Euro. Jnl of Applied Mathematics (2002), *vol.* 13, *pp.* 497–516. © 2002 Cambridge University Press 497
DOI: 10.1017/S0956792502004862 Printed in the United Kingdom

Dynamics of the interface between two immiscible liquids with nearly equal densities under gravity

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(*Received 19 April 1999; revised 26 October 2001*)

We consider the two-dimensional Rayleigh–Taylor problem for the dynamics of the free interface Γ between two layers of immiscible viscous liquids. For a slow flow model (which corresponds to the case of a small relative jump of density) and under sufficiently wide assumptions on the geometry of Γ, we analyze the time dynamics of Γ. In particular, we prove that its increase in time t is bounded by an exponential function with exponent independent of Γ.

1 Introduction

The problem of studying the Rayleigh–Taylor instability has drawn the attention of many researchers. The properties of gravity flows in which a light (low-density) liquid is overlain by a heavier liquid have been well studied experimentally [13]. Theoretical results are mainly obtained for the asymptotic stage of instability evolution (at which bubbles of the light liquid and narrow jets of the heavier liquid are formed) by using both stationary [5, 15, 29] and nonstationary [7, 17–19] approaches.

In the present paper, we study the dynamics of the interface between two liquids taking into account dissipation effects. Our main assumption is that the relative jump of densities is small. For example, such a situation occurs in seismology problems when flows in the uppermost mantle are studied [20–22, 25]. Since in this case the flows are slow, we use the Stokes model. We restrict ourselves to the two-dimensional case and use the method of asymptotic expansion with respect to smoothness. As a direct result, we derive the equations

$$\dot{\zeta}_{\pm} = -(\rho_2 - \rho_1) \int_{\Gamma_t} G(\zeta - \zeta_{\pm}, \xi - \xi_{\pm}) \frac{\partial \phi}{\partial \xi} \omega, \tag{1.1}$$

which describe the motion of the maximal (ζ_+) and minimal (ζ_-) vertical coordinate of the interface $\Gamma_t = \{\phi(\zeta, \xi, t) = 0\}$ between the liquids under sufficiently general assumptions on the geometry of the curve Γ_t (see Figures 1 and 4). Here ρ_2 and ρ_1 are normed densities of the upper and lower liquids, $\xi_{\pm}$ are the horizontal coordinates of the maximum and minimum points, $G(\zeta, \xi)$ is the derivative with respect to ξ of the Green's function for the biharmonic operator in a strip that is periodic in ξ and decreases as $\zeta \to \pm\infty$, ω is the Leray measure on Γ_t, i.e. a 1-form satisfying the equation $d\phi \wedge \omega = d\xi \wedge d\zeta$.

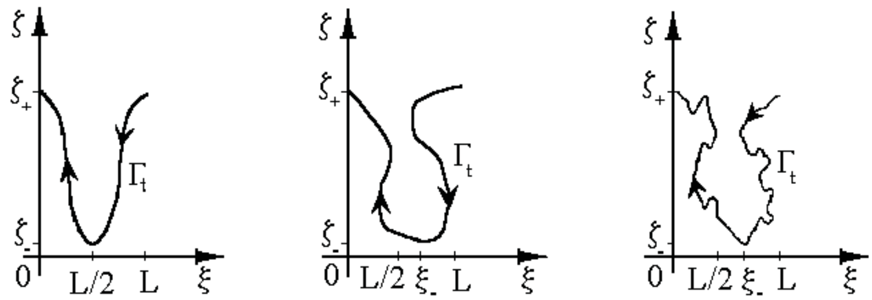

FIGURE 1. Admissible forms of the free interface between liquids with different densities.

In particular, a qualitative analysis of (1.1) shows that in the equation

$$\dot{\sigma} = (\rho_2 - \rho_1) f(\sigma, t)$$

for the width $\sigma = \zeta_+ - \zeta_-$ of the intermediate (fingering) zone, the right-hand side f can be estimated as

$$0 < f(\sigma, t) < c\sigma, \tag{1.2}$$

where the constant c is independent of the geometry of the curve Γ_t.

Note that Otto obtained a similar result for the two-fluid case of Hele–Shaw flow in a weak formulation [27].

Now let us derive the basic mathematical model. Consider the Navier–Stokes equations

$$\rho \frac{du}{dt} + \nabla p + \rho \vec{g} = \eta \Delta u, \qquad \operatorname{div} u = 0, \tag{1.3}$$

which describes flows in two-layered liquids under the action of a gravity force $\rho\vec{g}$, $\vec{g} = g\vec{e}_3$. We assume that the liquids separated by the surface $\Gamma_t = \{\phi(x,t) = 0\}$ are immiscible and of constant density. The density is ρ_2 in the upper layer ($\phi > 0$) and ρ_1 in the lower layer ($\phi < 0$). As usual, (1.3) is supplemented with the kinematic condition

$$\frac{dX}{dt} = u, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \langle u, \nabla \rangle, \tag{1.4}$$

where $X = X(x,t)$ are points on the surface Γ_t.

We denote the mean density by $\rho_0 = (\rho_1 + \rho_2)/2$, write the pressure p in the form

$$p = \text{const} - g\rho_0 x_3 + P,$$

and pass to the dimensionless coordinates

$$x = L_0 x', \qquad t = Tt'$$

and the dimensionless functions

$$u = Uu', \qquad \rho = \rho_0(1 + \Delta_0 \rho'), \qquad P = p_0 p',$$

where $\Delta_0 = (\rho_2 - \rho_1)/\rho_0$ is the relative jump in the density and the characteristic scales $U = L_0/T$ and p_0 correspond to the motion driven by gravity. Let

$$p_0 = g\rho_0 L_0 \,\mathrm{Fr},$$

where $\mathrm{Fr} = U_0/\sqrt{L_0 g}$ is the Froude number, and assume that $\Delta_0 = \mathrm{Fr}$. Then the equation of motion can be rewritten as

$$\mathrm{Fr}(1 + \Delta_0\rho')\frac{du'}{dt'} + \nabla p' + e_3\rho' = \frac{\mathrm{Fr}}{\mathrm{Re}}\Delta_{x'}u', \tag{1.3'}$$

where, as usual, $\mathrm{Re} = \rho_0 U L_0/\eta$ is the Reynolds number. We assume that

$$\mathrm{Fr} \ll 1. \tag{1.5}$$

Then we see that the coefficient Fr of the inertial terms in (1.3′) is small, while the coefficient $\varepsilon^2 = \mathrm{Fr}/\mathrm{Re}$ characterizing the viscous stress is, in general, not small. Thus assumption (1.5) allows us to pass from (1.3′) to the Stokes model. Next, note that for a medium formed by two layers of liquids with constant densities, which we study here, the kinematic condition (1.4) can be written in the following equivalent form:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho u) = 0, \tag{1.4'}$$

which is more convenient for us, since it allows us to pass to the global problem of finding a weak solution of a Stokes type system.

Restricting ourselves to the two-dimensional case, we write $\xi = x_1'/\varepsilon$ (the horizontal variable), $\zeta = x_3'/\varepsilon$ (the vertical variable), $p'' = p'/\varepsilon$, $t'' = t'/\varepsilon$, and preserve the notation ρ_2 and ρ_1 for the normed density in the upper and lower layers. Then, omitting the superscripts on the new variables and functions, we obtain the following basic system of equations:

$$\frac{\partial p}{\partial \xi} = \Delta_{\xi,\zeta}u, \tag{1.6}$$

$$\frac{\partial p}{\partial \zeta} + \rho = \Delta_{\xi,\zeta}v, \tag{1.7}$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \zeta} = 0, \tag{1.8}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \xi}(\rho u) + \frac{\partial}{\partial \zeta}(\rho v) = 0, \tag{1.9}$$

for the scalar functions u (the horizontal velocity), v (the vertical velocity), the pressure correction p, and the function $\phi = \phi(\zeta,\xi,t)$ describing the position of the interface $\Gamma_t = \{\phi = 0\}$ between the layers.

Note that (1.6)–(1.9) (and equations close to them) are used in seismology problems in modeling sedimentary basin formation [20, 21] and magma migration [22, 25]. The solvability of the initial boundary value problem for (1.6)–(1.9) was established elsewhere [1, 2].

Assume that the initial position of the interface Γ_0 is described by the function $\phi^0(\zeta,\xi)$ that is L-periodic in ξ, i.e. $\Gamma_0 = \{\phi^0 = 0\}$. Then, in view of the obvious translational properties of (1.6)–(1.9), it suffices to study these equations in the strip $\Pi = \{(\xi,\zeta), 0 < \xi < L, \zeta \in \mathbf{R}^1\}$ supplementing them with the boundary and initial conditions

$$\mathbf{u}\Big|_{\zeta\to\pm\infty} \to 0, \qquad \frac{\partial^k \mathbf{u}}{\partial \xi^k}\Big|_{\Sigma_1} = \frac{\partial^k \mathbf{u}}{\partial \xi^k}\Big|_{\Sigma_2}, \qquad k = 0, 1, \tag{1.10}$$

$$\rho\big|_{t=0} = \rho_1 + (\rho_2 - \rho_1)H\big(\phi^0(\zeta,\xi)\big), \tag{1.11}$$

where Σ_1 and Σ_2 are the lateral sides of the strip Π (for $\xi = 0$ and $\xi = L$), $\mathbf{u} = (u, v)$,

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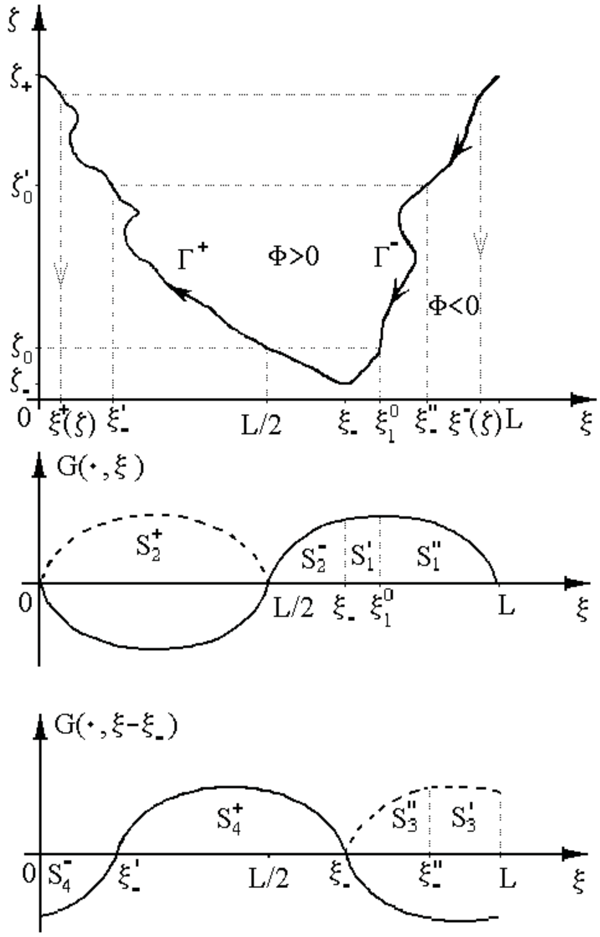

FIGURE 2. Free interface in the case of a unique minimum point and the corresponding Green's functions.

and $H(\tau)$ is the Heaviside function: $H(\tau) = 0$ for $\tau < 0$ and $H(\tau) = 1$ for $\tau > 0$. In what follows, we omit the indices ξ, ζ of the Laplace operator in the variables ξ and ζ.

We shall study the above problem by using an asymptotic expansion of the solution with regard to smoothness. For linear equations, the idea to expand the solution with respect to smoothness is well known. The development of this idea applied to quasilinear equations and special solutions lying in $\mathscr{D}'$ and admitting multiplication was performed by many authors [9, 10, 28], starting from Maslov's paper [24]. A possibility to study a more general class of nonsmooth solutions of quasilinear equations [3, 4, 6, 8, 14, 26] appeared after the construction of algebras of generalized function, which include distributions from $\mathscr{D}'$.

The paper is organized as follows. In § 1 we rewrite (1.6)–(1.9) in a more convenient form and introduce the basic equation (1.1) for the dynamics of critical points of the free interface $\Gamma_t = \{(\zeta, \xi) : \phi(\zeta, \xi, t) = 0\}$.

In § 2, we assume that, uniformly in t, the curve Γ_t has only one minimum point

$(\zeta_-(t), \xi_-(t))$, $\xi_- \in (0, L)$, (that is, $\partial\phi(\zeta_-, \xi_-, t)/\partial\xi = 0$) and prove the estimate (1.2). This means that the relation (1.2) obtained earlier [11] for symmetric curves (see plot 1 in Figure 1 and Figure 4) continues to hold under the perturbations shown in Figure 1, plot 2, and in Figure 2.

In § 3 we perform a similar study under the assumption that the function ϕ differs from that in Section 2 by small additional localized perturbations (see Figure 1, plot 3, and Figure 3).

In § 4 we consider the possibilities of a more detailed analysis for special shapes of Γ_t and the possibility of further development of the method proposed here.

Finally, in the Appendix, we prove some auxiliary formulas which we need for averaging the Heaviside function $H(\phi(\zeta, \xi, t))$ with respect to ξ and for establishing the basic equation for the dynamics of critical points.

2 Derivation of the basic model equations

We write

$$\rho = \rho_1 + (\rho_2 - \rho_1)H\big(\phi(\zeta, \xi, t)\big), \tag{2.1}$$

and assume that the following assumption is satisfied.

Assumption A *Let the function ϕ be L-periodic with respect to ξ, and let $\phi(\zeta, \xi, t) \in C^1(0, T; C^{2+\nu}(\Pi))$, where $\nu > 0$ is an arbitrary number. Suppose that $\Gamma_t = \{(\zeta, \xi) \in \Pi, \phi(\zeta, \xi, t) = 0)\}$ is a nondegenerate curve with finitely many critical points of the first order. Let $\phi > 0$ for all points (ξ, ζ) lying over the curve Γ_t, and $\phi < 0$ for all points lying under the curve Γ_t. We assume that the maximum value of the ζ-coordinate of the points $(\xi, \zeta) \in \Gamma_t$ for $\xi \in [0, L]$ is attained only at $\xi = 0$ and $\xi = L$ (we denote such points by $(0, \zeta_+)$ and (L, ζ_+)), while the minimum value is attained only at the point (ξ_-, ζ_-), $\xi_- \in (0, L)$.*

We now note that (1.6)–(1.9) are invariant with respect to the change of variables $\xi \to \xi^* - \xi$, $u(\zeta, \xi, t) \to -u(\zeta, \xi^* - \xi, t)$, $v(\zeta, \xi, t) \to v(\zeta, \xi^* - \xi, t)$, $\rho(\zeta, \xi, t) \to \rho(\zeta, \xi^* - \xi, t)$, and $p(\zeta, \xi, t) \to p(\zeta, \xi^* - \xi, t)$, where ξ^* is an arbitrary constant. If the function $\phi^0(\zeta, \xi)$ is even with respect to ξ^*, then the initial value $\rho\big|_{t=0}$ defined in (1.11) is also an even function. This fact implies the following statement.

Lemma 1 *Suppose that $\mathbf{u}$, ρ, and p form the solution of problem* (1.6)–(1.11) *and ρ has the form* (2.1). *Suppose also that the L-periodic $C^{2+\nu}$-function $\phi^0(\xi)$ is even with respect to the point $\xi = 0$. Then u is an odd function and φ, v, ρ and p are even functions with respect to this point.*

It follows from this lemma that the curve $\Gamma_t = \{(\xi, \zeta), \phi(\zeta, \xi, t) = 0\}$ attains its maximum at $\xi = 0$ if the curve $\Gamma_0 = \{(\xi, \zeta), \phi^0(\zeta, \xi) = 0\}$ has the same property.

To simplify the notation, we omit the argument t in all functions.

Let us transform (1.6)–(1.9). To this end, we introduce the stream function F, $v = \partial F/\partial\xi$, $u = -\partial F/\partial\zeta$ such that $F = F(\zeta, \xi)$ is L-periodic with respect to ξ, has zero average, and satisfies the problem

$$\Delta^2 F = \frac{\partial \rho}{\partial \xi}, \qquad F\Big|_{\zeta\to\pm\infty} \to 0. \tag{2.2}$$

It follows from the general theory of elliptic equations (e.g. see Lions & Magenes [23]) that if ϕ is a sufficiently smooth function, then $F \in H_p^{3-\nu}(\Pi)$ uniformly in t, where $\nu > 0$ is any number and H_p^s is the Sobolev–Slobodetskii space of functions L-periodic in ξ. Hence, in view of the embedding theorems, we have $F \in C^{2-\nu}(\Pi)$.

Obviously, by taking into account representation (2.1), we immediately arrive at the following equation for ϕ:

$$\frac{\partial \phi}{\partial t} - \frac{\partial F}{\partial \zeta}\Big|_{\Gamma} \frac{\partial \phi}{\partial \xi} + \frac{\partial F}{\partial \xi}\Big|_{\Gamma} \frac{\partial \phi}{\partial \zeta} = 0. \tag{2.3}$$

Theoretically, the solution of the quasilinear equation (2.3) can be constructed by the method of characteristics. Furthermore, with the help of the Fourier expansion, we can explicitly calculate the function F as a series. However, the formulas obtained in this way are very complicated and do not allow an effective analysis of the free interface dynamics.

We restrict our analysis to the consideration of the dynamics of critical points $(\xi_i, \zeta_i) \in \Gamma$ such that $\partial\phi(\zeta_i, \xi_i)/\partial\xi = 0$. It will be proved in the Appendix that the dynamics of ζ_i is described by the equation

$$\frac{d\zeta_i}{dt} = \frac{\partial F(\zeta_i, \xi)}{\partial \xi}\Big|_{\xi=\xi_i}. \tag{2.4}$$

Remark 1 Equation (2.4) can be derived from the system of equations of characteristics

$$\frac{\partial \zeta}{\partial t}(\zeta_0, \xi_0, t) = v(\zeta, \xi, \zeta, t) \qquad \zeta\big|_{t=0} = \varphi(\xi_0, 0)$$
$$\frac{\partial \xi}{\partial t}(\zeta_0, \xi_0, t) = u(\zeta, \xi, t) \qquad \xi\big|_{t=0} = \xi_0$$

only in the case in which the maximum (minimum) of the profile of the free boundary (the function $\varphi(\xi, t)$) moves along the characteristics. This means that if at $t = 0$ the extremum was at the point ζ_0^*, ξ_0^*, then for $t > 0$, it will be at the point $\xi(\zeta_0^*, \xi_0^*, t)$, $\zeta(\zeta_0^*, \xi_0^*, t)$. If the assumptions on the free boundary are more rigid than those considered in this paper, e.g. if the initial function $\phi^0(\zeta, \xi)$ is even in ξ with respect to the minima and maxima, then this property is obvious.

We do not study this problem in the general case but present a simple derivation of (2.4) which is based on the method of asymptotic expansions with respect to smoothness.

For a further reduction, we introduce an auxiliary function $G(\zeta, \xi)$ such that G is L-periodic with respect to ξ, has zero average, decreases as $\zeta \to \pm\infty$, and satisfies the equation

$$\Delta^2 G(\zeta, \xi) = \delta'(\xi)\delta(\zeta).$$

Obviously, $G \in H_p^{2-\nu}(\Pi)$. We shall also use Green's formula

$$(\Delta^2 g, f) = (g, \Delta^2 f). \tag{2.5}$$

For smooth functions g, f that are L-periodic with respect to ξ and decrease as $\zeta \to \pm\infty$, formula (1.5) can be readily obtained by integrating by parts. We can continuously extend this formula to the case where $g \in H_p^{2-\nu}(\Pi)$ and $f \in H_p^{3-\nu}(\Pi)$. Therefore, we have

$$\begin{aligned}\frac{\partial F}{\partial \xi}(\zeta_i, \xi_i) &= -\big(\delta(\zeta - \zeta_i)\delta'(\xi - \xi_i), F(\zeta, \xi)\big)\\ &= -\big(G(\zeta - \zeta_i, \xi - \xi_i), \Delta^2 F(\zeta, \xi)\big).\end{aligned}$$

Since $F = \Delta^{-2}\partial\rho/\partial\xi$ and (2.1) holds, we obtain the formula

$$\frac{\partial F}{\partial \xi}(\zeta_i, \xi_i) = -(\rho_2 - \rho_1)\left(G(\zeta - \zeta_i, \xi - \xi_i), \frac{\partial \phi}{\partial \xi}\delta(\phi)\right),$$

where $\delta(\phi)$ is the δ-function on the curve Γ [16], i.e. the distribution such that

$$\big(\varphi(\zeta, \xi), \delta(\phi)\big) = \int_\Gamma \varphi\omega \tag{2.6}$$

for all $\varphi \in \mathscr{D}(\Pi)$, and the 1-form ω satisfies the equation $d\phi \wedge \omega = d\xi \wedge d\zeta$.

Hence we arrive at our basic equation

$$\frac{d\zeta_i}{dt} = -(\rho_2 - \rho_1)\int_\Gamma G(\zeta - \zeta_i, \xi - \xi_i)\frac{\partial \phi}{\partial \xi}\omega. \tag{2.7}$$

It is easy to prove (e.g. see Danilov & Omel'yanov [11, 12]) that, uniformly in ζ, the function $G(\zeta, \xi)$ is odd with respect to the point $\xi = L/2$ and negative for $\xi \in (0, L/2)$. Moreover, G is even with respect to ζ and $|G(\zeta, \xi)|$ decreases with increasing $|\zeta|$ (for a fixed ξ). These properties are decisive in the analysis of the right-hand side in (2.7).

3 Estimates for the width of the intermediate (fingering) zone

Suppose that the curve Γ satisfies Assumption A and there exists only one minimum point (ξ_-, ζ_-). Then Γ can be divided into two parts (Γ^- from (L, ζ_+) to (ξ_-, ζ_-) and Γ^+ from (ξ_-, ζ_-) to $(0, \zeta_+)$), each of which can be uniquely projected on the ζ (see plot 2 in Figures 1 and 2). By taking into account the local representation of the form ω, in view of (2.7), we obtain

$$\frac{d\zeta_+}{dt} = -(\rho_2 - \rho_1)\int_{\zeta_-}^{\zeta_+} \big(G(\zeta - \zeta_+, \xi^+(\zeta)) - G(\zeta - \zeta_+, \xi^-(\zeta))\big)\, d\zeta, \tag{3.1}$$

$$\begin{aligned}\frac{d\zeta_-}{dt} = -(\rho_2 - \rho_1)\int_{\zeta_-}^{\zeta_+} &\big(G(\zeta - \zeta_-, \xi^+(\zeta) - \xi_-)\\ &-G(\zeta - \zeta_-, \xi^-(\zeta) - \xi_-)\big)d\zeta,\end{aligned} \tag{3.2}$$

where $\xi^\pm = \xi^\pm(\zeta)$ is the solution of the equation $\phi(\zeta, \xi) = 0$, which corresponds to $\Gamma^\pm$.

We write $\sigma = \zeta_+ - \zeta_-$. Then, after this change of variable, we obtain the following equation for the width σ:

$$\begin{aligned}\frac{d\sigma}{dt} = (\rho_2 - \rho_1)\int_0^\sigma \Big\{ &-G(z - \sigma, \xi^+(z + \zeta_-)) + G(z - \sigma, \xi^-(z + \zeta_-))\\ &+G(z, \xi^+(z + \zeta_-) - \xi_-) - G(z, \xi^-(z + \zeta_-) - \xi_-)\Big\}\, dz.\end{aligned} \tag{3.3}$$

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To analyze the expression on the right-hand side in (3.3) in the simplest way, we need the following additional assumption.

Assumption B *Let Γ have only one minimum point, and let the following condition be satisfied at all critical points (ξ_k, ζ_k) such that $\partial\phi(\xi_k, \zeta_k)/\partial\zeta = 0$:*

$$0 \leqslant \xi_k \leqslant \frac{L}{2} \quad \text{if} \quad (\xi_k, \zeta_k) \in \Gamma^+, \qquad \frac{L}{2} \leqslant \xi_k \leqslant L \quad \text{if} \quad (\xi_k, \zeta_k) \in \Gamma^-. \tag{3.4}$$

Now we show that under this assumption the sign of the right-hand side in (3.1) coincides with the sign of $(\rho_2 - \rho_1)$, while the sign of the right-hand side in (3.2) is opposite to the sigh of $(\rho_2 - \rho_1)$. Obviously, this implies that the width σ of the mushy region increases with time for $\rho_2 > \rho_1$ (the heavy liquid is over the light liquid) and decreases for $\rho_2 < \rho_1$.

First, we discuss a special case of the curve Γ in which $\xi_- = L/2$. We have $G(\cdot, \xi) < 0$ for $\xi \in (0, L/2)$ and $G(\cdot, \xi) > 0$ for $\xi \in (L/2, L)$. In this special case $\xi^+(\zeta) \in (0, L/2)$ and $\xi^-(\zeta) \in (L/2, L)$ for all $\zeta \in (\zeta_-, \zeta_+)$. Hence we have

$$G(\zeta - \zeta_+, \xi^+(\zeta)) < 0, \qquad G(\zeta - \zeta_+, \xi^-(\zeta)) > 0,$$

and hence the sign of the right-hand side in (3.1) coincides with the sign of $\rho_2 - \rho_1$. In a similar way, it readily follows from the inequalities

$$G(\cdot, \xi - L/2) > 0 \text{ for } \xi \in (0, L/2) \quad \text{and} \quad G(\cdot, \xi - L/2) < 0 \text{ for } \xi \in (L/2, L)$$

in this special case that the sign of the right-hand side in (3.2) is opposite to the sign of $(\rho_2 - \rho_1)$.

Let us consider the general case.

For definiteness, we assume that $\xi_- \geqslant L/2$. Then, in view of Assumption A and the properties of the function $G(\zeta, \xi))$ mentioned in § 1, we have

$$\int_{\zeta_-}^{\zeta_+} G(\zeta - \zeta_+, \xi^-(\zeta))\, d\zeta > 0, \qquad -\int_{\zeta_0}^{\zeta_+} G(\zeta - \zeta_+, \xi^+(\zeta))\, d\zeta > 0. \tag{3.5}$$

However,

$$-\int_{\zeta_-}^{\zeta_0} G(\zeta - \zeta_+, \xi^+(\zeta))\, d\zeta < 0. \tag{3.6}$$

Here ζ_0 satisfies the condition $\xi^+(\zeta_0) = L/2$ (see Figure 2).

We write $\xi_1^0 = \xi^-(\zeta_0)$ and note that in the integrals of $G(\zeta - \zeta_+, \xi^-(\zeta))$ and $G(\zeta - \zeta_+, \xi^+(\zeta))$ from ζ_- to ζ_0 the first arguments of these functions coincide, while the second arguments satisfy the conditions: $\xi^-(\zeta) \in (\xi_-, \xi_1^0)$ and $\xi^+(\zeta) \in (L/2, \xi_-)$. Thus the difference between these integrals is equivalent to the difference $|S_1'| - |S_2^-|$ between the areas of figures S_1' and S_2^- shown in the second plot in Figure 2. In the general case the difference $|S_1'| - |S_2^-|$ is not necessarily positive. However, we note that for $\zeta \in [\zeta_0, \zeta_+]$ we have $|G(\zeta - \zeta_+, \xi^\pm(\zeta))| \geqslant |G(\widetilde{\zeta} - \zeta_+, \xi^\pm(\zeta))|$, where $\widetilde{\zeta}$ is any number lying in the interval $[\zeta_-, \zeta_0]$. Thus we can use the areas $|S_2^+|$, $|S_1''|$ of figures S_2^+ and S_1'' as lower estimates of the integrals of the functions $G(\zeta - \zeta_+, \xi^-(\zeta))$ and $G(\zeta - \zeta_+, \xi^+(\zeta))$ from ζ_0 to ζ_+ (see Figure 2). Since $\xi_- < L$, obviously, we have $|S_2^+| + |S_1'| + |S_1''| > |S_2^-|$. This readily implies that the sum of integrals (3.5) and (3.6) is positive. Now it is clear that $\text{sign}(d\zeta_+/dt) = \text{sign}(\rho_2 - \rho_1)$.

In the study of (3.2) it is necessary to take into account that the second argument of the function G is shifted by ξ_- and the inequality $|G(\zeta-\zeta_-,\xi^{\pm}(\zeta))| \geqslant |G(\widetilde{\zeta}_1-\zeta_-,\xi^{\pm}(\zeta))|$ holds for all $\zeta \in [\zeta_-,\zeta_0']$ and $\widetilde{\zeta}_1 \in [\zeta_0',\zeta_+]$. Here ζ_0' satisfies the condition $\xi^+(\zeta_0') = \xi_-' \stackrel{\text{def}}{=} \xi_- - L/2$. One can readily see that the sign of the right-hand side in (3.2) is determined by the balance of the areas $|S_4^+|$, $|S_3'|$, $|S_3''|$, and $|S_4^-|$ of the corresponding figures shown in the lowest plot in Figure 2 (here $\xi_-'' = \xi^-(\zeta_0')$). Since necessarily $|S_4^+|+|S_3''|+|S_3'| > |S_4^-|$, we have $\mathrm{sign}(d\zeta_-/dt) = -\,\mathrm{sign}(\rho_2-\rho_1)$.

It remains to note that $G \in C^{1-\nu}(\Pi)$, and hence G is uniformly bounded. Thus, we arrive at the following statement.

Theorem 1 *Suppose that Assumptions* A *and* B *are satisfied. Then the width* σ *of the mushy region satisfies the equation*

$$\frac{d\sigma}{dt} = (\rho_2-\rho_1)f(\sigma,t), \tag{3.7}$$

where the function f *whose explicit form is given in* (3.3) *satisfies the estimates*

$$0 < f(\sigma,t) < c\sigma, \qquad c = 4\max_{\xi}|G(0,\xi)|. \tag{3.8}$$

Remark 2 Condition (3.4) becomes weaker if we assume that $-\mu \leqslant \xi_k \leqslant \frac{L}{2}+\mu$ for $(\xi_k,\zeta_k) \in \Gamma^+$ and $\frac{L}{2}-\mu \leqslant \xi_k \leqslant L+\mu$ for $(\xi_k,\zeta_k) \in \Gamma^-$, where $\mu > 0$ is a sufficiently small number. In this case the right-hand sides in (3.1), (3.2), besides of terms $\sim |S_2^-|$ and $\sim |S_4^-|$, contain another 'irregular' terms arising when we integrate over the parts Γ^+ lying to the right of the line $\xi = L/2$ and to the left of the line $\xi = 0$ and over the parts Γ^- lying to the left of the line $\xi = L/2$ and to the right of the line $\xi = L$. Since the number of critical points is finite, the contribution of all these terms is negative and of $O(\mu)$ as $\mu \to 0$. One can easily see that the statement of Theorem 1 remains valid for sufficiently small μ.

4 'Stability' of motion of a free boundary

In this case 'stability' means that the qualitative description of the motion of Γ obtained in § 2 remains the same under the action of small localized perturbations (similar to those shown in Figure 1, plot 3, and Figure 3).

Assumption C *Suppose that Assumption* A *is satisfied and*

$$\phi = \phi_0(\zeta,\xi) + \mu\phi_1(\zeta/\mu,\xi/\mu), \tag{4.1}$$

where ϕ_0 *satisfies Assumption* B *and* $\phi_1(\zeta/\mu,\xi/\mu))$ *has the same smoothness as* ϕ_0 *and is localized in a* μ*-neighbourhood of* N *points* (ξ_i,ζ_i)*, where* $N \ll 1/\mu$ *and* $\mu > 0$ *is a sufficiently small number.*

It follows from the above that it suffices to consider only the critical points $(\xi_{\pm,i},\zeta_{\pm,i})$ of the curve $\Gamma = \{(\xi,\zeta),\phi(\zeta,\xi,\mu) = 0\}$, where $\partial\phi/\partial\xi = 0$. By $(\xi_{+,i},\zeta_{+,i})$ and $(\xi_{-,i},\zeta_{-,i})$ we denote the coordinates of local maxima and minima of Γ and by (ξ_+,ζ_+), where $\xi_+ = 0$

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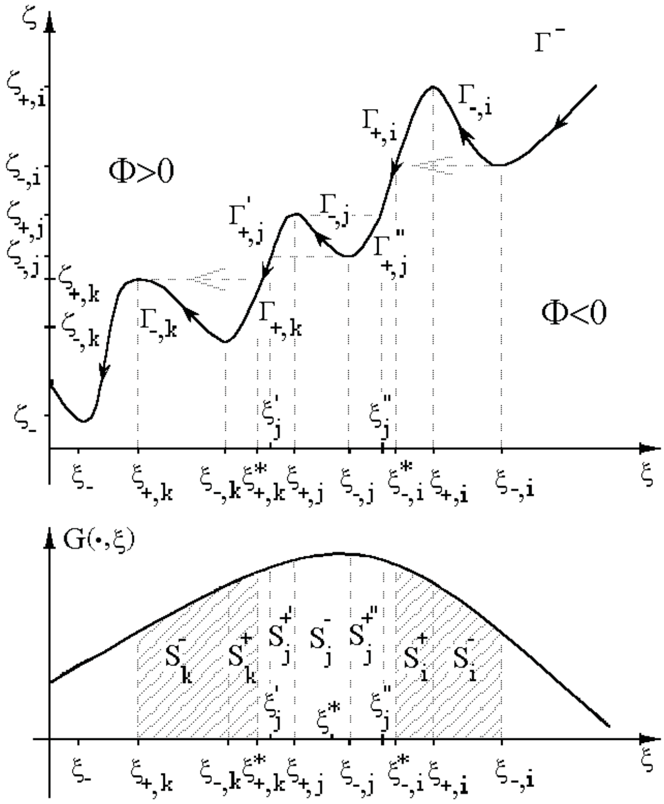

FIGURE 3. Small localized perturbations of the free interface.

or $\xi_+ = L$, and (ξ_-, ζ_-) we denote the coordinates of the global maximum and the global minimum (see Figure 3).

Let us study the right-hand side of (2.7) for $\zeta_i = \zeta_+$ in the situation of Assumption C. A distinction between this problem and that studied in § 2 (see (3.1)) is that each part $\Gamma^{\pm}$ of the curve Γ now contains parts of 'irregular' orientation. We restrict our consideration to Γ^- and denote such irregular parts by $\Gamma_{-,i}$. We choose a number i. Let ξ_G be the point at which $G(\cdot, \xi)$ attains its maximum, and let $\xi^* = \max_{\zeta \in [\zeta_{-,i}, \zeta_{+,i}]} \xi_G(\zeta)$. Suppose that the projection of $\Gamma_{-,i}$ on the ξ-axis lies to the right of the point ξ^*. We consider the part of Γ that follows $\Gamma_{-,i}$ (in accordance with the orientation of Γ) and lies between the points $(\xi_{+,i}, \zeta_{+,i})$ and $(\xi^*_{-,i}, \zeta_{-,i})$ (see Figure 3). We denote this part of Γ by $\Gamma_{+,i}$. By $\xi^*_{-,i}$ we denote the maximal point ξ such that $\xi < \xi_{-,i}$ and $\phi(\zeta_{-,i}, \xi, \mu)\big|_{\xi = \xi^*_{-,i}} = 0$. In addition, we assume that $\xi^*_{-,i} > \xi^*$. In the integral on the right-hand side of (2.7) we consider the part of the integral that corresponds to $\Gamma_{-,i} \cup \Gamma_{+,i}$. By using the local representation of the form ω, we rewrite this integral in the form

$$J_i = \int_{\zeta_{-,i}}^{\zeta_{+,i}} G\big(\zeta - \zeta_+, \xi_{+,i}(\zeta)\big) - G\big(\zeta - \zeta_+, \xi_{-,i}(\zeta)\big)\, d\zeta, \tag{4.2}$$

where $\xi_{\pm,i}(\zeta)$ are the solutions of the equation $\phi(\zeta, \xi, \mu) = 0$ corresponding to the parts $\Gamma_{\pm,i}$, respectively.

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Since $\xi^* < \xi^*_{-,i}(\zeta) < \xi_{+,i}(\zeta) < \xi_{-,i}(\zeta)$ for all $\zeta \neq \zeta_{+,i}$, we have $G(\cdot, \xi_{+,i}(\zeta)) > G(\cdot, \xi_{-,i}(\zeta))$ and hence $J_i > 0$. From the geometric viewpoint, this means that we can obtain the lower bound for the integral in (2.7) by calculating the integral over a curve in which $\Gamma_{-,i} \cup \Gamma_{+,i}$ is replaced by the line connecting the points $(\xi_{-,i}, \zeta_{-,i})$ and $(\xi^*_{-,i}, \zeta_{-,i})$ (see Figure 3) and by equating the integrand with zero on this line.

From another viewpoint, we compare the areas $|S_i^-|$ and $|S_i^+|$ of the figures S_i^- and S_i^+ shown in the lower plot in Figure 3. The above estimate means that we cut off the part $S_i^+ \cup S_i^-$ from the figure bounded by $G(\cdot, \xi)$ and the ξ-axis.

In a similar way, we consider the parts $\Gamma_{-,k}$ whose projections on the ξ-axis lie to the left from the maximum point of G. In this case, however, we assume that $\xi^* = \min_{\zeta \in [\zeta_{-,k}, \zeta_{+,k}]} \xi_G(\zeta)$ and perform the compensation by using the part $\Gamma_{+,k}$ lying between the points $(\xi^*_{+,k}, \zeta_{+,k})$ and $(\xi_{-,k}, \zeta_{-,k})$ (see Figure 3). Here $\xi^*_{+,k}$ is the minimal point ξ such that $\xi > \xi_{-,k}$ and $\phi(\zeta_{+,k}, \xi, \mu)\big|_{\xi=\xi^*_{+,k}} = 0$. By setting $\xi^*_{+,k} < \xi^*$, we obtain the lower bound for the integral in (2.7) by calculating the integral over a curve in which $\Gamma_{+,k} \cup \Gamma_{-,k}$ is replaced by the line connecting the points $(\xi^*_{+,k}, \zeta_{+,k})$ and $(\xi_{+,k}, \zeta_{+,k})$ and by equating the integrand with zero on this line.

However, in the general case there also exist parts $\Gamma_{-,j}$ whose projection on the ξ-axis either contains the maximum point of G or lies in the immediate neighbourhood of this point so that the assumptions that $\xi^*_{-,i} > \xi^*$ or $\xi^*_{+,k} < \xi^*$ are not satisfied. In this case, in general, the integrals over $\Gamma''_{+,j}$ (from $(\xi''_j, \zeta_{+,j})$, $\xi''_j \stackrel{\text{def}}{=} \xi''_{+,j}(\zeta_{+,j})$, to $(\xi_{-,j}, \zeta_{-,j})$) and over $\Gamma'_{+,j}$ (from $(\xi_{+,j}, \zeta_{+,j})$ to $(\xi'_j, \zeta_{-,j})$, $\xi'_j \stackrel{\text{def}}{=} \xi'_{+,j}(\zeta_{-,j})$) do not compensate the integral over $\Gamma_{-,j}$. In other words, it is possible that $|S_j^{+\prime}| + |S_j^{+\prime\prime}| < |S_j^-|$ (see Fig. 3). However, we must take into account that $G(\zeta - \zeta_+, \xi) > G(\zeta_{+,j} - \zeta_+, \xi)$ for $\zeta > \zeta_{+,j}$. Thus the problem of finding the sign of the right-hand side in (2.7) is reduced to comparing the area $\Sigma_- = |S_j^-| - (|S_j^{+\prime}| + |S_j^{+\prime\prime}|)$ with the area Σ_+ of all unshaded parts of the figure bounded by the line $\xi = \xi''_j$, the ξ-axis, and the curve $G(\cdot, \xi)$ (see Figure 3). Now we must take into account the fact that all intervals $[\xi^*_{-,i}, \xi_{-,i}]$ are of length $O(\mu)$. Thus the total area Σ'_- of all shaded parts is small, $\Sigma'_- \sim N(|S_i^+| + |S_i^-|) \ll 1$. For the same reason, we have $\Sigma_- = O(\mu)$ and hence $\Sigma_+ - \Sigma_- > 0$ for sufficiently small μ.

In a similar way, considering the part Γ^- and (2.7) with $\zeta_i = \zeta_-$, we obtain

$$\operatorname{sign}(d\zeta_+/dt) = \operatorname{sign}(\rho_2 - \rho_1), \qquad \operatorname{sign}(d\zeta_-/dt) = -\operatorname{sign}(\rho_2 - \rho_1). \tag{4.3}$$

Now we write the equation for σ in the form (3.7) with the right-hand side

$$f = \int_\Gamma \{-G(\zeta - \zeta_+, \xi) + G(\zeta - \zeta_-, \xi - \xi_-)\} \frac{\partial \phi}{\partial \xi} \omega, \tag{4.4}$$

and see that $f > 0$. The second inequality in (3.8) can easily be proved by using a rough estimate of G by the maximum of its absolute value. In this case it should be noted that the lengths of the part $\Gamma_{-,i}$ and of the adjacent part $\Gamma_{+,i}$ or of the adjacent parts $\Gamma'_{+,i}$ and $\Gamma''_{+,i}$ are of the order of $O(\mu)$ and the total number of such parts is $N \ll 1/\mu$. Thus we arrive at the following generalization of the statement of Theorem 1.

Theorem 2 *Suppose that Assumption* C *is satisfied and* μ *is a sufficiently small number. Then the width* σ *satisfies* (3.7) *and the estimate* (3.8) *with the constant* $c = (4 + O(\mu N)) \max_\xi |G(0, \xi)|$.

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Now we present the following result on estimation of the rate of growth of the perturbation amplitude $\sigma_i = \zeta_{+,i} - \zeta_{-,i}$ in the unstable case.

Theorem 3 *Suppose that Assumption* C *is satisfied and* μ *is a sufficiently small number. Then* σ_i *satisfies an equation of the form* (3.7) *whose right-hand side satisfies the estimate*

$$|f(\sigma_i, t)| \leqslant c\sigma_i + O(\mu + \sigma_i^2), \qquad c = \text{const}. \tag{4.5}$$

Since $\sigma_i\big|_{t=0} = O(\mu)$*, inequality* (4.5) *means that* $\sigma_i(t) = O(\mu)$ *for all finite* t*.*

Let us outline the proof of Theorem 3. By Γ_i we denote the part of the curve Γ on which the ith perturbation is localized mod$O(\mu^2)$ and write $\Gamma' = \Gamma \setminus \Gamma_i$. By using (2.7) we write the equation for σ_i and represent the right-hand side as the sum of integrals over Γ' and Γ_i. In the first integral the functions $G(\zeta - \zeta_{\pm,i}, \xi - \xi_{\pm,i})$ are smooth, and we can use the Taylor formula. Hence on Γ' we have

$$|G(\zeta - \zeta_{+,i}, \xi - \xi_{+,i}) - G(\zeta - \zeta_{-,i}, \xi - \xi_{-,i})| \leqslant c_1\sigma_i \tag{4.6}$$

with a constant c_1 independent of μ. Then we estimate the integral over Γ_i by using the fact that $G(\zeta - \zeta_{\pm,i}, \xi - \xi_{\pm,i})$ satisfies the Hölder condition and the distance $|\xi_{+,i} - \xi_{-,i}|$ is small. Thus we again arrive at an estimate of the form (4.6). An accurate realization of this scheme only slightly differs from the proof carried out in Danilov & Omel'yanov [11].

5 Conclusion

In this paper we propose a new method of investigation of the free boundary motion by using asymptotic expansions with respect to smoothness and apply it for studying flows arising due to the Rayleigh–Taylor instability. We restrict our considerations to a comparatively simple two-dimensional case of a small relative jump of density (1.5), which, nevertheless, corresponds to a concrete physical problem, and obtain (1.1) for vertical velocities of the interface motion. These 'explicit' formulas allow us to carry out a sufficiently detailed qualitative analysis without numerical simulation. In this case the shape of the curve Γ_t is assumed to be sufficiently arbitrary, which, in fact, corresponds to the initial and transient stages of instability evolution. Obviously, by specifying the shape of the curve Γ_t, the results obtained above can be made more precise.

For example, suppose that Γ_t is of the shape that is typical of the late state of evolution (see Figure 4). In this case the interface Γ_t can be considered as the union of curves of the following three types: segments of the curve Γ^1 slowly varying w.r.t. ξ, which correspond to the sides of large 'bubbles' of the light liquid, the fast varying curve Γ^3, which corresponds to the bottom of the heavy liquid jet, and the curves Γ^2 slowly varying w.r.t ζ which correspond to the jet sides. Let us compare the velocity $\dot{\zeta}_+$ of bubble floating and the velocity $\dot{\zeta}_-$ of jet falling. One can easily verify (e.g. by substituting $\omega = (\partial\phi/\partial\xi)^{-1}\, d\zeta$ and replacing Γ^2 by vertical lines) that the integral in (1.1) taken over the curves Γ^2 provides approximately the same contribution to $\dot{\zeta}_+$ and $\dot{\zeta}_-$. Next, we can assume that $|\partial\phi/\partial\xi| \ll 1$ on Γ^1, while we have $|\partial\phi/\partial\xi| \gg 1$ on Γ^3 (outside a small

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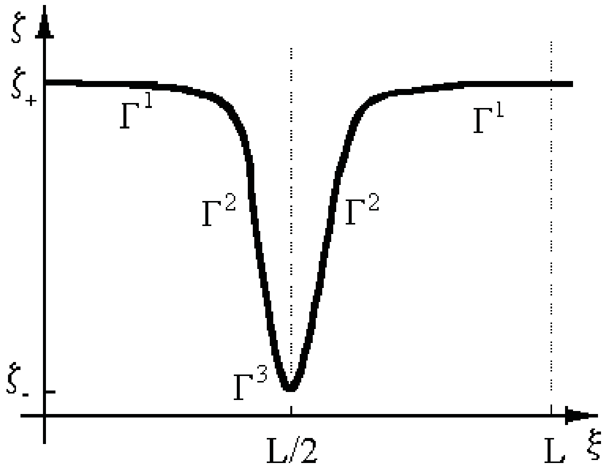

FIGURE 4. Bubble-type free interface.

neighbourhood of the point $(\zeta_-, L/2)$). Therefore, for the integrals

$$\int_{\Gamma^1} G(\zeta-\zeta_+,\xi)\frac{\partial\phi}{\partial\xi}\omega \quad \text{and} \quad \int_{\Gamma^3} G(\zeta-\zeta_+,\xi)\frac{\partial\phi}{\partial\xi}\omega$$

contained in $\dot{\zeta}_+$, we can readily conclude that the first integral is small due to the fact that $\partial\phi/\partial\xi$ is small and the second integral is small due to the fact that $G(\zeta,\xi)$ is (exponentially) small for $\zeta \sim \sigma = \zeta_+ - \zeta_-$ and sufficiently large σ. Conversely, to calculate $\dot{\zeta}_-$, we need to consider the integrals

$$\int_{\Gamma^1} G(\zeta-\zeta_-,\xi-L/2)\frac{\partial\phi}{\partial\xi}\omega \quad \text{and} \quad \int_{\Gamma^3} G(\zeta-\zeta_-,\xi-L/2)\frac{\partial\phi}{\partial\xi}\omega.$$

The first integral is small due to $\partial\phi/\partial\xi$ and G, since we have $\zeta-\zeta_- \sim \sigma$ on Γ^1. However, the second integral gives an essential contribution to $\dot{\zeta}_-$, since here we have $|\partial\phi/\partial\xi| \gg 1$ and $|G| \sim 1$. Thus, after a simple qualitative reasoning, we obtain the well-known result: $|\dot{\zeta}_-| > |\dot{\zeta}_+|$.

At the same time, it should be noted that the 'exact' equation (1.1) can be used only for a qualitative analysis of the dynamics of the free surface. For example, this follows from the study of the simplest case in which Γ_t is a small symmetric perturbation of the horizontal straight line $\Gamma_t = \{\zeta = \nu\psi(\xi,t), \nu \ll 1, \psi'_\xi < 0$ for $\xi \in (0, L/2)$ and $\psi'_\xi > 0$ for $\xi \in (L/2, L)\}$.

It is easy to verify that in this case the linearization of the equation leads to the relation

$$\frac{\partial\sigma}{\partial t} = 4(\rho_2-\rho_1)\nu\int_0^{L/2} G(0,\xi)\psi'_\xi(\xi,t)\,d\xi + O(\nu^2).$$

With precision up to $O(\nu^2)$ the obtained expression is linear with respect to ψ but implies only the estimate

$$\nu\int_0^{L/2} G(0,\xi)\psi'_\xi(\xi,t)\,d\xi \leqslant \max_\xi |G(0,\xi)|\sigma,$$

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which is exactly the same as that in the general case. Of course, this fact shows that the problem under study is essentially nonlocal.

In conclusion, we note that our simplifying assumptions (the two-dimensional case and condition (1.5)) are not absolutely necessary. It is rather obvious that the results obtained here can be transferred to the three-dimensional case provided that condition (1.5) is satisfied. In this case a more detailed analysis of the derivative of the Green's function in the strip $\xi_i \in (0, L_i)$, $i = 1, 2$, is only required. If we abandon condition (1.5), the problem becomes more complicated, since in this case one cannot exclude the velocity and pressure as it was done in § 1. Nevertheless, progress in this problem is also possible if we expand all the functions contained in (1.3), (1.4) simultaneously with respect to smoothness.

Acknowledgements

The authors are very grateful to Professor S. Gertsenshtein and Dr S. Howison for their useful advice on the physical interpretation of the model. The research was partially supported by PRAXIS XXI, 2/2.1/MAT/53/94 and by the Russian Foundation for Basic Research under grant No. 99-01-01074.

Auxiliary formulas and derivation of (2.4)

We derive (2.4) without using the assumption that each critical point (ζ_i, ξ_i) of the free interface Γ remains lying on the same characteristic for all $t \geqslant 0$.

To this end, we define the weighted average of ρ by the formula

$$\overline{\rho}_{f'} = \frac{1}{L}\int_0^L \frac{\partial f(\zeta, \xi, t)}{\partial \xi} \rho(\zeta, \xi, t)\, d\xi, \tag{A.1}$$

assuming that the kernel $\partial f/\partial \xi$ is a function uniformly continuous with respect to ξ, ζ, and t.

To make the computations of the right-hand side of (A.1) for ρ of the form (1.1) more constructive, we obtain the following simple formula.

Lemma 2 *Suppose that Assumption* A *is satisfied. Then* $\overline{\rho}_{f'}(\zeta)$ *is a continuous function of the form*

$$\overline{\rho}_{f'}(\zeta) = \frac{1}{L}\{f(\zeta, L) - f(\zeta, 0)\}\rho\Big|_{\xi=0} - \frac{\rho_2 - \rho_1}{L}\int f\phi'_\xi \delta(\phi)\, d\xi, \tag{A.2}$$

where $\int g\delta(\phi)\, d\xi$ *is a distribution such that the following relation holds for all test functions* $\psi(\zeta) \in \mathscr{D}$:

$$\left(\int g\delta(\phi)\, d\xi, \psi\right) = \int_\Gamma g\psi\omega, \tag{A.3}$$

where the 1*-form* ω *satisfies the equation* $d\phi \wedge \omega = d\xi \wedge d\zeta$.

Here and in what follows, we use the notation $\phi'_\xi = \frac{\partial\phi}{\partial\xi}$, $\phi'_\zeta = \frac{\partial\phi}{\partial\zeta}$, $\phi''_\xi(\zeta) = \frac{\partial^2\phi}{\partial\xi\partial\zeta}$, and so on. We also omit the argument t in all the functions.

By specifying the geometric properties of Γ, we rewrite the right-hand side in (A.2) in a simpler form. In particular, we have the following assertion.

Corollary 1 *Suppose that only* $(0,\zeta_+)$, (L,ζ_+), *and* (ξ_-,ζ_-) *are critical points of* Γ *such that* $\phi'_\xi\big|_\Gamma = 0$. *Then*

$$\overline{\rho}_{f'}(\zeta) = \frac{\rho^*}{L}\{f(\zeta,L) - f(\zeta,0)\} \qquad \text{for } \zeta \notin (\zeta_-,\zeta_+), \tag{A.4}$$

$$\overline{\rho}_{f'}(\zeta) = \frac{\rho_1}{L}\{f(\zeta,L) - f(\zeta,0)\}$$

$$-\frac{\rho_2-\rho_1}{L}\{f(\zeta,\xi^+(\zeta)) - f(\zeta,\xi^-(\zeta))\} \qquad \text{for } \zeta \in (\zeta_-,\zeta_+).$$

Here $\rho^* = \rho_1$ *for* $\zeta \leqslant \zeta_-$ *and* $\rho^* = \rho_2$ *for* $\zeta \geqslant \zeta_+$, $\xi^+(\zeta)$ *is a solution of the equation* $\phi(\zeta,\xi) = 0$ *for* $\xi \in (0,\xi_-)$ *and* $\xi^-(\zeta)$ *is a solution of this equation for* $\xi \in (\xi_-,L)$.

Further, we show that at the critical points the average $\overline{\rho}_{f'}(\zeta)$ has singularity of branch point type.

Lemma 3 *Suppose that Assumption* **A** *is satisfied and* (ξ_j,ζ_j) *are critical points such that* $\phi'_\xi(\zeta_j,\xi_j) = 0$. *Then* $\overline{\rho}_{f'}$ *has first-order continuous derivatives for all* $\zeta \neq \zeta_j$. *In neighbourhoods of the general points* ζ_j *the following relations are satisfied*:

$$\overline{\rho}_{f'}(\zeta) = \frac{1}{L}\{f(\zeta_j,L) - f(\zeta_j,0)\}\rho(\zeta,0)$$

$$-2\frac{\rho_2-\rho_1}{L}\kappa_j R_j \frac{\partial f(\zeta_j,\xi_j)}{\partial \xi}|\zeta-\zeta_j|^{1/2} + O\big(|\zeta-\zeta_j|\big). \tag{A.5}$$

Here $\kappa_j = 1$ *if* (ξ_j,ζ_j) *is the minimum point on* Γ *and* $\kappa_j = -1$ *if* (ξ_j,ζ_j) *is the maximum point. The value of* $R_j > 0$ *depends on* ϕ'' *calculated at intermediate points.*

Before we start proving these assertions, we show how (2.4) can be derived by using (A.5).

Let us consider (1.9). We average it over ξ and take into account the fact that $v = \partial F/\partial \xi$. We obtain

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}_{F'}}{\partial \zeta} = 0, \tag{A.6}$$

where we write $\overline{\rho}$ instead of $\overline{\rho}_1$ (i.e. $f = \xi$).

Suppose that $(\xi_i,\zeta_i) \in \Gamma$ is a critical point of the function ϕ such that $\phi'_\xi(\zeta_i,\xi_i) = 0$. By Lemma 3, in a neighbourhood of the point ζ_i we represent $\overline{\rho}$ and $\overline{\rho}_{F'}$ as asymptotic expansions with respect to smoothness [9, 10, 28]:

$$\overline{\rho}(\zeta) = A_i|\zeta-\zeta_i|^{1/2} + B_i|\zeta-\zeta_i| + O(|\zeta-\zeta_i|^{3/2}),$$
$$\overline{\rho}_{F'}(\zeta) = A_i\frac{\partial F}{\partial \xi}(\zeta_i,\xi_i)|\zeta-\zeta_i|^{1/2} + C_i|\zeta-\zeta_i| + O(|\zeta-\zeta_i|^{3/2}),$$

where A_i, B_i and C_i are some coefficients.

By substituting these expansions into (A.6) and equating the coefficients of $|\zeta-\zeta_i|^{-1/2}$ with zero, we readily obtain (2.4).

Proof of Lemma 2

By $\psi = \psi(\zeta)$ we denote a test function from $\mathscr{D}(\mathbf{R}^1)$ and consider the inner product $(\overline{\rho}_{f'},\psi)$.

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Taking into account (A.1) and integrating by parts, we obtain

$$(\overline{\rho}_{f'}, \psi) = \frac{1}{L}\big(\{f(\zeta, L) - f(\zeta, 0)\}\rho(\zeta, 0), \psi(\zeta)\big)$$
$$-\frac{\rho_2 - \rho_1}{L}\big(f(\zeta, \xi)\phi'_\xi(\zeta, \xi)\psi(\zeta), \delta(\phi)\big). \tag{A.7}$$

Here the inner product in the second term is calculated over $\Pi = \mathbf{R}^1_\zeta \times (0, L)$ and $\delta(\phi)$ is the δ-function on the curve Γ [16]. This implies formula (A.2).

We show how to calculate the functional $\int g\delta(\phi)\, d\xi$ in the general case and, in particular, we prove Corollary 1.

We divide the curve Γ into the parts $\Gamma_{i,j}$, $i = 1, \dots, I$, $j = 0, 1$, so that

$$\overline{\Gamma} = \bigcup_{i=1}^{I}(\overline{\Gamma}_{i,0} \cup \overline{\Gamma}_{i,1}), \qquad \phi'_\xi \neq 0 \quad \text{on } \Gamma_{i,0}, \qquad \phi'_\zeta \neq 0 \quad \text{on } \Gamma_{i,1}.$$

Then we have the following local representation for the form ω:

$$\omega = \frac{1}{\phi'_\xi}\, d\zeta \quad \text{on } \Gamma_{i,0} \qquad \text{and} \qquad \omega = -\frac{1}{\phi'_\zeta}\, d\xi \quad \text{on } \Gamma_{i,1}. \tag{A.8}$$

By $\{(\xi_{\pm,i_\pm}, \zeta_{\pm,i_\pm})\}$ we denote the set of points on the curve Γ, at which the derivative $\phi'_\xi = 0$. We assume that $(\xi_{+,i_+}, \zeta_{+,i_+})$, $i_+ = 1, \dots, I_+$, are the maximum points and $(\xi_{-,i_-}, \zeta_{-,i_-})$, $i_- = 1, \dots, I_-$, are the minimum points. For definiteness, we set

$$0 = \xi_{+,1} < \xi_{-,1} < \xi_{+,2} < \dots < \xi_{-,I_-} < \xi_{+,I_+} = L.$$

Since Γ is L-periodic, we have $I_+ = I_- + 1$ and $\zeta_{+,1} = \zeta_{+,I_+}$.

For all critical points of finite order, the function $1/\phi'_\xi\big|_\Gamma$ has an integrable singularity (see below). Therefore, we can write

$$\int_\Gamma \varphi\omega = \sum_{i=1}^{I_-}\Big\{\int_{\Gamma_i^+} \varphi\omega + \int_{\Gamma_i^-} \varphi\omega\Big\}, \tag{A.9}$$

where Γ_i^+ are the parts of the curve Γ from $(\xi_{+,i}, \zeta_{+,i})$ to $(\xi_{-,i}, \zeta_{-,i})$ where $\phi'_\xi > 0$ and Γ_i^- are the parts from $(\xi_{-,i}, \zeta_{-,i})$ to $(\xi_{+,i+1}, \zeta_{+,i+1})$ where $\phi'_\xi < 0$.

By taking into account the orientation of the curve Γ (see Figure 5 and the upper plot in Figure 2, which corresponds to the case $I_- = 1$), we have

$$\int_{\Gamma_i^+} \varphi\omega = \int_{\zeta_{-,i}}^{\zeta_{+,i}} \varphi(\phi'_\xi)^{-1}\Big|_{\xi = \xi_i^+(\zeta)} d\zeta, \tag{A.10}$$
$$\int_{\Gamma_i^-} \varphi\omega = -\int_{\zeta_{-,i}}^{\zeta_{+,i+1}} \varphi(\phi'_\xi)^{-1}\Big|_{\xi = \xi_i^-(\zeta)} d\zeta,$$

where $\xi_i^\pm(\zeta)$ are solutions of the equation $\phi(\zeta, \xi) = 0$ for the corresponding values of ξ.

If there is only one minimum point, i.e. if $I_- = 1$, then, writing $\zeta_\pm = \zeta_{\pm,1}$ and $\xi^\pm(\zeta) = \xi_1^\pm(\zeta)$, we derive the following relation from (A.9) and (A.10):

$$\big(\varphi(\zeta, \xi), \delta(\phi)\big) = \int_{\zeta_-}^{\zeta_+} \left\{ \frac{\varphi}{\phi'_\xi}\Big|_{\xi = \xi^+(\zeta)} - \frac{\varphi}{\phi'_\xi}\Big|_{\xi = \xi^-(\zeta)} \right\} d\zeta. \tag{A.11}$$

If $I_- > 1$, then the integral in the right-hand side in (A.11) is taken from $\zeta_- = \min_{i_-} \zeta_{-,i_-}$ to $\zeta_+ = \max_{i_+} \zeta_{+,i_+}$, and the integrand is the sum of the expressions

$$\frac{\varphi}{\phi'_\xi}\bigg|_{\xi=\xi_i^+(\zeta)} H(\zeta-\zeta_{-,i})H(\zeta_{+,i}-\zeta) \quad \text{and} \quad -\frac{\varphi}{\phi'_\xi}\bigg|_{\xi=\xi_i^-(\zeta)} H(\zeta-\zeta_{-,i})H(\zeta_{+,i+1}-\zeta).$$

By setting $\varphi = f(\zeta,\xi)\phi'_\xi(\zeta,\xi)\psi(\zeta)$, we obtain a constructive formula for calculating $\int f\phi'_\xi\delta(\phi)\,d\xi$. In the general case this formula is rather cumbersome (see Danilov & Omel'yanov [11] and formula (A.13) below). In the special case studied in Corollary 1, this formula becomes easier and takes the form (A.4).

To prove that $\overline{\rho}_{f'}$ is continuous, it suffices to consider neighbourhoods of the points $\zeta_\pm$. We assume that $\zeta = \zeta_- - \mu$, where $\mu \geqslant 0$. Since $\rho(\zeta_- - \mu, 0) = \rho_1$, we readily obtain the first formula in (A.4) for $\zeta < \zeta_-$. Next, for sufficiently small μ, we have

$$\begin{aligned}\overline{\rho}_{f'}(\zeta_- + \mu) = \bigg(&\frac{\rho_1}{L}\{f(\zeta,L) - f(\zeta,0)\}\\ &-\frac{\rho_2-\rho_1}{L}\{f(\zeta,\xi_{i'}^+(\zeta)) - f(\zeta,\xi_{i'}^-(\zeta))\}\bigg)\bigg|_{\zeta=\zeta_-+\mu},\end{aligned}$$

where i' is the number of the critical point at which the absolute minimum is attained. Since $\xi_{i'}^\pm(\zeta) \to \xi_-$ as $\mu \to 0$, we again arrive at the first formula in (A.4) calculated for $\zeta = \zeta_- + 0$.

In a similar way, we have

$$\begin{aligned}\overline{\rho}_{f'}(\zeta_+ - \mu) = \bigg(&\frac{\rho_1}{L}\{f(\zeta,L) - f(\zeta,0)\}\\ &-\frac{\rho_2-\rho_1}{L}\{f(\zeta,\xi_1^+(\zeta)) - f(\zeta,\xi_1^-(\zeta))\}\bigg)\bigg|_{\zeta=\zeta_+-\mu}.\end{aligned}$$

However, $\xi_1^+(\zeta_+ - \mu) \to 0$ and $\xi_1^-(\zeta_+ - \mu) \to L$ as $\mu \to 0$. Hence we again arrive at the first expression in formula (A.4), which was calculated for $\zeta \geqslant \zeta_+$. Lemma 2 is thereby proved. □

Note that if the global maximum is attained at the point $(\xi_{+,i''}, \zeta_{+,i''})$ and $\xi_{+,i''} \neq 0$, then we have the following relations as $\mu \to 0$: $\rho(\zeta_{+,i''} - \mu, 0) \to \rho_2$ and $\xi_{i''}^\pm(\zeta_{+,i''} - \mu) \to \xi_{+,i''}$. These relations also imply that $\overline{\rho}_{f'}$ is continuous at the point $\zeta_{+,i''}$.

Proof of Lemma 3

Using the Taylor formula, we obtain

$$\begin{aligned}\phi(\zeta,\xi) = \phi'_\zeta(\zeta_j,\xi_j)(\zeta-\zeta_j) &+ \frac{1}{2}\phi''_{\xi\xi}(\zeta_1^*,\xi_1^*)(\xi-\xi_j)^2\\ &+\phi''_{\zeta\xi}(\zeta_2^*,\xi_2^*)(\zeta-\zeta_j)(\xi-\xi_j) + O\big((\zeta-\zeta_j)^2\big),\end{aligned}$$

where $\zeta_{1,2}^*$ and $\xi_{1,2}^*$ are intermediate points. Considering this relation on the curve Γ and taking into account the fact that $\nabla\phi\big|_\Gamma \neq 0$, we obtain the relation

$$\zeta - \zeta_j = -\frac{1}{2}\frac{\phi''_{\xi\xi}(\zeta_1^*,\xi_1^*)}{\phi'_\zeta(\zeta_j,\xi_j)}(\xi-\xi_j)^2 + O\big((\xi-\xi_j)^3\big),$$

https://doi.org/10.1017/S0956792502004862 Published online by Cambridge University Press

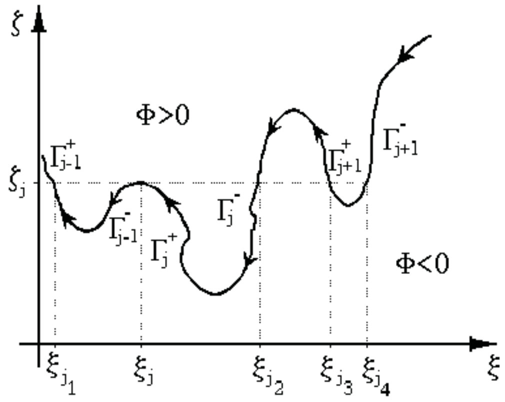

FIGURE 5. Calculation of $\overline{\rho}_{f'}(\zeta)$ in the general situation.

which implies the following local formulas for the inverse functions:

$$\xi_j^{\pm}(\zeta) = \xi_j \pm R_j|\zeta - \zeta_j|^{1/2} + O\big(|\zeta - \zeta_j|\big). \tag{A.12}$$

Assume that $(\xi_j, \zeta_j) \in \Gamma$ is the maximum point. By setting $\zeta \sim \zeta_j - 0$ and using Lemma 2, we obtain the formula

$$\overline{\rho}_{f'}(\zeta) = \frac{1}{L}\{f(\zeta_j, L) - f(\zeta_j, 0)\}\rho(\zeta, 0) - \frac{\rho_2 - \rho_1}{L}\sum_{i=1}^{I_j}\sum_{+,-}(-1)^{\kappa_{j_i}^{\pm}} f(\zeta_j, \xi_{j_i}^{\pm}(\zeta)), \tag{A.13}$$

where I_j is the number of points $(\xi_{j_i}, \zeta_j) \in \Gamma$ that can be projected at the point $\zeta = \zeta_j$. We choose $\kappa_{j_i}^{\pm} = 2$ if the orientation of the corresponding piece of Γ coincides with the orientation of the ζ-axis and $\kappa_{j_i}^{\pm} = 1$ in the opposite case (see Figure 5).

The general situation means that there is only one critical point (ξ_j, ζ_j) among all points (ξ_{j_i}, ζ_j). In this case, for $j_i \neq j$ the inverse functions $\xi_{j_i}^{\pm}(\zeta)$ are smooth, while for $\xi_j^{\pm}(\zeta)$ we have formulas (A.12). Thus (A.13) can be rewritten in the following form:

$$\begin{aligned}\overline{\rho}_{f'}(\zeta) &= \frac{1}{L}\{f(\zeta_j, L) - f(\zeta_j, 0)\}\rho(\zeta, 0) \\ &- \frac{\rho_2 - \rho_1}{L}\{f(\zeta_j, \xi_j^{+}(\zeta)) - f(\zeta_j, \xi_j^{-}(\zeta))\} \\ &+ \text{smooth functions.}\end{aligned} \tag{A.14}$$

By using (A.12) and the Taylor formula, one can easily see that

$$f(\zeta_j, \xi_j^{+}(\zeta)) - f(\zeta_j, \xi_j^{-}(\zeta)) = 2R_j \frac{\partial f}{\partial \xi}(\zeta_j, \xi_j)|\zeta - \zeta_j|^{1/2} + O(|\zeta - \zeta_j|). \tag{A.15}$$

This fact readily implies formula (A.5) corresponding to the maximum point. Repeating these calculations in the case in which (ξ_j, ζ_j) is the minimum point and taking into account the fact that for $\zeta \neq \zeta_j$ the differentiability of $\overline{\rho}_{f'}$ follows from the smoothness of the inverse functions $\xi_{j_i}^{\pm}(\zeta)$, we arrive at the statement of the lemma. □

In specific cases, for instance, in the case in which Γ is even with respect to the point $\xi = L/2$ and there are other critical points (ξ_j, ζ_j) with $\xi_j \neq L/2$ and $\xi_j \neq L$, formula (A.5) preserves its structure but becomes more cumbersome. For instance, let the set $\{(\xi_j, \zeta_j)\}$ contain, among other critical points, two maximum points (ξ_j, ζ_j) and (ξ_{j_1}, ζ_j). Then the right-hand side in (A.14) must contain the following additional expression:

$$-\frac{\rho_2 - \rho_1}{L}\{f(\zeta_j, \xi^+_{j_1}(\zeta)) - f(\zeta_j, \xi^-_{j_1}(\zeta))\}.$$

Applying a formula of the form (A.15) to this expression, we obtain an analog of formula (A.5), where the coefficient of $|\zeta - \zeta_j|^{1/2}$ is equal to the sum $R_j f'_\xi(\zeta_j, \xi_j) + R_{j_1} f'_\xi(\zeta_j, \xi_{j_1})$.

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