of a_{j+1}) that $a_{j+1} \leq c_{j+1}$ in order to obtain

$$\begin{aligned} W_j^{(j)}(x_1^*) &= W_j(x_1^*) \\ &= \mathbb{P}(X_{j+1} \leq x_1^*)W_{j+1}(x_1^*) + \int_{x_1^*}^{\infty} U_{j+1}(x) \, dF_{k+1}(x) \\ &\geq \mathbb{P}(X_{j+1} \leq x_1^*)W_{j+1}(x_1^*) + \mathbb{P}(X_{j+1} > x_1^*)U_{j+1}(a_{j+1}) \\ &= \mathbb{P}(X'_{j+1} = b_{j+1})W_{j+1}^{(j+1)}(x_1^*) + \mathbb{P}(X'_{j+1} = a_{j+1})U_{j+1}^{(j+1)}(a_{j+1}) \\ &= W_j^{(j+1)}(x_1^*), \end{aligned}$$

where we used (3.7) in the third equality, and (3.11) in the last equality. This completes the backward induction. Finally, setting $j = 1$ in (3.9), and using (3.8) and

$$V_n^{(k)} = \max\{U_1^{(k)}(x_1^*), W_1^{(k)}(x_1^*)\}, \quad k = 1, 2, \dots, n,$$

it follows that $V_n^{(k)} \geq V_n^{(k+1)}$ for $k = 1, \dots, n - 1$. This proves the lemma. □

Proof of Theorem 1.1. The inequality (1.2) is an immediate consequence of Lemma 3.5 and Lemma 3.1. The bound is attained by replacing the V -sequence that attains the bound in Lemma 3.1 with a continuous sequence, as follows. Let X_1, \dots, X_n be independent random variables such that X_1 has the uniform distribution on $(0, 1)$, and, for $i = 2, \dots, n$, X_i has density function

$$f_i(x) := \begin{cases} \frac{1}{n} & \text{if } i < x < i + 1, \\ \frac{n-1}{n} & \text{if } -i < x < -i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, since the supports of X_1, \dots, X_n do not overlap, the optimal win probability $V_n(X_1, \dots, X_n)$ is the same as the optimal win probability for the V -sequence that attains the bound in Lemma 3.1. That is, $V_n(X_1, \dots, X_n) = (1 - 1/n)^{n-1}$. \square

Acknowledgement

The authors thank the anonymous referee for several helpful suggestions that led to an improved presentation of the paper.

References

- [1] BRUSS, F. T. (2000). Sum the odds to one and stop. *Ann. Prob.* **28**, 1384–1391.
- [2] BRUSS, F. T. (2003). A note on bounds for the odds theorem of optimal stopping. *Ann. Prob.* **31**, 1859–1861.
- [3] FERGUSON, T. S. (1989). Who solved the secretary problem? With comments and a rejoinder by the author. *Statist. Sci.* **4**, 282–296.
- [4] GILBERT, J. P. AND MOSTELLER, F. (1966). Recognizing the maximum of a sequence. *J. Amer. Statist. Assoc.* **61**, 35–73.
- [5] SAMUELS, S. M. (1991). Secretary problems. In *Handbook of Sequential Analysis*, Dekker, New York, pp. 381–405.