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INVARIANT VECTORS OF A NONRECURRENT MARKOV MATRIX

BY

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ABSTRACT. Let P be an irreducible Markov matrix. Assume:

$$p_{ij} = 0 \text{ if } |i-j| > K \quad -\infty < i < \infty.$$

We study conditions for such a matrix to be nonrecurrent. If P is nonrecurrent we study the invariant vectors of P (invariant column vectors and invariant row vectors).

1. **Notation.** Let $P = (p_{i,j}), -\infty < i, j < \infty$, be a Markov matrix, namely:

$$p_{i,j} \geqq 0; \sum_{j=-\infty}^{\infty} p_{i,j} = 1.$$

Let us denote column vectors by f, g, h and row vectors by u, v, w. Thus

$$Pf(i) = \sum_{j=-\infty}^{\infty} p_{i,j}f(j), uP(j) = \sum_{i=-\infty}^{\infty} u(i)p_{i,j}.$$

If P is non-recurrent then:

$$\sum_{n=0}^{\infty} P^n 1_X \in \ell_\infty, \text{ whenever } X \text{ is finite};$$

$$\sum_{n=0}^{\infty} uP^n < \infty, \text{ whenever } 0 \leqq u \in \ell_1.$$

P is called irreducible if given i, j there exists an $n > 0$ with $p_{i,j}^{(n)} > 0$. If P is recurrent and irreducible (ergodic) then:

$$f \geqq 0, Pf \leqq f \Rightarrow f = \text{Const}.$$

If P is irreducible then P is either recurrent or nonrecurrent. An elementary discussion of these notions is given in [2, Chapter II].

Our main results are:

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(1) Changing a finite number of rows does not effect nonrecurrence.

(2) If the changed rows have a finite number of non-zero terms then the two cones of invariant vectors are isomorphic.

(3) Computations of the cones of invariant vectors for some simple cases.

Put

$$\langle u,f\rangle = \sum_{i=-\infty}^{\infty} u(i)f(i)$$

Let $0 \leqq h$, $h(i) \leqq 1$; $0 \leqq w$, $\sum w(i) \leqq 1$. Then $h \otimes w$ is the Markov matrix $(h \otimes w)_{i,j} = h(i)w(j)$.

Thus: $(h \otimes w)f = \langle w,f\rangle h$; $u(h \otimes w) = \langle u,h\rangle w$; $P(h \otimes w) = (Ph) \otimes w$; $(h \otimes w)P = h \otimes (wP)$.

2. **Invariant column vectors.** Given a Markov matrix P choose

$$h_i(j) = \delta_{i,j}; w_i(j) = p_{i,j}; -r \leqq i \leqq r.$$

Then

$$P = \sum_{i=-\infty}^{r} h_i \otimes w_i + Q$$

where: Q is the matrix defined by

$$q_{i,j} = 0 \quad |i| \leqq r; \quad q_{i,j} = p_{i,j}|i| > r.$$

Denote

$$I(P) = \{f : \quad 0 \leqq f < \infty, Pf = f\}.$$
$$I(Q) = \{g : \quad 0 \leqq g < \infty, Qg = g\}.$$

If $f \in I(P)$ then $Qf \leqq Pf = f$ hence:

$$Q_\infty f = \lim_{n\to\infty} Q^n f \leqq f; Q_\infty f \in I(Q).$$

On the other hand, if $g \in I(Q)$ then $Pg \geqq Qg = g$ hence $g \leqq P_\infty g = \lim_{n\to\infty} P^n g \leqq \infty$; *if* $P_\infty g < \infty$ *then* $P_\infty g \in I(P)$.

LEMMA 2.1. *If P is nonrecurrent then*

$$\sum_{n=0}^{\infty} P^n h_i \in \ell_\infty.$$

If $f \geqq 0$ satisfies $\langle w_i,f\rangle < \infty, -r \leqq i \leqq r$ then

$$\sum_{n=0}^{\infty} P^n(P - Q)f \in \ell_\infty.$$

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PROOF. Fix i and choose a finite set X with $\langle w_i, 1_X\rangle \neq 0$ then:

$$\langle w_i, 1_X\rangle \sum_{n=0}^{\infty} P^n h_i = \sum_{n=0}^{\infty} P^n(h_i \otimes w_i)1_X \leqq \sum_{n=0}^{\infty} P^{n+1} 1_X \in \ell_\infty.$$

Now $(P-Q)f = \sum_{i=-r}^{r} \langle w_i, f\rangle h_i$.

THEOREM 2.2. *Let P be nonrecurrent. If*

$$f \in I(P) \text{ then } P_\infty Q_\infty f = f.$$

PROOF. $\langle w_i, f\rangle = f(i)$ since $Pf = f$. Thus

$$\sum_{n=0}^{\infty} P^n(P-Q)f \in \ell_\infty.$$

Now

$$\sum_{n=0}^{N} Q^n(P-Q)f = \sum_{n=0}^{N} Q^n(I-Q)f = f - Q^{N+1}f$$

therefore

$$f = Q_\infty f + \sum_{n=0}^{\infty} Q^n(P-Q)f.$$

Now $P^k Q_\infty f \leqq P^k f = f$ thus $P_\infty Q_\infty f \leqq f$. On the other hand

$$\begin{aligned} f = P^k f &= P^k Q_\infty f + P^k \sum_{n=0}^{\infty} Q^n(P-Q)f \\ &\leqq P_\infty Q_\infty f + P^k \sum_{n=0}^{\infty} P^n(P-Q)f \\ &= P_\infty Q_\infty f + \sum_{n=k}^{\infty} P^n(P-Q)f \;\underset{k\to\infty}{\downarrow}\; P_\infty Q_\infty f. \end{aligned}$$

NOTE. $\langle w_i, Q_\infty f\rangle \leqq \langle w_i, f\rangle < \infty$.

THEOREM 2.3. *Let P be nonrecurrent. If $g \in I(Q)$ and $\langle w_i, g\rangle < \infty$, $-r \leqq i \leqq r$, then:*

$$P_\infty g < \infty; P_\infty g \in I(P); Q_\infty P_\infty g = g.$$

PROOF. By Lemma 2.1

$$\sum_{n=0}^{\infty} P^n(P-Q)g \in \ell_\infty.$$

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Now

$$\sum_{n=0}^{N} P^n(P-Q)g = \sum_{n=0}^{N} P^n(P-I)g = P^{N+1}g - g.$$

Therefore

$$P_\infty g = g + \sum_{n=0}^{\infty} P^n(P-Q)g < \infty$$

hence $P_\infty g \in I(P)$. Finally

$$g \leqq P_\infty g \Rightarrow g \leqq Q^k P_\infty g \Rightarrow g \leqq Q_\infty P_\infty g.$$

On the other hand

$$Q^k P_\infty g = g + Q^k \sum_{n=0}^{\infty} P^n(P-Q)g \leqq g + \sum_{n=k}^{\infty} P^n(P-Q)g \underset{k\to\infty}{\downarrow} g.$$

We have established: $I(P)$ and $I(Q) \cap \{g : \langle g, w_i \rangle < \infty, -r \leqq i \leqq r\}$ are isomorphic. The isomorphism is given by Q_∞ and P_∞.

Let P be nonrecurrent. Then $P_\infty Q_\infty 1 = 1$ thus $Q_\infty 1 \neq 0$ On the other hand if P is recurrent and irreducible then $PQ_\infty 1 \geqq Q_\infty 1 \Rightarrow Q_\infty 1 = \text{Const}, (P-Q)Q_\infty 1 = 0$. Therefore $Q_\infty 1 = 0$.

Let us summarize:

THEOREM 2.4. *Let P be irreducible. P is nonrecurrent if and only if $Q_\infty 1 \neq 0$.*

NOTE. $Q_\infty 1 \in I(Q) \cap \ell_\infty$. If $0 \leqq g \leqq 1$, $Qg = g$ then $g = Q^n g \leqq Q^n 1 \Rightarrow g \leqq Q_\infty 1 : Q_\infty 1 \neq 0 \Leftrightarrow I(Q) \cap \ell_\infty \neq \{0\}$.

COROLLARY: *Let P, P_1 be irreducible. If $p_{i,j} = p_{i,j}^{(1)}, |i| > r$, then they are nonrecurrent together.*

PROOF. $Q(P) = Q(P_1)$. □

From Theorems 2.2 and 2.3 we conclude:

THEOREM 2.5. *Let P, P_1 be nonrecurrent. If $p_{i,j} = p_{i,j}^{(1)}, |i| > r$; $p_{i,j} = p_{i,j}^{(1)} = 0, |i| \leqq r$ and $|j| > K$. Then $I(P)$ and $I(P_1)$ are isomorphic.*

PROOF. $Q(P) = Q(P_1)$ and $I(P) \sim I(Q) \quad I(P_1) \sim I(Q)$. □

3. **Invariant row vectors.** Let P be a Markov matrix, choose

$$h_i^1(j) = p_{j,i}, \quad |i| \leqq r;$$
$$w_i^1(j) = \delta_{i,j}, \quad |i| \leqq r.$$
$$P = \sum_{i=-r}^{r} h_i^1 \otimes w_i^1 + S$$

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where

$$s_{i,j} = 0, |j| \leqq r;$$
$$s_{i,j} = p_{i,j}, |j| > r.$$

Denote

$$J(P) = \{u : \quad 0 \leqq u < \infty, uP = u\}$$
$$J(S) = \{v : \quad 0 \leqq v < \infty, vS = v\}.$$

If $u \in J(P)$ then $uS \leqq uP = u$ hence:

$$uS_\infty = \lim_{n\to\infty} uS^n \leqq u; uS_\infty \in J(S).$$

On the other hand, if $v \in J(S)$ then $vP \geqq vS = v$ hence:

$$v \leqq vP_\infty = \lim_{n\to\infty} vP^n \leqq \infty.$$

if $vP_\infty < \infty$ then $vP_\infty \in J(P)$.

LEMMA 3.1. *Let P be nonrecurrent. If $u \geqq 0$ and $\langle u, h_i^1\rangle < \infty, |i| \leqq r$, then*

$$\sum_{n=0}^{\infty} u(P-S)P^n < \infty.$$

PROOF.

$$u(P-S) = \sum_{i=-r}^{r} \langle u, h_i^1\rangle w_i^1$$

and

$$\sum_{n=0}^{\infty} w_i^1 P^n < \infty, \text{ since } w_i^1 \in \ell_1.$$

THEOREM 3.2. *Let P be nonrecurrent. If $u \in J(P)$ then $\langle u, h_i^1\rangle < \infty, |i| \leqq r$, and $uS_\infty P_\infty = u$.*

PROOF.

$$\sum_{n=0}^{N} u(P-S)S^n = \sum_{n=0}^{N} u(I-S)S^n = u - uS^{N+1}$$

hence

$$u = uS_\infty + \sum_{n=0}^{\infty} u(P-S)S^n.$$

Therefore $(uS_\infty)P^k \leqq uP^k = u$ hence

$$uS_\infty P_\infty \leqq u.$$

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On the other hand

$$u = uP^k = (uS_\infty)P^k + \sum_{n=0}^{\infty} u(P-S)S^n)P^k$$
$$\leqq uS_\infty P_\infty + \sum_{n=k}^{\infty} u(P-S)P^n \underset{k\to\infty}{\downarrow} uS_\infty P_\infty$$

by Lemma 3.1 since

$$\langle u, h_i^1 \rangle = (uP)(i) = u(i) < \infty.$$

THEOREM 3.3. *Let P be nonrecurrent. If $v \in J(S)$ and $\langle v, h_i^1 \rangle < \infty, |i| \leqq r$, then $vP_\infty < \infty$; $vP_\infty \in J(P)$; $vP_\infty S_\infty = v$.*

PROOF.

$$\sum_{n=0}^{N} v(P-S)P^n = \sum_{n=0}^{N} v(P-I)P^n = vP^{N+1} - v$$

hence, by Lemma 3.1.,

$$vP_\infty = v + \sum_{n=0}^{\infty} v(P-S)P^n < \infty.$$

Clearly $vP_\infty \in J(P)$ also $v \leqq vP_\infty$ hence

$$v \leqq vP_\infty S_\infty.$$

Finally

$$(vP_\infty)S^k = v + \left(\sum_{n=0}^{\infty} v(P-S)P^n \right) S^k$$
$$\leqq v + \sum_{n=k}^{\infty} v(P-S)P^n \underset{k\to\infty}{\downarrow} v$$

by Lemma 3.1. □

We have established that $J(P)$ and $J(S) \cap \{v : \langle v, h_i^1 \rangle < \infty\}$ are isomorphic.

COROLLARY. *Let P be non recurrent. If $p_{i,j} = 0, |j| \leqq r, |i| > K$ then $J(P)$ and $J(S)$ are isomorphic.*

Another way of putting this is

THEOREM 3.4. *Let P, P_1 be nonrecurrent. If $p_{i,j} = p_{i,j}^{(1)}, |j| > r$; $p_{i,j} = p_{i,j}^{(1)} = 0$, $|j| \leqq r$ and $|i| > K$. Then $J(P)$ and $J(P_1)$ are isomorphic.*

4. **Examples.**

(a) 3 *diagonals:* Let P satisfy $p_{i,j} = 0 \quad |i-j| > 1$; $p_{i,i-1} > 0 \quad p_{i,i+1} > 0$. Thus: P is irreducible.

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Let us compute $I(Q)$: $g_0 = 0$, $g_1 = 1$ (normalization),

$$g_n = p_{n,n-1}g_{n-1} + p_{n,n}g_n + p_{n,n+1}g_{n+1}, n > 1.$$

Thus

$$\begin{aligned} g_{n+1} &= \frac{1}{p_{n,n+1}}\left(g_n(1-p_{n,n}) - g_{n-1}p_{n,n-1}\right) \\ &= g_n + \frac{p_{n,n-1}}{p_{n,n+1}}(g_n - g_{n-1}) \\ g_n &= g_1\left(1 + \sum_{i=1}^{n-1}\frac{p_{1,0}}{p_{1,2}}\cdots\frac{p_{i,i-1}}{p_{i,i+1}}\right) n > 1. \end{aligned}$$

A similar calculation will show that

$$g_{-n} = g_{-1}\left(1 + \sum_{i=1}^{n-1}\frac{p_{-1,0}}{p_{-1,-2}}\cdots\frac{p_{-i,-i+1}}{p_{-i,-i-1}}\right) n > 1$$

Thus P is *recurrent* $\Leftrightarrow$ *both series diverge*. (See [1, p. 67–69] for a similar result). *If P is nonrecurrent then $I(P)$ is generated by two non negative invariant vectors. If P is recurrent then we found $g \geqq 0$ with $Pg \geqq g$ and $Pg \neq g$.*

Let us compute $J(S)$: $v_0 = 0$; $v_1 = 1$ (normalization).

$$\begin{aligned} (*) v_{n+1} &= \frac{1}{p_{n+1,n}}\left(v_n(1-p_{n,n}) - v_{n-1}p_{n-1,n}\right) \\ &= \frac{1}{p_{n+1,n}}(v_n p_{n,n-1} + v_n p_{n,n+1} - v_{n-1}p_{n-1,n}) \end{aligned}$$

Thus

$$v_2 = \frac{1}{p_{2,1}}(p_{1,0} + v_1 p_{1,2})$$

Let us prove by induction that

$$n > 2 \quad v_n = \frac{1}{P_{n,n-1}}(p_{1,0} + v_{n-1}p_{n-1,n}) :$$

$$v_n p_{n,n-1} - v_{n-1}p_{n-1,n} = p_{1,0} \text{ hence, by } (*),$$

$$v_{n+1} = \frac{1}{p_{n+1,n}}(p_{1,0} + v_n p_{n,n+1}).$$

Therefore $v_n > 0$.

A similar calculation for $n < 0$ will show that:

If P is nonrecurrent then $J(P)$ is generated by two non negative invariant vectors. If P is recurrent we found $v \geqq 0$ with $vP \geqq v$ but $vP \neq v$.

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(b) 5 *diagonals;* Let P satisfy $p_{i,j} = 0 \; |i-j| > 2$; $p_{i,i-2} \neq 0$, $p_{i,i-1} \neq 0$, $p_{i,i+1} \neq 0$, $p_{i,i+2} \neq 0$. Thus P is irreducible.

Let us compute $I(Q)$: $g_{-1} = g_0 = g_1 = 0$; $g_2 + g_3 = 1$ (normalization)

$$g_{n+2} = \frac{1}{p_{n,n+2}}\big[p_{n,n-2}(g_n - g_{n-2}) + p_{n,n-1}(g_n - g_{n-1}) + p_{n,n+1}(g_n - g_{n+1})\big] + g_n$$

Thus g_n is a linear function of g_2, g_3. $I(Q)$ is isomorphic to the set

$$\bigcap_{n=3}^{\infty}\{g_n \geqq 0\}.$$

This is a compact convex subset of an interval and is either empty, or a point, or an interval. Similar calculations apply to $n < 0$:

$I(P)$ is generated by k vectors, $1 \leqq k \leqq 4$ and at least one vector is bounded.

(c) $2k+1$ *diagonals:* Let P satisfy $p_{i,j} = 0$, $|i-j| > k$; $p_{i,j} \neq 0 \; j \neq i, |i-j| \leqq k$. Thus P is irreducible and $P1 = 1$.

Let us compute $I(Q)$: $g_{-k+1} = \cdots = g_0 = \cdots = g_{k-1} = 0$; $g_k + \cdots + g_{2k-1} = 1$ (normalization). Then g_n is a linear function of $g_k, \ldots, g_{2k-1}$. Thus $I(P) = aA + bB$ $a, b \geqq 0 A, B$ convex compact subsets of

$$\left\{(g_k, \ldots, g_{2k-1}) : \quad g_j \geqq 0, \sum_{j=k}^{2k-1} g_j = 1\right\}.$$

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