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A NEW COARSELY RIGID CLASS OF BANACH SPACES

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Abstract We prove that the class of reflexive asymptotic-c_0 Banach spaces is coarsely rigid, meaning that if a Banach space X coarsely embeds into a reflexive asymptotic-c_0 space Y, then X is also reflexive and asymptotic-c_0. In order to achieve this result, we provide a purely metric characterization of this class of Banach spaces. This metric characterization takes the form of a concentration inequality for Lipschitz maps on the Hamming graphs, which is rigid under coarse embeddings. Using an example of a quasi-reflexive asymptotic-c_0 space, we show that this concentration inequality is not equivalent to the non-equi-coarse embeddability of the Hamming graphs.

Keywords: Coarse embeddings; coarsely rigid classes of banach spaces; asymptotic-c_0 spaces; Hamming graphs

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Contents

1. Introduction

The concept of rigidity for a class of mathematical objects has permeated mathematical fields. A prime example of a rigidity problem arose in geometric group theory. Take a finitely generated group Γ which is an algebraic object. One can apprehend Γ in the category of metric spaces by looking at its Cayley graph. Then, a fundamental aspect of Gromov's geometric group theory program [10] is to understand how much of the algebraic properties of a group one can recover knowing solely the large-scale shape of its Cayley graph. A class $\mathcal{G}$ of groups is said to be quasi-isometrically rigid if every group that is quasi-isometric to a group in $\mathcal{G}$ is actually virtually isomorphic to a group in $\mathcal{G}$. A quasi-isometric embedding is what we call a coarse-Lipschitz embedding in this paper (see all the relevant definitions of non-linear embeddings in § 2.3). It is quite remarkable that many classes of groups are known to be quasi-isometrically rigid: free groups, hyperbolic groups and amenable groups, and we refer to [15] for a detailed list.

In this work, we provide a Banach space analogue of this type of results. A class $\mathcal{C}$ of Banach spaces is called *coarsely rigid* if it follows from Y being a member of $\mathcal{C}$ and X being coarsely embedded into Y, that X is also in $\mathcal{C}$. Let us insist on the fact that coarse embeddings are very weak embeddings. Indeed, it is classical that L_1 coarsely embeds into ℓ_2 (while it does not coarse-Lipschitz embed). On the other hand, Nowak [20] showed that for any $p \in [1, \infty)$, ℓ_2 coarsely embeds into ℓ_p. This was extended by Ostrovskii [22] who proved that ℓ_2 coarsely embeds into any Banach space with an unconditional basis and of non-trivial cotype. On a more elementary level, note that $\mathbb{R}$ coarse-Lipschitz embeds into $\mathbb{Z}$. Therefore, coarsely rigid classes are rare. The class of spaces that coarsely embeds into a fixed metric space (M, d) or the class of spaces in which a fixed (M, d) does not coarsely embed is clearly coarsely rigid. It is, for instance, rather simple to see that a Banach space X has dimension less than $n \in \mathbb{N}$ if and only if the integer grid $\mathbb{Z}^n$ equipped with the ℓ_1 metric does not coarsely embed into X. Besides such simple coarsely rigid classes, very few rigidity results have been obtained so far. Let us describe three important examples. Randrianarivony showed in [23] that a quasi-Banach space X coarsely embeds into a Hilbert space if and only if there is a probability space (Ω, B, μ) such that X is linearly isomorphic to a subspace of $L_0(\Omega, B, \mu)$. This clearly describes a class of quasi-Banach spaces that is coarsely rigid. Then, a major achievement by Mendel and Naor [19] was a purely metric extension

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of the linear notion of Rademacher cotype. Using that notion of *metric cotype*, they were able to show that within the class of Banach spaces with non-trivial type, the class $\{X\colon \inf\{q' \geqslant 2\colon X \text{ has Rademacher cotype } q'\} \leqslant q\}$ is coarsely rigid. It is still unclear and important to understand whether the non-trivial type restriction is necessary. Another important rigidity result was obtained by Kalton [14]. Indeed, he showed that, within the class of Banach spaces that do not have ℓ_1-spreading models (or, equivalently, spaces with the alternating Banach–Saks property), the class of reflexive Banach spaces is coarsely rigid. It then follows from an ultraproduct argument that, within the class of Banach spaces with non-trivial type, the class of super-reflexive Banach spaces is coarsely rigid. Since ℓ_1 coarsely embeds into ℓ_2, we need at least to exclude spaces which contain ℓ_1 to obtain both conclusions. The last papers of N. Kalton ([14] among others, and see also the survey [9] and references therein) show that asymptotic structures of Banach spaces often provide linear properties that are invariant under coarse or coarse-Lipschitz embeddings. Our work follows this program, studying the links between asymptotic structures and large-scale geometry of Banach spaces.

In this article, we exhibit a new example of an unrestricted class of infinite-dimensional Banach spaces which is coarsely rigid. The notion of an *asymptotic-c_0 space* will be recalled in § 3.

Theorem A. *Let Y be a reflexive asymptotic-c_0 Banach space. If X is a Banach space that coarsely embeds into Y, then X is also reflexive and asymptotic-c_0.*

Since there are reflexive asymptotic-c_0 spaces, like Tsirelson's original space T^* [24], which will be recalled later, Theorem A immediately implies the main result from [3], where the existence of an infinite-dimensional Banach space that does not coarsely contain ℓ_2 is proved. Our proof of Theorem A, which is carried out in § 4, follows from the following purely metric characterization of the linear property of being 'reflexive and asymptotic-c_0' in terms of a concentration inequality for Lipschitz maps on the Hamming graphs (see the definition and notation in § 2.2).

Theorem B. *A Banach space X is reflexive and asymptotic-c_0 if and only if there exists $C \geqslant 1$ such that for every $k \in \mathbb{N}$ and every Lipschitz map $f : ([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)}) \to X$, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ so that*

$$\operatorname{diam}(f([\mathbb{M}]^k)) \leqslant C \operatorname{Lip}(f).$$

This concentration inequality was introduced in [3] where it was shown to hold for maps taking values into Tsirelson's original space T^*. The space T^* is the prototypical example of a separable reflexive asymptotic-c_0 Banach space, and the proof from [3] can be generalized to show that the same concentration inequality holds for maps with values into any reflexive asymptotic-c_0 Banach space. The more conceptual approach undertaken in this article to prove that any reflexive asymptotic-c_0 Banach space satisfies the above metric concentration inequality and requires the central notion of asymptotic structure from [18] which is described in § 3. In order to prove the converse, the crucial step is to show that if a Banach space X satisfies the metric concentration inequality, then all its asymptotic models generated by weakly null arrays are isomorphic to c_0. The notion of asymptotic models was introduced by Halbeisen and Odell in [11]. Then the conclusion

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follows from an unexpected link between the notions *asymptotic structure* and *asymptotic models* (see § 3). Indeed, it was proved by Freeman *et al.* [8] that a separable Banach space which does not contain a copy of ℓ_1 is asymptotic-c_0 whenever all its asymptotic models generated by weakly null arrays are isomorphic to c_0.

The concentration inequality in Theorem B clearly prevents the equi-coarse embeddability of the sequence of Hamming graphs. We show in § 5 that the converse is not true. More precisely, we provide an example of a non-reflexive Banach space in which the Hamming graphs do not equi-coarsely embed.

2. Preliminaries

2.1. Trees

For $k \in \mathbb{N}$, we put $[\mathbb{N}]^{\leqslant k} = \{S \subset \mathbb{N} : |S| \leqslant k\}$, $[\mathbb{N}]^k = \{S \subset \mathbb{N} : |S| = k\}$, $[\mathbb{N}]^{<\omega} = \bigcup_{k\in\mathbb{N}} [\mathbb{N}]^{\leqslant k}$, $[\mathbb{N}]^\omega = \{S \subset \mathbb{N} : S \text{ infinite}\}$ and $[\mathbb{N}] = \{S : S \subset \mathbb{N}\}$. We always list the elements of some $\bar{m} \in [\mathbb{N}]^{<\omega}$ or in $[\mathbb{N}]^\omega$ in increasing order, i.e., if we write $\bar{m} = \{m_1, m_2, \ldots, m_l\}$ or $\bar{m} = \{m_1, m_2, m_3, \ldots\}$, we tacitly assume that $m_1 < m_2 < \cdots$. For $\bar{m} = \{m_1, m_2, \ldots, m_r\} \in [\mathbb{N}]^{\leqslant k}$ and $\bar{n} = \{n_1, n_2, \ldots, n_s\} \in [\mathbb{N}]^{\leqslant k}$, we write $\bar{m} \prec \bar{n}$, if $r < s \leqslant k$ and $m_i = n_i$, for $i = 1, 2, \ldots, r$, and we write $\bar{m} \preceq \bar{n}$ if $\bar{m} \prec \bar{n}$ or $\bar{m} = \bar{n}$. Note that $[\mathbb{N}]^{\leqslant k}$, $k \in \mathbb{N}$ with $\prec$, are *rooted trees*, i.e., partial orders with a unique minimal element, namely $\emptyset$, and the property that for each $\bar{n} \in [\mathbb{N}]^{\leqslant k}$, the set of *predecessors of* $\bar{n}$ $\{\bar{m} : \bar{m} \prec \bar{n}\}$ is finite and linearly ordered.

In this paper, we will only consider trees of finite height. For a set X, we will call a family $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$, for $k \in \mathbb{N}$, a *tree of height* k. Sometimes we are also considering *unrooted trees of height* k, which are families of the form $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k} \setminus \emptyset)$. We call for $\bar{n} \in [\mathbb{N}]^k$ a sequence of the form $(x_{\bar{m}} : \bar{m} \preceq \bar{n}) = (x_{\{n_1,n_2,\ldots,n_l\}})_{l=0}^k$ *a branch of the tree* $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$, and $(x_{\bar{m}} : \emptyset \prec \bar{m} \preceq \bar{n}) = (x_{\{n_1,n_2,\ldots,n_l\}})_{l=1}^k$ *a branch of the unrooted tree* $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k} \setminus \{\emptyset\})$. Sequences of the form $(x_{\bar{n}\cup\{i\}})_{i>\max(\bar{n})}$, where $\bar{n} \in [\mathbb{N}]^{\leqslant k-1}$ (for a tree of height k), are called *nodes* of the tree $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$.

If $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$ is a tree in X and $\mathbb{M} = \{m_1, m_2, \ldots\} \in [\mathbb{N}]^\omega$, we call $(x_{\bar{m}} : \bar{m} \in [\mathbb{M}]^{\leqslant k})$ a *refinement of* $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$. By relabeling $\tilde{x}_{\bar{n}} = x_{\{m_i : i \in \bar{n}\}}$, for $\bar{n} \in [\mathbb{N}]^{\leqslant k}$, the family $(\tilde{x}_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$ is a tree which we also call a refinement of $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$.

If X is a Banach space, we call a tree $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$ in X *normalized* if $x_{\bar{n}} \in S_X$, for all $\bar{n} \in [\mathbb{N}]^{\leqslant k}$, and weakly convergent or weakly null if all its nodes are weakly converging or weakly null, respectively. Here S_X denotes the unit sphere in X, while B_X denotes the closed unit ball.

2.2. Hamming graph on $[\mathbb{N}]^k$

For $k \in \mathbb{N}$ and $\bar{m} = \{m_1, m_2, \ldots, m_k\}$ and $\bar{n} = \{n_1, n_2, \ldots, n_k\}$ in $[\mathbb{N}]^k$, we define the *Hamming distance* by

$$d_{\mathbb{H}}^{(k)}(\bar{m}, \bar{n}) = |\{i \in \{1, 2, \ldots, k\} : m_i \neq n_i\}| \tag{1}$$

and put $\mathbb{H}_k^\omega = ([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)})$.

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2.3. Embeddings

Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$. One defines

$$\rho_f(t) = \inf\{d_Y(f(x), f(y)) : d_X(x, y) \geqslant t\}$$

and

$$\omega_f(t) = \sup\{d_Y(f(x), f(y)) : d_X(x, y) \leqslant t\}.$$

Note that for every $x, y \in X$,

$$\rho_f(d_X(x, y)) \leqslant d_Y(f(x), f(y)) \leqslant \omega_f(d_X(x, y)). \tag{2}$$

The moduli ρ_f and ω_f will be called the *compression modulus* and the *expansion modulus* of the map f, respectively. We adopt the convention $\sup(\emptyset) = 0$ and $\inf(\emptyset) = +\infty$. The map f is a *coarse embedding* if $\lim_{t\to\infty} \rho_f(t) = \infty$ and $\omega_f(t) < \infty$ for all $t > 0$. A map $f : X \to Y$ is said to be a *uniform embedding* if $\lim_{t\to 0} \omega_f(t) = 0$ and $\rho_f(t) > 0$ for all $t > 0$, i.e., f is an injective uniformly continuous map whose inverse is uniformly continuous.

If one is given a family of metric spaces $(X_i)_{i\in I}$, one says that $(X_i)_{i\in I}$ *equi-coarsely (resp. equi-uniformly)* embeds into Y if there exist non-decreasing functions $\rho, \omega : [0, \infty) \to [0, \infty)$ and for all $i \in I$, maps $f_i : X_i \to Y$ such that $\rho \leqslant \rho_{f_i}$, $\omega_{f_i} \leqslant \omega$, and $\lim_{t\to\infty} \rho(t) = \infty$ and $\omega(t) < \infty$ for all $t > 0$ (resp. $\lim_{t\to 0} \omega(t) = 0$ and $\rho(t) > 0$ for all $t > 0$).

We call a map $f : X \to Y$ *Lipschitz continuous* if

$$\mathrm{Lip}(f) = \sup\left\{\frac{d(f(x), f(y))}{d(x, y)} : x, y \in X, d(x, y) > 0\right\} < \infty,$$

and we call it a *bi-Lipschitz embedding* if it is injective and if f and f^{-1} are both Lipschitz continuous.

A *coarse-Lipschitz embedding* is a map $f : X \to Y$, for which there are numbers $\theta \geqslant 0$, and $0 < c_1 < c_2$, so that

$$c_1 d_X(x, y) \leqslant d_Y(f(x), f(y)) \leqslant c_2 d_X(x, y), \quad \text{whenever } x, y \in X \text{ and } d(x, y) \geqslant \theta. \tag{3}$$

3. Asymptotic properties of Banach spaces and their interplay

For two basic sequences (x_i) and (y_i) in some Banach spaces X and Y, respectively, and $C \geqslant 1$, we say that (x_i) and (y_i) are *C-equivalent*, and we write $(x_i) \sim_C (y_i)$ if there are positive numbers A and B, with $C = A \cdot B$, so that for all $(a_j) \in c_{00}$, the vector space of all sequences $x = (\xi_j)$ in $\mathbb{R}$ for which the support $\mathrm{supp}(x) = \{j : \xi_j \neq 0\}$ is finite, we have

$$\frac{1}{A}\left\|\sum_{i=1}^{\infty} a_i x_i\right\| \leqslant \left\|\sum_{i=1}^{\infty} a_i y_i\right\| \leqslant B\left\|\sum_{i=1}^{\infty} a_i x_i\right\|.$$

In that case, we say that $\frac{1}{A}$ is the *lower estimate* and B the *upper estimate of* (y_i) *with respect to* (x_i). Note that (x_i) and (y_i) are C-equivalent if and only if $C \geqslant \|T\| \cdot \|T^{-1}\|$, where the linear operator $T : \mathrm{span}(x_i : i \in \mathbb{N}) \to \mathrm{span}(y_i : i \in \mathbb{N})$ is defined by $T(x_i) = y_i$, $i \in \mathbb{N}$.

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If (e_i) is a Schauder basis of a Banach space X, we recall that (x_n) is a *block sequence* in X with respect to the basis (e_i) if $\max(\mathrm{supp}(x_1)) < \min(\mathrm{supp}(x_2)) \leqslant \max(\mathrm{supp}(x_2)) < \cdots \leqslant \max(\mathrm{supp}(x_{n-1})) < \min(\mathrm{supp}(x_n)) \leqslant \cdots$.

For $k \in \mathbb{N}$, we denote by $\mathcal{E}_k$ the set of all norms on $\mathbb{R}^k$, for which the unit vector basis $(e_i)_{i=1}^k$ is a normalized monotone basis. With an easily understood abuse of terminology, this can also be referred to as the set of all pairs $(E, (e_j)_{j=1}^k)$, where E is a k-dimensional Banach space and $(e_j)_{j=1}^k$ is a monotone basis of E.

We define a metric δ_k on $\mathcal{E}_k$ as follows: for two spaces $E = (\mathbb{R}^k, \|\cdot\|_E)$ and $F = (\mathbb{R}^k, \|\cdot\|_F)$, we let $\delta_k(E, F) = \log(\|I_{E,F}\| \cdot \|I_{E,F}^{-1}\|)$, $I_{E,F} : E \to F$, be the formal identity. It is also well known and easy to show that $(\mathcal{E}_k, \delta_k)$ is a compact metric space. The following definition is due to Maurey *et al.* [18].

Definition 3.1 (The kth asymptotic structure of X [18]). Let X be a Banach space. For $k \in \mathbb{N}$, we define the *kth asymptotic structure of X* to be the set, denoted by $\{X\}_k$, of spaces $E = (\mathbb{R}^k, \|\cdot\|) \in \mathcal{E}_k$ for which the following is true:

$$\forall \varepsilon > 0\, \forall X_1 \in \mathrm{cof}(X)\, \exists x_1 \in S_{X_1}\, \forall X_2 \in \mathrm{cof}(X)\, \exists x_2 \in S_{X_2} \cdots \forall X_k \in \mathrm{cof}(X)\, \exists x_k \in S_{X_k} \qquad (4)$$
$$(x_j)_{j=1}^k \sim_{1+\varepsilon} (e_j)_{j=1}^k.$$

For $1 \leqslant p \leqslant \infty$ and $c \geqslant 1$, we say that X *is c-asymptotically ℓ_p*, if for all $k \in \mathbb{N}$ and all spaces $E \in \{X\}_k$, with monotone normalized basis $(e_j)_{j=1}^k$, $(e_j)_{j=1}^k$ is c-equivalent to the ℓ_p^k unit vector basis. We say that X *is asymptotically ℓ_p*, if it is c-asymptotically ℓ_p for some $c \geqslant 1$. In case that $p = \infty$, we say that the space X is c-asymptotically c_0 or asymptotically c_0.

We denote by T^* the Banach space originally constructed by Tsirelson in [24]. It was the first example of a Banach space that does not contain any isomorphic copies of ℓ_p nor c_0. Since it is the archetype of a reflexive asymptotic-c_0 space, we explain shortly its construction (we will also use it at the end of § 5). Soon after, in [7], it became clear that the more natural space to define is T, the dual of T^*, because the norm of this space is more conveniently described. It has since become common to refer to T as Tsirelson space instead of T^*. Figiel and Johnson in [7] gave an implicit formula that describes the norm of T as follows. For $E, F \in [\mathbb{N}]^{<\omega}$ and $n \in \mathbb{N}$, we mean by $n \leqslant E$ that $n \leqslant \min E$ and by $E < F$ that $\max(E) < \min(F)$. We call a sequence $(E_j)_{j=1}^n$ of finite subsets of $\mathbb{N}$ *admissible* if $n \leqslant E_1 < E_2 < \cdots < E_n$. For $x = \sum_{j=1}^\infty \lambda_j e_j \in c_{00}$ and $E \in [\mathbb{N}]^{<\omega}$, we write $Ex = \sum_{j\in E} \lambda_j e_j$. As it was observed in [7], if $\|\cdot\|_T$ denotes the norm of T, then for every $x \in c_{00}$,

$$\|x\|_T = \max\left\{\|x\|_\infty, \frac{1}{2} \sup \sum_{j=1}^n \|E_j x\|_T\right\}, \qquad (5)$$

where the supremum is taken over all $n \in \mathbb{N}$ and admissible sequences $(E_j)_{j=1}^n$. The space T is the completion of c_{00} with this norm and the unit vector basis is a 1-unconditional basis. Then it was proven in [7] that T does not contain a subspace isomorphic to ℓ_1, which, together with the easy observation that T certainly does not contain a subspace isomorphic to c_0, yields by James' theorem [12, Theorem 2] that T must be reflexive.

The following property of T^* (see [24, Lemma 4]) is essential:

$$\Big\|\sum_{j=1}^n x_j\Big\|_{T^*} \leqslant 2 \max_{1\leqslant j\leqslant n} \|x_j\|_{T^*} \text{ if } (x_j)_{j=1}^n \text{ is a block sequence, with } n \leqslant \text{supp}(x_1). \quad (6)$$

The fact that T^* is 2-asymptotic-c_0 is an easy consequence of the above estimate. This well-known fact is hard to track down in the literature and follows from the fact that every weakly null tree admits a refinement for which all branches are arbitrary small perturbations of blocks.

Remarks 3.2. Let us recall some easy facts about the asymptotic structure of a Banach space which can be found in [16, 18] or [21].

(a) Let $E = (\mathbb{R}^k, \|\cdot\|)$, with $\|\cdot\|$ being a norm on $\mathbb{R}^k$, for which (e_j) is a normalized basis (but not necessarily monotone). If (e_j) satisfies (4) for some infinite-dimensional Banach space X, then $(e_j)_{j=1}^k$ is automatically a monotone basis of E (by using the ideas of Mazur's proof that normalized weakly null sequences have basic subsequences with a basis constant which is arbitrarily close to 1). Therefore, the above-introduced definition of asymptotic structure coincides with the original one given in [18].

(b) For any infinite-dimensional Banach space X and $k \in \mathbb{N}$, $\{X\}_k$ is a closed and thus compact subset of $\mathcal{E}_k$ with respect to the above-introduced metric δ_k on $\mathcal{E}_k$.

(c) For a k-dimensional space E with a monotone normalized basis $(e_j)_{j=1}^k$ to be in the k-asymptotic structure can be equivalently described by having a winning strategy in the following game between two players: We fix $\varepsilon > 0$. Player I (the 'space chooser') chooses a space $X_1 \in \text{cof}(X)$, then player II (the 'vector chooser') chooses a vector $x_1 \in S_{X_1}$, and then player I and player II repeat these moves to obtain spaces $X_1, X_2, \ldots, X_k$ in $\text{cof}(X)$ and vectors $x_1, x_2, \ldots, x_k$, with $x_i \in S_{X_i}$. The space E being in $\{X\}_k$ means that for every $\varepsilon > 0$, player II has a winning strategy, if his or her goal is to obtain a sequence $(x_j)_{j=1}^k$ which is $(1+\varepsilon)$-equivalent to $(e_j)_{j=1}^k$.
For $E \in \mathcal{E}_k$ with monotone basis $(e_j)_{j=1}^k$ and $\varepsilon > 0$, a winning strategy for the vector chooser can then be defined to be a *tree family*

$$\mathcal{F} = (x(X_1, X_2, \ldots, X_l) : 1 \leqslant l \leqslant k, X_1, X_2, \ldots, X_l \in \text{cof}(X)) \subset S_X$$

with the property that for any choice of $X_1, X_2, \ldots, X_k \in \text{cof}(X)$, and any $l \leqslant k$, $x(X_1, X_2, \ldots, X_l) \in S_{X_l}$ so that the sequence $(x(X_1, X_2, \ldots, X_l))_{l=1}^k$ is $(1+\varepsilon)$-equivalent to $(e_j)_{j=1}^k$.
Since the game has finitely many steps, it is *determined,* meaning that either the vector chooser or the space chooser has a winning strategy. Using the language of the game and its determinacy, it is then easy to see that the set $\{X\}_k$ is the smallest compact subset for which the space chooser has a winning strategy if for a given $\varepsilon > 0$, his or her goal is the resulting sequence $(x_j)_{j=1}^k$ at a distance at most ε to $\{X\}_k$ (with respect to the metric δ_k). In particular, a Banach space is asymptotically ℓ_p, $1 \leqslant p < \infty$ or asymptotically c_0 if and only if there is a $c > 0$ so that for each

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$k \in \mathbb{N}$, the space chooser has a winning strategy to get a sequence $(x_j)_{j=1}^k$ which is c-equivalent to the unit vector basis in ℓ_p^k or ℓ_∞^k, respectively.

(d) Assume that X is a space with a separable dual. Then we can replace in the definition of $\{X\}_k$ the set $\mathrm{cof}(X)$ by a countable subset of $\mathrm{cof}(X)$, namely by the set

$$\{F^\perp : F \subset \{x_j^* : j \in \mathbb{N}\} \text{ finite}\}, \quad \text{where } \{x_j^* : j \in \mathbb{N}\} \subset S_{X^*} \text{ is dense.}$$

In that case, normalized weakly null trees in X indexed by $[\mathbb{N}]^{\leqslant k}$ can be used to describe the kth asymptotic structure: if X^* is separable and $k \in \mathbb{N}$, a space $E \in \mathcal{E}_k$ with monotone basis $(e_j)_{j=1}^k$ is in $\{X\}_k$ if and only if for every $\varepsilon > 0$, there is an unrooted weakly null tree $\mathcal{T} = (x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k} \setminus \{\emptyset\})$ in S_X for which all branches are $(1+\varepsilon)$-equivalent to $(e_j)_{j=1}^k$.
It follows, therefore, from (c) and Ramsey's theorem that X is asymptotically ℓ_p, for $1 \leqslant p < \infty$, or asymptotically c_0 if there is a $C \geqslant 1$ so that for every $k \in \mathbb{N}$, every unrooted normalized weakly null tree of height k has a refinement (as introduced in § 2.1) all of whose branches are C-equivalent to the ℓ_p^k-unit vector basis.

The following observation will reduce the proof of the main results to the separable case.

Proposition 3.3. *Let X be a reflexive Banach space. Then there exists a separable subspace Y of X so that for all $k \in \mathbb{N}$, we have $\{X\}_k = \{Y\}_k$.*

We will need the following two lemmas first.

Lemma 3.4. *Let X be an infinite-dimensional Banach space and let E be a k-dimensional Banach space with a normalized monotone Schauder basis $(e_i)_{i=1}^k$. If for every $\varepsilon > 0$, there exists a weakly null tree $\{x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k} \setminus \{\emptyset\}\} \subset S_X$ so that for every $\bar{m} = \{m_1, \dots, m_k\} \in [\mathbb{N}]^k$, the sequence $(x_{\{m_1,\dots,m_i\}})_{i=1}^k$ is $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^k$. Then $(e_i)_{i=1}^k$ is in $\{X\}_k$.*

Proof. Recall that if $Y \in \mathrm{cof}(X)$ and $(z_i)_{i=1}^\infty$ is a normalized weakly null sequence, then $\lim_i \mathrm{dist}(z_i, S_Y) = 0$. Fixing $\varepsilon > 0$ and $k \in \mathbb{N}$, we will show that the vector player can choose a sequence that is $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^k$. Take a weakly null tree $(x_{\bar{m}} : \bar{m} \in [\mathbb{N}]^{\leqslant k}) \subset S_X$ so that for all $\bar{m} = \{m_1, \dots, m_k\}$, the sequence is $(x_{\{m_1,\dots,m_i\}})_{i=1}^k$ is $(1+\delta)$-equivalent to $(e_i)_{i=1}^k$, where we will choose $\delta > 0$ later. For each turn $1 \leqslant i \leqslant k$ of the game when the subspace player chooses $Y_i \in \mathrm{cof}(X)$, the vector player picks $m_i > m_{i-1}$ (where $m_0 = 0$) so that there is $x_i \in S_{Y_i}$ with $\|x_i - x_{\{m_1,\dots,m_i\}}\| \leqslant \delta$. For δ sufficiently small, this strategy for choosing x_i in S_{Y_i} ensures that the sequence $(x_i)_{i=1}^k$ is $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^k$. □

Lemma 3.5. *Let X be a reflexive Banach space, $k \in \mathbb{N}$, $(e_i)_{i=1}^k \in \{X\}_k$, and let $\varepsilon > 0$. Then there exists a countably branching weakly null tree $\{x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k} \setminus \{\emptyset\}\}$ in S_X, all of whose branches are $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^k$.*

Proof. We recall that the Eberlein–Šmulyan theorem ensures that if W is a relatively weakly compact set in a Banach space and $x_0 \in \overline{W}^w$, then there exists a sequence $(x_i)_{i=1}^\infty$

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in W with $x_i \xrightarrow{w} x_0$. Let $\varepsilon > 0$ and let $(x(Y_1, Y_2, \ldots, Y_i) : i = 1, 2, \ldots, k, Y_1, Y_2, \ldots, Y_i \in \mathrm{cof}(X))$ be a normalized tree with $x(Y_1, Y_2, \ldots, Y_i) \in S_{Y_i}$ and whose branches approximate $(e_j)_{j=1}^k$ up to $(1+\varepsilon)$-equivalence (see Remarks 3.2 (c)). By reflexivity, the set $\{x(Y) : Y \in \mathrm{cof}(X)\}$ (the first level of the tree) is relatively weakly compact. Also, $0 \in \overline{\{x(Y) : Y \in \mathrm{cof}(X)\}}^w$. Indeed, if $f_1, \ldots, f_d$ are in X^*, then $Y = \bigcap_{j=1}^d \ker(f_j)$ is in $\mathrm{cof}(X)$ and, hence, $f_j(x(Y)) = 0$ for $1 \leqslant j \leqslant d$. We may thus pick a sequence $(Y_l)_l$ in $\mathrm{cof}(X)$ with $x(Y_l) \xrightarrow{w} 0$.

Assume that for some $i \in \mathbb{N}$, we have assigned for each $\{m_1, \ldots, m_i\} \in [\mathbb{N}]^i$, a vector $x_{\{m_1,\ldots,m_i\}}$ of the form $x(Y_{m_1}, Y_{m_2}, \ldots, Y_{m_i})$. As before, we may pick a sequence $(Y_l^{(i+1)})_l$ so that $x(Y_{m_1}, Y_{m_2}, \ldots, Y_{m_i}, Y_l^{(i+1)}) \xrightarrow{w} 0$. For $j > m_i$, we define

$$x_{\{m_1,\ldots,m_i,j\}} = x(Y_{m_1}, Y_{m_2}, \ldots, Y_{m_i}, Y_l^{(i+1)}),$$

for some large enough l. Thus, every $(x_{\{m_1,\ldots,m_i\}})_{i=1}^k$ is of the form $(x(Y_1, \ldots, Y_i))_{i=1}^k$, and, thus, $(1+\varepsilon)$-equivalent to $(e_j)_{j=1}^k$. ☐

Proof of Proposition 3.3. Since for every $k \in \mathbb{N}$, the k-asymptotic structure $\{X\}_k$ is separable (with respect to the metric introduced in § 3 (b)), we can find a countable set $\{(e_j^{(l)})_{j=1}^k : l \in \mathbb{N}\} \subset \{X\}_k$ which is dense in $\{X\}_k$ and, using Lemma 3.5, a countable collection of weakly null trees $\{(x_{\bar{n}}^{(r)} : \bar{n} \in [\mathbb{N}]^{\leqslant k}) : r \in \mathbb{N}\}$ in S_X so that for each $\varepsilon > 0$ and each $l \in \mathbb{N}$, there is a $r \in \mathbb{N}$, so that for all $\bar{n} \in [\mathbb{N}]^k$, the sequence $(x_{\bar{m}}^{(r)} : \bar{m} \preceq \bar{n})$ is $(1+\varepsilon)$-equivalent to $(e_j^{(l)})_{j=1}^k$. We define Y_k to be the closed linear span of $\{x_{\bar{n}}^{(r)} : r \in \mathbb{N}, \bar{n} \in [\mathbb{N}]^{\leqslant k}\}$. Since $\{Y_k\}_k$ and $\{X\}_k$ are compact (see (b) in § 3), it follows that $\{Y_k\}_k = \{X\}_k$. Finally, we conclude our proof by setting Y to be the closed linear span of $\bigcup_{k\in\mathbb{N}} Y_k$ and deduce our claim. ☐

We now turn to 'sequential asymptotic properties' of Banach spaces. These are properties which involve sequences and their subsequences, as opposed to trees and their refinements.

Let X be a Banach space and $k \in \mathbb{N}$. A family $(x_j^{(i)} : i = 1, 2, \ldots, k, j \in \mathbb{N}) \subset X$ is called an *array of height* k *in* X. An *array of infinite height* in X is a family $(x_j^{(i)} : i, j \in \mathbb{N}) \subset X$.

For (finite or infinite) arrays $(x_j^{(i)} : i = 1, 2, \ldots, k, j \in \mathbb{N})$, or $(x_j^{(i)} : i, j \in \mathbb{N})$, respectively, we call the sequence $(x_j^{(i)})_{j\in\mathbb{N}}$ *the* i*th row of the array*. We call an array weakly null if all rows are weakly null. A *subarray* of a finite array $(x_j^{(i)} : i = 1, 2, \ldots, k, j \in \mathbb{N}) \subset X$, or an infinite array $(x_j^{(i)} : i \in \mathbb{N}, j \in \mathbb{N}) \subset X$, is an array of the form $(x_{j_s}^{(i)} : i = 1, 2, \ldots, k, s \in \mathbb{N})$ or $(x_{j_s}^{(i)} : i \in \mathbb{N}, s \in \mathbb{N})$, respectively, where $(j_s) \subset \mathbb{N}$ is a subsequence. Thus, for a subarray, we are taking the same subsequence in each row.

The following notion was introduced by Halbeisen and Odell [11].

Definition 3.6 [11]. A basic sequence (e_i) is called an *asymptotic model* of a Banach space X if there exist an infinite array $(x_j^{(i)} : i, j \in \mathbb{N}) \subset S_X$ and a null sequence $(\varepsilon_n) \subset (0, 1)$ so

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that for all n, all $(a_i)_{i=1}^n \subset [-1,1]$ and $n \leqslant k_1 < k_2 < \cdots < k_n$, it follows that

$$\left| \left\| \sum_{i=1}^n a_i x_{k_i}^{(i)} \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| < \varepsilon_n.$$

In [11], the following was shown.

Proposition 3.7 [11, Proposition 4.1 and Remark 4.7.5]. *Assume that $(x_j^{(i)} : i, j \in \mathbb{N}) \subset S_X$ is an infinite array, all of whose rows are normalized and weakly null. Then there is a subarray of $(x_j^{(i)} : i, j \in \mathbb{N})$ which has a 1-suppression unconditional asymptotic model (e_i).*

We call a basic sequence (e_i) *c-suppression unconditional*, for some $c \geqslant 1$, if for any $(a_i) \subset c_{00}$ and any $A \subset \mathbb{N}$,

$$\left\| \sum_{i \in A} a_i e_i \right\| \leqslant c \left\| \sum_{i=1}^\infty a_i e_i \right\|.$$

We call (e_i) c-unconditional if for any $(a_i) \subset c_{00}$ and any $(\sigma_i) \in \{\pm 1\}^{\mathbb{N}}$,

$$\left\| \sum_{i=1}^\infty a_i e_i \right\| \leqslant c \left\| \sum_{i=1}^\infty \sigma_i a_i e_i \right\|.$$

Note that a c-unconditional basic sequence is c-suppression unconditional.

The following important result was shown in [8] and it is an integral ingredient of the proof of Theorem B.

Theorem 3.8 [8, Theorem 4.6]. *If a separable Banach space X does not contain any isomorphic copy of ℓ_1 and all the asymptotic models generated by normalized weakly null arrays are equivalent to the c_0 unit vector basis, then X is asymptotically c_0.*

Asymptotic models can be seen as a generalization of *spreading models*, a notion which was introduced much earlier by Brunel and Sucheston [6]. Spreading models are asymptotic models for arrays with identical rows.

Definition 3.9 [6]. Let E be a Banach space with a normalized basis (e_i) and let (x_i) be a basic sequence in a Banach space X. We say that E with its basis (e_i) is a *spreading model of* (x_i) if there is a null sequence $(\varepsilon_n) \subset (0,1)$ so that for all n, all $(a_i)_{i=1}^n \subset [-1,1]$ and $n \leqslant k_1 < k_2 < \cdots < k_n$, it follows that

$$\left| \left\| \sum_{i=1}^n a_i x_{k_i} \right\|_X - \left\| \sum_{i=1}^n a_i e_i \right\|_E \right| < \varepsilon_n$$

or, in other words, if

$$\lim_{k_1 \to \infty} \lim_{k_2 \to \infty} \cdots \lim_{k_n \to \infty} \left\| \sum_{j=1}^n a_j x_{k_j} \right\|_X = \left\| \sum_{j=1}^n a_j e_j \right\|_E.$$

Using Ramsey's theorem, it is easy to see that every normalized basic sequence has a subsequence which admits a spreading model, which, of course, also follows form the above cited result in [11]. A spreading model E with basis (e_i) generated by a normalized weakly null sequence is *1-suppression unconditional* [4, Proposition 1, p. 24].

Let $k \in \mathbb{N}$ and let $(x_j^{(i)} : i = 1, 2, \dots, k, j \in \mathbb{N}) \subset S_X$ be a normalized weakly null array of height k. We extend this array to an infinite array $(x_j^{(i)} : i \in \mathbb{N}, j \in \mathbb{N})$ by letting

$$x_j^{(sk+i)} = x_j^{(i)}, \quad \text{for } s \in \mathbb{N} \text{ and } i = 1, 2, \dots, k.$$

By Proposition 3.7, we can pass to a subarray $(z_j^{(i)} : i \in \mathbb{N}, j \in \mathbb{N})$ of $(x_j^{(i)} : i \in \mathbb{N}, j \in \mathbb{N})$ which admits an asymptotic model (e_j). Now letting $e_j^{(i)} = e_{(j-1)k+i}$, for $i = 1, 2, \dots, k$ and $j \in \mathbb{N}$, we observe that the array $(e_j^{(i)})_{i,j\in\mathbb{N}}$ is the *joint spreading model of* $(z_j^{(i)} : i \in \mathbb{N}, j \in \mathbb{N})$, a notion introduced and discussed in [1]. We recall the definition of *joint spreading models* and will first recall the definition of *plegmas*.

Definition 3.10 [2, Definition 3]. Let $k, m \in \mathbb{N}$ and $s_i = (s_1^{(i)}, s_2^{(i)}, \dots, s_m^{(i)}) \subset \mathbb{N}$ for $i = 1, \dots, k$. The family $(s_i)_{i=1}^k$ is called a *plegma* if

$$s_1^{(1)} < s_1^{(2)} < \cdots < s_1^{(k)} < s_2^{(1)} < s_2^{(2)} < \cdots < s_2^{(k)} < \cdots < s_m^{(1)} < s_m^{(2)} < \cdots < s_m^{(k)}.$$

Definition 3.11 [1, Definition 3.1]. Let $(x_j^{(i)} : 1 \leqslant i \leqslant k, j \in \mathbb{N})$ and $(e_j^{(i)} : 1 \leqslant i \leqslant k, j \in \mathbb{N})$ be two normalized arrays in the Banach spaces X and E, respectively, whose rows are normalized and basic. We say that $(x_j^{(i)} : 1 \leqslant i \leqslant k, j \in \mathbb{N})$ *generates* $(e_j^{(i)} : 1 \leqslant i \leqslant k, j \in \mathbb{N})$ *as a joint spreading model* if there exists a null sequence of positive real numbers $(\varepsilon_m)_{m=1}^\infty$ so that for every $m \in \mathbb{N}$, every plegma $(s_i)_{i=1}^k$, $s_i = (s_j^{(i)} : j = 1, 2, \dots, m)$ for $1 \leqslant i \leqslant k$, with $\min(s_1) = s_1^{(1)} \geqslant m$, and scalars $((a_j^{(i)})_{j=1}^m)_{i=1}^k$ in $[-1, 1]$, we have

$$\left| \left\| \sum_{j=1}^m \sum_{i=1}^k a_j^{(i)} x_{s_j^{(i)}}^{(i)} \right\|_X - \left\| \sum_{j=1}^m \sum_{i=1}^k a_j^{(i)} e_j^{(i)} \right\|_E \right| < \varepsilon_m.$$

Remark 3.12. Note that if $(x_j^{(i)} : 1 \leqslant i \leqslant k, j \in \mathbb{N})$ generates $(e_j^{(i)} : 1 \leqslant i \leqslant k, j \in \mathbb{N})$ as a joint spreading model, then $(e_j^{(i)})_{j=1}^\infty$ is a spreading model of $(x_j^{(i)})_{j=1}^\infty$, for $i = 1, 2, \dots, k$.

In the next remark, we discuss the differences between asymptotic and sequential asymptotic properties.

Remark 3.13. Assume that X is a separable reflexive space. Then, by observation (d) in Remarks 3.2, the property that X is asymptotically ℓ_p, for some $1 \leqslant p \leqslant \infty$ (as usual replace ℓ_∞ by c_0 if $p = \infty$), is equivalent to the property that there is a $C \geqslant 1$ so that for every $k \in \mathbb{N}$, every weakly null tree $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$ of height k can be refined (as defined in § 2.1) to a tree $(x_{\bar{m}} : \bar{m} \in [\mathbb{M}]^{\leqslant k})$, $\mathbb{M} \in [\mathbb{N}]^\omega$, which has the property that each branch is C-equivalent to the ℓ_p^k unit vector basis.

Second, we consider the property of a Banach space X that every asymptotic model generated by a weakly null array is C-equivalent to the ℓ_p-unit vector basis, for some $1 \leqslant p < \infty$, or the c_0-unit vector basis. For a normalized weakly null array $(x_j^{(i)} : i, j \in \mathbb{N})$, we put $x_{\bar{m}} = x^{(i)}_{\max(\bar{m})}$ for $\bar{m} \in [\mathbb{N}]^i$ and call for $k \in \mathbb{N}$, the tree $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$, *the tree of height k generated by the array $(x_j^{(i)} : i, j \in \mathbb{N})$*. Note that $x_{\bar{n}}$ for $\bar{n} \in [\mathbb{N}]^{\leqslant k}$ only depends on $\max(\bar{n})$ and the cardinality of $\bar{n}$ but not on the predecessors of $\bar{n}$. Then, by a straightforward diagonalization argument, one shows that the property that every asymptotic model generated by a weakly null array is C-equivalent to the ℓ_p-unit vector basis for some $C \geqslant 1$ is equivalent with the property that every tree of height k, generated by a normalized weakly null array, has a refinement, all of whose branches are C-equivalent to the ℓ_p^k-unit vector basis, for some $C \geqslant 1$.

Thus, the property that the asymptotic models generated by normalized weakly null arrays are C-equivalent to the ℓ_p-unit vector basis is a property of specific weakly null trees. Theorem 3.8 is, therefore, a surprising result, and its proof relies on the fact that the c_0-norm is somewhat extremal. Usually, it is not possible to deduce from a sequentially asymptotic property of a Banach space an asymptotic property. For example, in a forthcoming paper, we build a reflexive space X, all of whose asymptotic models are isometrically equivalent to the ℓ_2-unit vector basis, but for a given $1 \leqslant p \leqslant \infty$, $p \neq 2$, X has ℓ_p^n in its nth asymptotic structure.

4. Proof of Theorems A and B

This section is devoted to proving Theorem B and then obtaining Theorem A as a corollary. The proof is based on the main argument of [3] and on the above cited result in [8] (see Theorem 3.8 in our paper) that connects asymptotic properties with properties of arrays.

The following lemma includes a well-known refinement argument which is crucial for the proof of the main result. For completeness, we include a proof.

Lemma 4.1. *Let X be a reflexive Banach space, $k \in \mathbb{N}$, and $f : [\mathbb{N}]^k \to X$ have a bounded image. Then there exist $\mathbb{M} \in [\mathbb{N}]^\omega$ and a weakly null tree $(y_{\bar{m}} : \bar{m} \in [\mathbb{M}]^{\leqslant k})$ so that $f(\bar{m}) = y_\emptyset + \sum_{i=1}^k y_{\{m_1,\dots,m_i\}}$, for all $\bar{m} \in [\mathbb{M}]^k$.*

Moreover, if we equip $[\mathbb{N}]^k$ with $d_{\mathbb{H}}^{(k)}$, then for all $\bar{m} \in [\mathbb{M}]^{\leqslant k} \setminus \{\emptyset\}$, we have $\|y_{\bar{m}}\| \leqslant \mathrm{Lip}(f)$.

Proof. We prove the claim by induction for all $k \in \mathbb{N}$. If $k = 1$, we can take a subsequence (x_n) of $(f(\{n\}))_{n\in\mathbb{N}}$ which converges to some $y_\emptyset \in X$. Then put $y_{\{n\}} = x_n - y_\emptyset$.

Assume our claim to be true for $k-1$, with $k \in \mathbb{N}$, and let $f : [\mathbb{N}]^k \to X$ have a bounded image. We put $l_i = i$, for $i = 1, 2, \dots, k-1$, and choose $\mathbb{L}_{k-1} \in [\{k, k+1, \dots\}]^\omega$ so that $x_{\{1,2,\dots,k-1\}} = w - \lim_{l\to\infty, l\in\mathbb{L}_{k-1}} f(\{1, 2, \dots, k-1\} \cup \{l\})$ exists. Then we can recursively choose for each $n \geqslant k$, $l_n \in \mathbb{N}$, $\mathbb{L}_n \in [\mathbb{L}_{n-1}]^\omega$, with $l_n \in \mathbb{L}_{n-1}$ and $l_n < \min(\mathbb{L}_n)$, so that for each $\bar{m} \subset \{l_1, l_2, \dots, l_n\}$, with $\#\bar{m} = k-1$, $x_{\bar{m}} = w - \lim_{l\to\infty, l\in\mathbb{L}_n} f(\bar{m} \cup \{l\})$ exists. Let $\mathbb{L} = \{l_j : j \in \mathbb{N}\}$ and put $y_{\bar{m}} = f(\bar{m}) - x_{\{m_1,m_2,\dots,m_{k-1}\}}$ for $\bar{m} = \{m_1, m_2, \dots, m_k\} \in [\mathbb{L}]^k$.

Finally, we apply the induction hypothesis to $f' : [\mathbb{L}]^{k-1} \to X$, $\bar{m} \mapsto x_{\bar{m}}$, which provides

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us with an infinite $\mathbb{M} \subset \mathbb{L}$ and a weakly null tree $(y_{\bar{m}} : \bar{m} \in [\mathbb{M}]^{k-1})$ so that $x_{\bar{m}} = \sum_{i=0}^{k} y_{\{m_1,m_2,\ldots,m_i\}}$ for all $\bar{m} = \{m_1, m_2, \ldots, m_{k-1}\} \in [\mathbb{M}]^{k-1}$, and, thus,

$$f(\bar{m}) = y_{\bar{m}} + x_{\{m_1,m_2,\ldots,m_{k-1}\}} = \sum_{i=0}^{k} y_{\{m_1,m_2,\ldots,m_i\}} \text{ for all } \bar{m} = \{m_1, m_2, \ldots, m_k\} \in [\mathbb{M}]^k.$$

To prove the second part of the statement, let $\bar{m} = \{m_1, m_2, \ldots, m_i\}$ in $[\mathbb{M}]^k \setminus \{\emptyset\}$ and put $\bar{m}' = \{m_1, m_2, \ldots, m_{i-1}\}$. It follows from the lower semicontinuity of the norm with respect to the weak topology that

$$\begin{aligned}\|y_{\bar{m}}\| &= \Big\| w\text{-}\lim_{n_i\to\infty} \lim_{n_{i+1}\to\infty} \cdots \lim_{n_k\to\infty} \big(f(\bar{m} \cup \{n_{i+1}, \ldots, n_k\}) - f(\bar{m}' \cup \{n_i, n_{i+1}, \ldots, n_k\})\big)\Big\| \\ &\leqslant \limsup_{n_i\to\infty} \limsup_{n_{i+1}\to\infty} \cdots \limsup_{n_k\to\infty} \|f(\bar{m} \cup \{n_{i+1}, \ldots, n_k\}) - f(\bar{m}' \cup \{n_i, n_{i+1}, \ldots, n_k\})\| \\ &\leqslant \limsup_{n_i\to\infty} \limsup_{n_{i+1}\to\infty} \cdots \limsup_{n_k\to\infty} \mathrm{Lip}(f) d_{\mathbb{H}}^{(k)}(\bar{m} \cup \{n_{i+1}, \ldots, n_k\}, \bar{m}' \cup \{n_i, n_{i+1}, \ldots, n_k\}) \\ &= \mathrm{Lip}(f).\end{aligned}$$

□

For the proof of Theorem B, a slightly weaker version of the next result would be sufficient, but its proof would not be significantly easier.

Lemma 4.2. *Let X be a C-asymptotic-c_0 Banach space for some $C \geqslant 1$, $k \in \mathbb{N}$, and let also $(x_{\bar{n}} : \bar{n} \in [\mathbb{N}]^{\leqslant k})$ be a bounded weakly null tree. Then for every $\varepsilon > 0$, there exists $\mathbb{L} \in [\mathbb{N}]^{\omega}$ so that for every $\bar{m} = \{m_1, \ldots, m_k\} \in [\mathbb{L}]^k$ and every $F \subset \{1, \ldots, k\}$, we have*

$$\Big\|\sum_{i\in F} x_{\{m_1,\ldots,m_i\}}\Big\| \leqslant (C + 1 + \varepsilon) \max_{i\in F} \|x_{\{m_1,\ldots,m_i\}}\|.$$

Proof. We will just find one such $\bar{m}$. This is sufficient by Ramsey's theorem since such a set $\bar{m}$ could be found in each infinite subset of $\mathbb{N}$. Let us play a k-round vector game in which the subspace player follows a winning strategy to force the vector player to choose a sequence $(C+\varepsilon)$-equivalent to the unit vector basis of ℓ_∞^k. In each step i, the subspace player picks a subspace Y_i of finite codimension according to his or her winning strategy. The vector player picks $y_i \in Y_i$ according to the following scheme: recursively pick $m_1 < \cdots < m_k$ so that one of the following holds:

(a) If $\limsup_n \|x_{\{m_1,\ldots,m_{i-1},n\}}\| > 0$, pick m_i (with $m_i > m_{i-1}$ if $i > 1$) and y_i in the unit sphere of Y_i so that

$$\left\| y_i - \frac{x_{\{m_1,\ldots,m_i\}}}{\|x_{\{m_1,\ldots,m_i\}}\|} \right\| < \varepsilon 2^{-i}.$$

In the above argument, we have used the following corollary of the Hahn–Banach Theorem. If $Y \in \mathrm{cof}(X)$ and $(z_i)_{i=1}^{\infty}$ is a weakly null sequence, then $\lim_i \mathrm{dist}(z_i, Y) = 0$. If, in particular, $(z_i)_i$ is normalized, then $\lim_i \mathrm{dist}(z_i, S_Y) = 0$.

(b) If $\lim_n \|x_{\{m_1,\ldots,m_{i-1},n\}}\| = 0$, we distinguish between the following subcases:

(b1) if $i = 1$ or $x_{\{m_1,\ldots,m_j\}} = 0$, for all $1 \leqslant j < i$, pick arbitrary m_i so that $m_i > m_{i-1}$ if $i > 1$ and arbitrary y_i in the unit sphere of Y_i, and (b2) if $i > 1$ and $x_{\{m_1,\dots,m_j\}} \neq 0$, for some $1 \leqslant j < i$, pick $m_i > m_{i-1}$ so that

$$\|x_{\{m_1,\dots,m_i\}}\| < (\varepsilon 2^{-i}) \min\{\|x_{\{m_1,\dots,m_j\}}\| : 1 \leqslant j < i \text{ with } x_{\{m_1,\dots,m_j\}} \neq 0\}$$

and pick an arbitrary y_i in the unit sphere of Y_i.

It follows that the sequence $(y_i)_{i=1}^k$ is $(C+\varepsilon)$-equivalent to the unit vector basis of ℓ_∞^k.

Let now $F \subset \{1, 2, \dots, k\}$. Set

$$F_1 = \{i \in F : \text{ (a) is satisfied}\} \text{ and } \bar{F}_2 = \{i \in F : \text{ (b) is satisfied and } x_{\{m_1,\dots,m_i\}} \neq 0\}.$$

Set $i_0 = \min(\bar{F}_2)$ and $F_2 = \bar{F}_2 \setminus \{i_0\}$ if $\bar{F}_2$ is non-empty, otherwise, let $F_2 = \emptyset$. We now calculate

$$\begin{aligned}
\Big\|\sum_{i\in F} x_{\{m_1,\dots,m_i\}}\Big\| &\leqslant \Big\|\sum_{i\in F_1} x_{\{m_1,\dots,m_i\}}\Big\| + \|x_{\{m_1,\dots,m_{i_0}\}}\| + \Big\|\sum_{i\in F_2} x_{\{m_1,\dots,m_i\}}\Big\| \\
&\leqslant \Big\|\sum_{i\in F_1} \|x_{\{m_1,\dots,m_i\}}\| y_i\Big\| + \Big\|\sum_{i\in F_1} x_{\{m_1,\dots,m_i\}} - \|x_{\{m_1,\dots,m_i\}}\| y_i\Big\| \\
&\quad + \|x_{\{m_1,\dots,m_{i_0}\}}\| + \sum_{i\in F_2} \frac{\varepsilon}{2^i} \|x_{\{m_1,\dots,m_{i_0}\}}\| \\
&\leqslant (C+\varepsilon) \max_{i\in F_1} \|x_{\{m_1,\dots,m_i\}}\| + \sum_{i\in F_1} \frac{\varepsilon}{2^i} \|x_{\{m_1,\dots,m_i\}}\| + (1+\varepsilon)\|x_{\{m_1,\dots,m_{i_0}\}}\| \\
&\leqslant (C+1+3\varepsilon) \max_{i\in F} \|x_{\{m_1,\dots,m_i\}}\|.
\end{aligned}$$

An adjustment of ε yields the desired estimate. □

The following is one of the main statements presented in this paper.

Theorem 4.3 (Theorem B). *A Banach space X is reflexive and asymptotic-c_0 if and only if there exists $C \geqslant 1$ satisfying the following: for every $k \in \mathbb{N}$ and Lipschitz map $f : ([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)}) \to X$, there exists $\mathbb{L} \in [\mathbb{N}]^\omega$ so that*

$$\operatorname{diam}(f([\mathbb{L}]^k)) \leqslant C \operatorname{Lip}(f). \tag{7}$$

Proof. We first assume that X is reflexive and B-asymptotically c_0. Let $k \in \mathbb{N}$ and let $f : ([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)}) \to X$ be a Lipschitz map. By Lemma 4.1, there exist $\mathbb{M} \in [\mathbb{N}]^\omega$ and a weakly null tree $(y_{\bar{m}} : \bar{m} \in [\mathbb{M}]^{\leqslant k})$ so that $f(\bar{m}) = \sum_{\bar{l}\preceq\bar{m}} y_{\bar{l}}$, for all $\bar{m} \in [\mathbb{M}]^k$, and $\|y_{\bar{m}}\| \leqslant \operatorname{Lip}(f)$, for all $\bar{m} \in [\mathbb{M}]^{\leqslant k} \setminus \{\emptyset\}$. By Lemma 4.2, we find $\mathbb{L} \in [\mathbb{M}]^\omega$ so that

$$\Big\|\sum_{i\in F} y_{\{m_1,\dots,m_i\}}\Big\| \leqslant (B+2) \max_{i\in F} \|y_{\{m_1,\dots,m_i\}}\|,$$

for all $\bar{m} = \{m_1, m_2, \dots, m_k\} \in [\mathbb{L}]^k$ and $F \subset \{1, \dots, k\}$. Thus, for $\bar{m}, \bar{n}$ in $[\mathbb{L}]^k$, we have

$$\|f(\bar{m}) - f(\bar{n})\| = \Big\|\sum_{\bar{u}\preceq\bar{m}} y_{\bar{u}} - \sum_{\bar{v}\preceq\bar{n}} y_{\bar{v}}\Big\| \leqslant \Big\|\sum_{\emptyset\prec\bar{u}\preceq\bar{m}} y_{\bar{u}}\Big\| + \Big\|\sum_{\emptyset\prec\bar{v}\preceq\bar{n}} y_{\bar{v}}\Big\| \leqslant 2(B+2)\operatorname{Lip}(f)$$

and so for $C = 2(B+2)$, the conclusion is satisfied.

To prove the converse, we show that if either X is not reflexive or X is reflexive and not asymptotic-c_0, then there exists a sequence (f_k), $f_k : ([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)}) \to X$, $\mathrm{Lip}(f_k) \leqslant 1$, for $k \in \mathbb{N}$, and

$$\inf_{\mathbb{M}\in[\mathbb{N}]^\omega} \mathrm{diam}(f_k([\mathbb{M}]^k)) \nearrow \infty, \quad \text{if } k \nearrow \infty. \tag{8}$$

Assume first that X is non-reflexive. By James' characterization of reflexive spaces [13], there exists a sequence $(x_n) \subset B_X$ such that for all $k \geqslant 1$ and $\bar{m} = \{m_1, m_2, \ldots, m_{2k}\} \in [\mathbb{N}]^{2k}$,

$$\left\| \sum_{i=1}^{k} x_{m_i} - \sum_{i=k+1}^{2k} x_{m_i} \right\| \geqslant \frac{k}{2}. \tag{9}$$

Define $f_k(\bar{m}) = \frac{1}{2}\sum_{i=1}^k x_{m_i}$, for $\bar{m} = \{m_1, \ldots, m_k\}$ in $[\mathbb{N}]^k$. This map is 1-Lipschitz with respect to $d_{\mathbb{H}}^{(k)}$ and (9) implies (8).

Second, assume that X is reflexive and not asymptotically-c_0. By Proposition 3.3, there is a separable subspace of X that is not asymptotically-c_0, so we can assume that X is separable. By Theorem 3.8, there exists a 1-suppression unconditional sequence $(e_i)_i$ that is not equivalent to the unit vector basis of c_0, and, hence, $\lambda_k = \|\sum_{i=1}^k e_i\| \nearrow \infty$ if $k \nearrow \infty$, and that is generated as an asymptotic model of a normalized weakly null array $(x_j^{(i)} : i, j \in \mathbb{N})$ in X. Fixing $k \in \mathbb{N}$ and $\delta > 0$ and after passing to appropriate subsequences of the array, we may assume that for any $k \leqslant j_1 < \cdots < j_k$ and any $a_1, \ldots, a_k$ in $[-1, 1]$, we have

$$\left| \left\| \sum_{i=1}^{k} a_i x_{j_i}^{(i)} \right\| - \left\| \sum_{i=1}^{k} a_i e_i \right\| \right| < \delta. \tag{10}$$

Define now $f_k(\bar{m}) = \frac{1}{2}\sum_{i=1}^k x_{m_i}^{(i)}$ for $\bar{m} = \{m_1, \ldots, m_k\} \in [\mathbb{N}]^k$. Note that f is 1-Lipschitz for the metric $d_{\mathbb{H}}^{(k)}$.

Then, if $\bar{m} = \{m_1, \ldots, m_k\}$, $\bar{n} = \{n_1, \ldots, n_k\}$ and $F = \{i : m_i \neq n_i\}$, we have

$$f_k(\bar{m}) - f_k(\bar{n}) = \frac{1}{2}\sum_{i\in F} x_{m_i}^{(i)} - \frac{1}{2}\sum_{i\in F} x_{n_i}^{(i)}.$$

Using the fact that the array is weakly null and the Hahn–Banach theorem, for all $\mathbb{M} \in [\mathbb{N}]^\omega$, all $\bar{m}$ in $[\mathbb{M}]^k$, we can find $x^* \in S_{X^*}$ and $\bar{n} \in [\mathbb{M}]^k$ such that

$$x^*\left(\sum_{i\in F} x_{m_i}^{(i)} - \sum_{i\in F} x_{n_i}^{(i)} \right) \geqslant \left\| \sum_{i\in F} x_{m_i}^{(i)} \right\| - \delta.$$

Using equation (10), we deduce that

$$\| f_k(\bar{m}) - f_k(\bar{n}) \| \geqslant \frac{1}{2}\lambda_k - \delta.$$

If δ was chosen small enough, we obtain that for all $\mathbb{M} \in [\mathbb{N}]^\omega$, $\mathrm{diam}(f_k([\mathbb{M}]^k)) \geqslant \frac{\lambda_k}{4}$, which proves our claim. □

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Corollary 4.4 (Theorem A). *Let Y be a reflexive asymptotic-c_0 Banach space. If X is a Banach space that coarsely embeds into Y, then X is also reflexive and asymptotic-c_0.*

Proof. Let $g : X \to Y$ be a coarse embedding with moduli $\rho_g, \omega_g : [0, \infty) \to [0, \infty)$. By Theorem 4.3, the space Y satisfies (7), for some constant $C \geqslant 1$. It is enough to show that the same is true for X and some $D \geqslant 1$ such that $\rho_g(D) > C\omega_g(1)$.

Let $f : [\mathbb{N}]^k \to X$ be a non-constant Lipschitz map. Take $h : [\mathbb{N}]^k \to Y$ with $h(\bar{m}) = g(\mathrm{Lip}(f)^{-1} f(\bar{m}))$. Because $d_{\mathbb{H}}^{(k)}$ is an unweighted graph metric, it follows that

$$\mathrm{Lip}(h) = \omega_h(1) \leqslant \omega_g(\mathrm{Lip}(f)^{-1}\omega_f(1)) = \omega_g(\mathrm{Lip}(f)^{-1}\,\mathrm{Lip}(f)) = \omega_g(1).$$

Pick $\mathbb{L} \in [\mathbb{N}]^\omega$ so that for all $\bar{m}, \bar{n} \in [\mathbb{L}]^k$, we have $\|h(\bar{m}) - h(\bar{n})\| \leqslant C\omega_g(1)$. On the other hand,

$$\begin{aligned} C\omega_g(1) &\geqslant \|h(\bar{m}) - h(\bar{n})\| = \|g(\mathrm{Lip}(f)^{-1} f(\bar{m})) - g(\mathrm{Lip}(f)^{-1} f(\bar{n}))\| \\ &\geqslant \rho_g(\mathrm{Lip}(f)^{-1}\|f(\bar{m}) - f(\bar{n})\|). \end{aligned}$$

Thus, $\mathrm{Lip}(f)^{-1}\|f(\bar{m}) - f(\bar{n})\| \leqslant D$ or $\|f(\bar{m}) - f(\bar{n})\| \leqslant D\,\mathrm{Lip}(f)$, for any $\bar{m}, \bar{n} \in [\mathbb{L}]^k$. □

A simple re-scaling argument (see the end of [3, § 4]) allows us to adapt the above proofs in order to show the following.

Corollary 4.5. *Let Y be a reflexive asymptotic-c_0 Banach space. If X is a Banach space such that B_X uniformly embeds into Y, then X is also reflexive and asymptotic-c_0.*

Remark 4.6. For $k \in \mathbb{N}$, the Johnson graph of height k is the set $[\mathbb{N}]^k$ equipped with the metric defined by $d_{\mathbb{J}}^{(k)}(\bar{m}, \bar{n}) = \frac{1}{2}\sharp(\bar{m}\Delta\bar{n})$ for $\bar{m}, \bar{n} \in [\mathbb{N}]^k$. It is proved in [3] that there is a constant $C \geqslant 1$ such that for any $k \in \mathbb{N}$ and $f : ([\mathbb{N}]^k, d_{\mathbb{J}}^{(k)}) \to T^*$ Lipschitz, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ so that $\mathrm{diam}(f([\mathbb{M}]^k)) \leqslant C\,\mathrm{Lip}(f)$. It is easily seen that the same is true if T^* is replaced by any reflexive asymptotic-c_0 space. It is also clear that this concentration property for Lipschitz maps from the Johnson graphs implies the reflexivity of the target space. We do not know if it implies that it is asymptotic-c_0. We do not know either whether the equi-coarse embeddability of the Johnson graphs and of the Hamming graphs are equivalent conditions for a Banach space. The reason is that canonical embeddings of the Johnson graphs are built on sequences and not arrays. This confirms the qualitative difference between asymptotic models and spreading models.

5. Quasi-reflexive asymptotic-c_0 spaces

Let us first recall that a Banach space is said to be *quasi-reflexive* if the image of its canonical embedding into its bidual is of finite codimension in this bidual. For an infinite subset $\mathbb{M}$ of $\mathbb{N}$, we denote $I_k(\mathbb{M})$ the set of strictly interlaced pairs in $[\mathbb{M}]^k$, namely

$$I_k(\mathbb{M}) = \{(\bar{m}, \bar{n}) \in [\mathbb{M}]^k \times [\mathbb{M}]^k, m_1 < n_1 < m_2 < n_2 < \cdots < m_k < n_k\}.$$

Note that for $(\bar{m}, \bar{n}) \in I_k(\mathbb{M})$, $d_{\mathbb{H}}^{(k)}(\bar{m}, \bar{n}) = k$. Our next result mixes arguments from Lemma 4.2 of this paper and of [17, Theorem 2.2].

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Theorem 5.1. *Let $C \geqslant 1$ and X be a quasi-reflexive C-asymptotic-c_0 Banach space. Then, for any Lipschitz map $f : ([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)}) \to X$, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ such that*

$$\forall (\bar{m}, \bar{n}) \in I_k(\mathbb{M}), \ \ \|f(\bar{m}) - f(\bar{n})\| \leqslant 3(C+1)\,\mathrm{Lip}(f).$$

In particular, the family $([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)})_{k\in\mathbb{N}}$ does not equi-coarsely embed into X.

Proof. Let us write $X^{**} = X \oplus E$, where E is a finite dimensional space. Let $f : ([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)}) \to X$ be a Lipschitz map. Since f is countably valued and X is quasi-reflexive, we may as well assume that X and, therefore, all its iterated duals are separable. We may also assume that $\mathrm{Lip}(f) > 0$. Then mimicking the proof of Lemma 4.1 and using weak* compactness instead of weak compactness, we infer the existence of $\mathbb{M} \in [\mathbb{N}]^\omega$ and of a weak* null tree $(z_{\bar{m}} : \bar{m} \in [\mathbb{M}]^{\leqslant k})$ in X^{**} so that $f(\bar{m}) = z_\emptyset + \sum_{i=1}^k z_{\{m_1,\dots,m_i\}}$, for all $\bar{m} \in [\mathbb{M}]^k$ and $\|z_{\bar{m}}\| \leqslant \mathrm{Lip}(f)$, for all $\bar{m} \in [\mathbb{M}]^{\leqslant k} \setminus \{\emptyset\}$. For any $\bar{m} \in [\mathbb{M}]^{\leqslant k} \setminus \{\emptyset\}$, we write $z_{\bar{m}} = x_{\bar{m}} + e_{\bar{m}}$ with $x_{\bar{m}} \in X$ and $e_{\bar{m}} \in E$.

Fix now $\eta > 0$. Since E is finite-dimensional, using Ramsey's theorem, we may assume after further extractions that

$$\forall i \in \{1, \dots, k\} \quad \forall \bar{m}, \bar{n} \in [\mathbb{M}]^i, \|e_{\bar{m}} - e_{\bar{n}}\| \leqslant \eta. \tag{11}$$

It follows from another Ramsey argument that it is enough to construct one pair $(\bar{m}, \bar{n}) \in I_k(\mathbb{M})$ such that $\|f(\bar{m}) - f(\bar{n})\| \leqslant 3(C+1)\,\mathrm{Lip}(f)$. We shall build $m_1 < n_1 < \cdots < m_i < n_i$ inductively as follows. Since X is C-asymptotic c_0, we shall play our usual k-round game. At each step i, the subspace player follows, as she may, a winning strategy to force the vector player to build a sequence which is $(C+1)$-equivalent to the canonical basis of ℓ_∞^k. So she picks X_i in $\mathrm{cof}(X)$ according to her winning strategy. Then the vector player picks $x_i \in S_{X_i}$, and 'we' choose $m_i < n_i$ in $\mathbb{M}$ according to the following scheme. The subspace player picks X_1 according to her strategy, the vector player picks $x_1 \in S_{X_1}$ and we just pick $m_1 < n_1$ in $\mathbb{M}$. Assume now that $X_1, \dots, X_{i-1}; x_1, \dots, x_{i-1}$ and $m_1 < n_1 < \cdots < m_{i-1} < n_{i-1}$ have been chosen for $2 \leqslant i \leqslant k$. For $n > n_{i-1}$, denote $y_n = x_{\{m_1,\dots,m_{i-1},n\}} - x_{\{n_1,\dots,n_{i-1},n+1\}}$ and $v_n = z_{\{m_1,\dots,m_{i-1},n\}} - z_{\{n_1,\dots,n_{i-1},n+1\}}$. The space player picks $X_i \in \mathrm{cof}(X)$ according to her strategy. Note that $X_i^\perp$ is a finite-dimensional weak* closed subspace of X^*.

(a) Assume first that $\liminf_{n\to\infty} \|y_n\| \leqslant \frac{1}{4k}\,\mathrm{Lip}(f)$.

Then we pick $n > n_{i-1}$ such that $\|y_n\| \leqslant \frac{1}{2k}\,\mathrm{Lip}(f)$, the vector player picks any $x_i \in S_{X_i}$ and we set $m_i = n$ and $n_i = n+1$.

(b) Assume now that $\liminf_{n\to\infty} \|y_n\| > \frac{1}{4k}\,\mathrm{Lip}(f)$.

Since (v_n) is weak*-null, we have that (v_n) tends uniformly to 0 on bounded subsets of $X_i^\perp$. It follows from (11) and the standard identification of $(X/X_i)^*$ with $X_i^\perp$ that $\limsup_{n\to\infty} d(y_n, X_i) \leqslant \eta$. So we can pick $n > n_{i-1}$ such that $\|y_n\| > \frac{1}{4k}\,\mathrm{Lip}(f)$ and $d(y_n, X_i) \leqslant 2\eta$, which implies the existence of $x_i \in S_{X_i}$ so that $\|\frac{y_n}{\|y_n\|} - x_i\| \leqslant \frac{16k\eta}{\mathrm{Lip}(f)}$. We set $m_i = n$ and $n_i = n+1$.

This concludes the description of our procedure and we recall that it ensures that $(x_i)_{i=1}^k$ is $(C+1)$-equivalent to the canonical basis of ℓ_∞^k. We now denote A as the set

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of i's such that procedure (a) has been followed and B as the complement of A. For simplicity, denote $u_i = x_{\{m_1,\dots,m_i\}} - x_{\{n_1,\dots,n_i\}}$. We clearly have

$$\Big\|\sum_{i\in A} u_i\Big\| \leqslant \frac{1}{2}\,\mathrm{Lip}(f).$$

On the other hand, we have

$$\begin{aligned}\Big\|\sum_{i\in B} u_i\Big\| &\leqslant \Big\|\sum_{i\in B}\|u_i\|x_i\Big\| + \Big\|\sum_{i\in B}\|u_i\|\Big(x_i - \frac{u_i}{\|u_i\|}\Big)\Big\| \\ &\leqslant (C+1)\max_{i\in B}\|u_i\| + k\max_{i\in B}\|u_i\|\frac{16k\eta}{\mathrm{Lip}(f)} \\ &\leqslant (2\mathrm{Lip}(f)+\eta)\Big(C+1+\frac{16k^2\eta}{\mathrm{Lip}(f)}\Big).\end{aligned}$$

Note that since f takes values in X, we also have that $f(\bar{m}) - f(\bar{n}) = \sum_{i=1}^k u_i$. Then, combining the above estimates with an initial choice of a small enough η, we get that $\|f(\bar{m}) - f(\bar{n})\| \leqslant 3(C+1)\,\mathrm{Lip}(f)$. □

We deduce the following.

Corollary 5.2. *There exists a Banach space X which is not reflexive but such that the family $([\mathbb{N}]^k, d_{\mathbb{H}}^{(k)})_{k\in\mathbb{N}}$ does not equi-coarsely embed into X.*

Proof. We only need to give an example of a quasi-reflexive, but not reflexive, asymptotic-c_0 Banach space. It is based on a construction due to Bellenot *et al.* [5]. For a given Schauder basis (u_i) of a Banach space X, the space $J[(u_i)]$ is defined to be the completion of c_{00} (the space of finitely supported sequences $(a_i)_{i=1}^\infty$ of real numbers) under the norm

$$\Big\|\sum a_i e_i\Big\| = \sup\Big\{\Big\|\sum_{i=1}^n\Big(\sum_{j\in s_i} a_j\Big)u_{p_i}\Big\|_X,\ n\in\mathbb{N}, s_1 < \cdots < s_n, \min s_i = p_i\Big\},$$

where $s_1,\dots,s_n$ are intervals in $\mathbb{N}$ and $(e_i)_{i=1}^\infty$ is the canonical basis of c_{00}.

It is proved in [5] that if (u_i) is the basis of a reflexive space, then $J[(u_i)]$ is quasi-reflexive of order one. Let now (u_i) be the unit vector basis of T^* (see the description of T^* in § 3). Since T^* is reflexive, $J[(u_i)]$ is quasi-reflexive of order one and, therefore, not reflexive. This particular space was first considered in [8] and estimates similar to those given in the proof of [8, Proposition 3.2] show that $J[(u_i)]$ is asymptotic-c_0. □

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