Euro. Jnl of Applied Mathematics (2016), *vol.* 27, *pp.* 1–22.
doi:10.1017/S0956792515000248

On convergence of the penalty method for a static unilateral contact problem with nonlocal friction in electro-elasticity

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(*Received 18 July 2014; revised 24 April 2015; accepted 6 May 2015; first published online 4 June 2015*)

In this paper, we consider the penalty method to solve the unilateral contact with friction between an electro-elastic body and a conductive foundation. Mathematical properties, such as the existence of a solution to the penalty problem and its convergence to the solution of the original problem, are reported. Then, we present a finite elements approximation for the penalised problem and prove its convergence. Finally, we propose an iterative method to solve the resulting finite element system and establish its convergence.

Key words: Piezoelectric; Variational inequality; Static friction contact; Nonlocal friction; Signorini condition; Penalty method; Fixed point process; Finite element approximation

1 Introduction

Penalty methods have recently obtained popularity to be applied in the field of numerical methods to solve constrained problems in mechanics such as unilateral contact problems and problems with Dirichlet boundary conditions. It is also widely admitted that inexact integration must be performed for evaluation of the penalty term in finite element approximation to get physically meaningful solution. The advantage of this approach is that standard methods can be used to solve the resulting nonlinear algebraic equations (see, e.g., [10, 20]). In a recent paper, Chouly and Hild [3] proved a convergence of the penalty method for unilateral contact problem in elasticity under the regularity hypotheses on the solution $\big(u$ belongs to $(H^{\frac{3}{2}+\nu})^d$ with $\nu \in (0, 1/2]\big)$.

In this work, we study a frictional contact problem for a piezoelectric body, when the foundation is electrically conductive. Unlike the models considered in [3], in the present paper we use both the electro-elastic constitutive law, the electrical contact conditions and we assume that the contact is modelled with Signorini condition, and the associated version of Coulomb's law of dry friction; as a consequence, the resulting variational formulation of the problem is different from that in [3] which is in the form of a coupled system involving as unknowns the displacement field and the electric potential. The static frictional contact problems for electro-elastic materials were studied in [12–14, 17], under the assumption that the foundation is insulated, and in [7, 15] under the assumption that the foundation is electrically conductive. Recent modelling, analysis and numerical simulations of static contact with or without friction for piezoelectric materials can be found in [1, 2, 9, 12] and references therein.

The main purpose of this work is to present a convergence analysis of the penalty method applied to unilateral contact with Coulomb's friction problem studied in [7]. The weak formulation of the problem consists of a variational inequality for the displacement field coupled with a nonlinear variational equation for the electric potential. It is very difficult to perform direct numerical solution of this problem, to overcome this difficulty we introduce the penalized problem by using a simultaneous penalization of unilateral conditions for imposition of contact constraint combined with a regularization of the frictional term [5, 6]. The approximate problem is formulated as a coupled system of nonlinear variational equations (depending on a penalization parameter). We prove the existence and uniqueness of the weak penalized solution by using fixed point arguments and establish its convergence to the solution of the initial problem. We then study the discrete problem and prove the convergence of its solution towards the solution of the penalized problem. Moreover, we describe an iterative method for the numerical solutions and obtain its convergence.

The paper is organized as follows. In Section 2, we present the classical and variational formulations of the mathematical model, we state the assumptions on the problem data and we recall the existence and uniqueness theorem obtained in [7]. Also in this section, we introduce the penalized problem and state our main results. The proofs are established in Section 3. Finally, in Section 4 we present a finite elements approximation for the penalized problem and prove its convergence [8]. Moreover, we propose an iterative method to solve the resulting finite element system and establish its convergence.

2 Formulation and main results

2.1 The physical setting and known Rresult

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be the reference domain occupied by the electro-elastic body which is supposed to be open, bounded, with a sufficiently regular boundary $\partial\Omega = \Gamma$. In the sequel, we decompose Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand, such that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. We assume that the body is fixed on Γ_1 where the displacement field vanishes. The body is acted upon by a volume force of density f_0 and volume electric charges of density q_0 in Ω and a surface traction of density f_2 on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. On Γ_3 the body is in unilateral contact with friction with a conductive obstacle, the so-called foundation. We model the contact with the Signorini condition and friction. The indices i, j, k, l run between 1 and d. The summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i,j} = \partial u_i/\partial x_j$. Everywhere below we use $\mathbb{S}^d$ to denote the space of second order symmetric tensors on $\mathbb{R}^d$ while " $\cdot$ " and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is for all $u, v \in \mathbb{R}^d$ and $\sigma, \tau \in \mathbb{S}^d$,

$$u \cdot v = u_i \cdot v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}}, \quad \text{and} \quad \sigma \cdot \tau = \sigma_{ij} \cdot \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}.$$

We shall adopt the usual notations for normal and tangential components of displacement vector and stress : $v_\nu = v\cdot\nu$, $v_\tau = v - v_\nu \nu$, $\sigma_\nu = (\sigma\nu)\cdot\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$, where ν denotes the outward normal vector on Γ. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, and " Div ", " div " denote, respectively, the divergence operators for tensor and vector valued functions, i.e., $\operatorname{Div}\sigma = (\sigma_{ij,j})$, $\operatorname{div} D = (D_{j,j})$. Under these conditions, the classical formulation of the mechanical problem is as follows

Problem P. Find a displacement field $u : \Omega \to \mathbb{R}^d$, a stress field $\sigma : \Omega \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \to \mathbb{R}$ and an electric displacement field $D : \Omega \to \mathbb{R}^d$ such that

$$\sigma = \mathfrak{F}\varepsilon(u) - \mathscr{E}^* E(\varphi) \quad \text{in} \quad \Omega, \tag{2.1}$$

$$D = \mathscr{E}\varepsilon(u) + \beta E(\varphi) \quad \text{in} \quad \Omega, \tag{2.2}$$

$$Div\sigma + f_0 = 0 \quad \text{in} \quad \Omega, \tag{2.3}$$

$$\text{div} D = q_0 \quad \text{in} \quad \Omega, \tag{2.4}$$

$$u = 0 \quad \text{on} \quad \Gamma_1, \tag{2.5}$$

$$\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2, \tag{2.6}$$

$$\sigma_\nu(u,\varphi) \leqslant 0, u_\nu \leqslant 0, \sigma_\nu(u,\varphi)u_\nu = 0 \quad \text{on} \quad \Gamma_3, \tag{2.7}$$

$$\left.\begin{array}{l} \|\sigma_\tau\| \leqslant \mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi)| \\ \|\sigma_\tau\| < \mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi)| \Rightarrow u_\tau = 0 \\ \sigma_\tau = -\mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi)|\frac{u_\tau}{\|u_\tau\|} \Rightarrow u_\tau \neq 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{2.8}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a, \tag{2.9}$$

$$D \cdot \nu = q_2 \quad \text{on} \quad \Gamma_b, \tag{2.10}$$

$$D \cdot \nu = \psi(u_\nu)\phi_L(\varphi - \varphi_0) \quad \text{on} \quad \Gamma_3. \tag{2.11}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \overline{\Omega}$. Equations (2.1) and (2.2) represent the electro-elastic constitutive law of the material in which $\mathfrak{F}$ denotes the elasticity operator, assumed to be nonlinear, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathscr{E}$ represents the third order piezoelectric tensor, $\mathscr{E}^*$ is its transpose and β denotes the electric permittivity tensor. Equations (2.3) and (2.4) represent the equilibrium equations for the stress and electric displacement fields, respectively. Relations (2.5) and (2.6) are the displacement and traction boundary conditions, respectively, and (2.9), (2.10) represent the electric boundary conditions. The unilateral boundary conditions (2.7) represent the Signorini law and (2.8) represent the Coulomb's friction law in which μ is the coefficient of friction and R is a regularisation operator. Finally, (2.11) represent the regularised electrical contact condition on Γ_3, which was considered in [11], where ψ and ϕ are a regularisation function and the truncation function, respectively, such that

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \leqslant s \leqslant L, \\ L & \text{if } s > L, \end{cases} \qquad \psi(r) = \begin{cases} 0 & \text{if } r < 0, \\ k\delta r & \text{if } 0 \leqslant r \leqslant 1/\delta, \\ k & \text{if } r > 1/\delta, \end{cases}$$

in which L is a large positive constant, $\delta > 0$ denotes a small parameter and $k \geqslant 0$ is the electrical conductivity coefficient. Note also that when $\psi \equiv 0$, then (2.11) leads to

$$D \cdot \nu = 0 \quad \text{on } \Gamma_3. \tag{2.12}$$

The condition (2.12) models the case when the obstacle is a perfect insulator.
Next, we introduce the notation and recall some definitions needed in the sequel. First, we introduce the following functional spaces:

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d,$$
$$\mathscr{H} = \{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \quad \mathscr{H}_1 = \{\sigma \in \mathscr{H} \mid \operatorname{Div} \sigma \in H\}.$$

These are real Hilbert spaces endowed with the inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \quad (u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathscr{H}},$$
$$(\sigma, \tau)_{\mathscr{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \, dx, \quad (\sigma, \tau)_{\mathscr{H}_1} = (\sigma, \tau)_{\mathscr{H}} + (\operatorname{Div} \sigma, \operatorname{Div} \tau)_{\mathscr{H}},$$

and the associated norms $\| \cdot \|_H$, $\| \cdot \|_{H_1}$, $\| \cdot \|_{\mathscr{H}}$ and $\| \cdot \|_{\mathscr{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation v to denote the trace γv of v on Γ.

Let H_Γ' be the dual of H_Γ and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between H_Γ' and H_Γ. For every $\sigma \in \mathscr{H}_1$, $\sigma\nu$ can be defined as the element in H_Γ' which satisfies

$$\langle \sigma\nu, \gamma v \rangle = (\sigma, \varepsilon(v))_{\mathscr{H}} + (\operatorname{Div} \sigma, v)_H, \quad \forall v \in H_1. \tag{2.13}$$

Moreover, if σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v \rangle = \int_\Gamma \sigma\nu \cdot v \, da, \tag{2.14}$$

for all $v \in H_1$, where da is the surface measure element. Keeping in mind the boundary condition (2.5), we introduce the closed subspace of H_1 defined by

$$V = \{v \in H_1 \mid v = 0 \text{ on } \Gamma_1\},$$

and let K be the set of admissible displacements

$$K = \{v \in V \mid v_\nu \leqslant 0 \text{ on } \Gamma_3\}.$$

Since $meas(\Gamma_1) > 0$ and Korn's inequality (see, e.g., [16]) holds,

$$\|\varepsilon(v)\|_{\mathscr{H}} \geqslant c_k \|v\|_{H_1}, \quad \forall v \in V, \tag{2.15}$$

where $c_k > 0$ is a constant which depends only on Ω and Γ_1. Over the space V, we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \quad \|u\|_V = (u, u)_V^{\frac{1}{2}}, \tag{2.16}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (2.15) that the norms $\|\cdot\|_{H_1}$and $\|\cdot\|_V$ are equivalent on V. Therefore $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, by the Sobolev trace theorem, (2.15) and (2.16) there exists a constant $c_0 > 0$ which only depends on the domain Ω, Γ_3 and Γ_1 such that

$$\|v\|_{L^2(\Gamma)^d} \leqslant c_0 \|v\|_V, \quad \forall v \in V. \tag{2.17}$$

We also introduce the spaces

$$W = \{\psi \in H^1(\Omega) \,|\, \psi = 0 \text{ on } \Gamma_a\},$$
$$\mathscr{W} = \{D = (D_i) \in L^2(\Omega)^d \,|\, \mathrm{div} D \in L^2(\Omega)\}.$$

The spaces W and $\mathscr{W}$ are real Hilbert spaces with the inner products

$$(\varphi, \psi)_W = (\varphi, \psi)_{H^1(\Omega)}, \quad (D, E)_{\mathscr{W}} = (D, E)_{L^2(\Omega)^d} + (\mathrm{div} D, \mathrm{div} E)_{L^2(\Omega)}.$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathscr{W}}$, respectively. Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs–Poincaré inequality holds:

$$\|\nabla\psi\|_{\mathscr{W}} \geqslant c_F \|\psi\|_W, \quad \forall \psi \in W, \tag{2.18}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. Moreover, by the Sobolev trace theorem, there exists a constant c_1, depending only on Ω, Γ_a and Γ_3, such that

$$\|\xi\|_{L^2(\Gamma_3)} \leqslant c_1 \|\xi\|_W, \quad \forall \xi \in W. \tag{2.19}$$

When $D \in \mathscr{W}$ is a sufficiently regular function, the following Green's type formula holds,

$$(D, \nabla\xi)_{L^2(\Omega)^d} + (\mathrm{div} D, \xi)_{L^2(\Omega)} = \int_\Gamma D \cdot \nu \xi \, da, \quad \forall \xi \in H^1(\Omega). \tag{2.20}$$

Recall also that the transposite $\mathscr{E}^*$ is given by

$$\mathscr{E}^* = (e^*_{ijk}), \quad \text{where} \quad e^*_{ijk} = e_{kij},$$
$$\mathscr{E}\sigma \cdot v = \sigma \cdot \mathscr{E}^* v, \quad \forall \sigma \in \mathbb{S}^d, v \in \mathbb{R}^d. \tag{2.21}$$

In the study of mechanical problem (2.1)–(2.11), we assume that the elasticity operator $\mathfrak{F}$

https://doi.org/10.1017/S0956792515000248 Published online by Cambridge University Press

satisfies the following conditions:

$$\begin{cases} \text{(a)} & \mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d; \\ \text{(b)} & \text{there exists } M_{\mathfrak{F}} > 0 \text{ such that} \\ & \|\mathfrak{F}(x,\xi_1) - \mathfrak{F}(x,\xi_2)\| \leqslant M_{\mathfrak{F}} \|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(c)} & \text{there exists } m_{\mathfrak{F}} > 0 \text{ such that} \\ & (\mathfrak{F}(x,\xi_1) - \mathfrak{F}(x,\xi_2))(\xi_1 - \xi_2) \geqslant m_{\mathfrak{F}} \|\xi_1 - \xi_2\|^2 \forall \xi_1, \xi_2 \in \mathbb{S}^d, \text{a.e.} x \in \Omega; \\ \text{(d)} & \text{the mapping } x \to \mathfrak{F}(x,\xi) \text{ is Lebesgue measurable on } \Omega, \forall \xi \in \mathbb{S}^d; \\ \text{(e)} & \text{the mapping } x \to \mathfrak{F}(x,0) \text{ belongs to } \mathscr{H}. \end{cases} \tag{2.22}$$

We note that condition (2.22) is satisfied in the case of the linear electro-elastic constitutive law, $\sigma = \mathfrak{F}\varepsilon(u) - \mathscr{E}^* E(\varphi)$ in which $\mathfrak{F}\xi = (f_{ijkl}\xi_{kl})$ provided that $f_{ijkl} \in L^\infty(\Omega)$ and there exists $\alpha > 0$ such that $f_{ijkl}(x)\xi_k\xi_l \geqslant \alpha\|\xi\|^2$, for all $\xi \in \mathbb{S}^d$, a.e. $x \in \Omega$. Examples of a nonlinear operator $\mathfrak{F}$ which satisfy condition (2.22) can be found in [17].

The piezoelectric tensor $\mathscr{E}$ and the electric permittivity tensor β satisfy

$$\begin{cases} \text{(a)} & \mathscr{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d; \\ \text{(b)} & e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases} \tag{2.23}$$

$$\begin{cases} \text{(a)} & \beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d; \\ \text{(b)} & \beta_{ij} = \beta_{ji} \in L^\infty(\Omega); \\ \text{(c)} & \exists m_\beta > 0 \text{ such that } \beta_{ij}E_iE_j \geqslant m_\beta \|E\|^2, \forall E \in R^d, \text{ a.e. } x \in \Omega. \end{cases} \tag{2.24}$$

The surface electrical conductivity function ψ and the coefficient of friction μ satisfy:

$$\begin{cases} \text{(a)} & \psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+; \\ \text{(b)} & \exists\, M_\psi > 0 \text{ such that } |\psi(x,u)| \leqslant M_\psi, \quad \forall u \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{(c)} & x \to \psi(x,u) \text{ is measurable on } \Gamma_3, \text{for all } u \in \mathbb{R}; \\ \text{(d)} & x \to \psi(x,u) = 0 \text{ for all } u \leqslant 0. \end{cases} \tag{2.25}$$

$$\begin{cases} \text{(a)} & \mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+; \\ \text{(b)} & \exists\, \mu^* > 0 \text{ such that } \mu(x,u) \leqslant \mu^*, \forall u \in \mathbb{R}_+, \text{ a.e. } x \in \Gamma_3; \\ \text{(c)} & \text{The mapping } x \to \mu(x,u) \text{ is measurable on } \Gamma_3, \text{for all } u \in \mathbb{R}+. \end{cases} \tag{2.26}$$

Moreover, we assume that ψ and μ are Lipschitz continuous functions in the following sense

$$\exists\, L_\psi > 0 \text{ such that } |\psi(.,u_1) - \psi(.,u_2)| \leqslant L_\psi |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}, \tag{2.27}$$

$$\exists\, L_\mu > 0 \text{ such that } |\mu(.,u) - \mu(.,v)| \leqslant L_\mu |u - v| \quad \forall u, v \in \mathbb{R}_+. \tag{2.28}$$

We assume that the body forces, the tractions, the volume and surface charge densities satisfy

$$f_0 \in L^2(\Omega)^d, \;\; f_2 \in L^2(\Gamma_3)^d, \tag{2.29}$$

$$q_0 \in L^2(\Omega), \;\; q_2 \in L^2(\Gamma_b). \tag{2.30}$$

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Also, the given potential is such that

$$\varphi_0 \in L^2(\Gamma_3). \tag{2.31}$$

Next, we use Riesz's representation theorem to consider the elements $f \in V$, and $q \in W$ given by

$$(f, v)_V = \int_\Omega f_0 \cdot v \, dx + \int_{\Gamma_2} f_2 \cdot v \, da, \forall v \in V, \tag{2.32}$$

$$(q, \xi)_W = \int_\Omega q_0 \xi \, dx - \int_{\Gamma_b} q_2 \xi \, da, \forall \xi \in W, \tag{2.33}$$

and, we define the mappings $j : V \times V \to \mathbb{R}$ and $\ell : V \times W \times W \to \mathbb{R}$, respectively, by

$$\ell(u, \varphi, \xi) = \int_{\Gamma_3} \psi(u_\nu) \phi_L(\varphi - \varphi_0) \xi \, da, \forall u \in V, \ \forall \varphi, \xi \in W, \tag{2.34}$$

$$j(u, v) = \int_{\Gamma_3} \mu(\|u_\tau\|) |R\sigma_\nu(u, \varphi)| \ \|v_\tau\| \, da, \forall u, v \in V. \tag{2.35}$$

Keeping in mind assumptions (2.25)–(2.31) it follows that the integrals in (2.32)–(2.35) are well defined. Finally, we assume that

$$R : H'_{\Gamma_3} \to L^\infty(\Gamma_3) \text{ is a linear and continuous mapping.} \tag{2.36}$$

Using Green's formula (2.13), (2.14) and (2.20) it is straightforward to see that if (u, σ, φ, D) are sufficiently regular functions which satisfy (2.3)–(2.11) then

$$(\sigma, \varepsilon(v) - \varepsilon(u))_{\mathscr{H}} + j(u, v) - j(u, u) \geqslant (f, v - u)_V, \forall v \in K, \tag{2.37}$$

$$(D, \nabla \xi)_{L^2(\Omega)^d} = \ell(u, \varphi, \xi) - (q, \xi)_W, \forall \xi \in W. \tag{2.38}$$

We plug (2.1) in (2.37), (2.2) in (2.38) and use the notation $E = -\nabla\varphi$ to obtain the following variational formulation of Problem P, in the terms of displacement field and electric potential.

Problem PV Find a displacement field $u \in K$ and an electric potential $\varphi \in W$ *such that:*

$$\begin{aligned} &(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathscr{H}} + (\mathscr{E}^* \nabla\varphi, \varepsilon(v) - \varepsilon(u))_{L^2(\Omega)^d} + j(u, v) - j(u, u) \\ &\geqslant (f, v - u)_V, \quad \forall v \in K, \end{aligned} \tag{2.39}$$

$$(\beta \nabla\varphi, \nabla\xi)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u), \nabla\xi)_{L^2(\Omega)^d} + \ell(u, \varphi, \xi) = (q, \xi)_W, \quad \forall \xi \in W. \tag{2.40}$$

The following existence and uniqueness of solution to Problem PV has been established in [7].

Theorem 2.1 *Assume that (2.22)–(2.26) and (2.29)–(2.31) hold. Then*

(1) *The Problem PV has at least one solution $(u, \varphi) \in K \times W$.*

(2) *Under the assumptions (2.27) and (2.28), there exists $L^* > 0$ such that if $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$ then the Problem PV has a unique solution.*

2.2 The penalty formulation of the contact problem

Let $\epsilon > 0$. We consider a penalised electro-elastic contact problem with a solution denoted by $(u_\epsilon, \varphi_\epsilon)$ verifying the given equations in Ω (2.1)–(2.4) and the boundary conditions on Γ (2.5)–(2.11) similar to the Problem (P) except the fact that the contact condition (2.7) on Γ_3 was replaced by $\sigma_\nu(u,\varphi) = -\frac{1}{\epsilon}u_\nu^+$, where $r^+ = max(r,0)$, on one hand, and the non-differentiable term $j(u,\cdot)$ was approximated by a family of differentiable ones $j_\epsilon(u,\cdot)$ (a regularisation method), on the other hand, where $\epsilon > 0$ is small penalisation parameter. Convergence of the method is obtained when $\epsilon \to 0$.

We consider the family of convex and differentiable function $\Psi_\epsilon : \mathbb{R}^d \to \mathbb{R}$ given by

$$\Psi_\epsilon(v) = \sqrt{\|v\|^2 + \epsilon^2},\ \forall v \in \mathbb{R}^d, \tag{2.41}$$

for all positive ϵ, we have

$$0 < \Psi_\epsilon(v) - \|v\| \leqslant \epsilon. \tag{2.42}$$

We approximate the functional j by $j_\epsilon : V \times V \to \mathbb{R}$, a family of regularised frictional functionals depending on $\epsilon > 0$,

$$j_\epsilon(u,v) = \int_{\Gamma_3} \mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi)|\ \Psi_\epsilon(v)\,da, \quad \forall u, v \in V. \tag{2.43}$$

The functional j_ϵ is Gâteaux-differentiable and we denote by j_ϵ' the derivative of j_ϵ given by

$$\langle j_\epsilon'(u_\epsilon, v), w\rangle = \int_{\Gamma_3} \mu(\|u_{\epsilon\tau}\|)|R\sigma_\nu(u_\epsilon,\varphi_\epsilon)| \frac{v_\tau w_\tau}{\sqrt{\epsilon^2 + \|v_\tau\|^2}}\,da, \tag{2.44}$$

for all $u_\epsilon, v, w \in V$. Let $\Phi : V \times V \to \mathbb{R}$

$$\Phi(u,v) = \int_{\Gamma_3} u_\nu^+ v_\nu\,da. \tag{2.45}$$

We can introduce now the following variational problem.

Problem PV_ϵ Find $u_\epsilon \in V$ and $\varphi_\epsilon \in W$ such that for all $v \in V$, $\xi \in W$

$$(\mathfrak{F}\varepsilon(u_\epsilon), \varepsilon(v))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_\epsilon, \varepsilon(v))_{L^2(\Omega)^d} + \frac{1}{\epsilon}\Phi(u_\epsilon, v) + \langle j_\epsilon'(u_\epsilon, u_\epsilon), v\rangle = (f, v)_V, \tag{2.46}$$

$$(\beta\nabla\varphi_\epsilon, \nabla\xi)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u_\epsilon), \nabla\xi)_{L^2(\Omega)^d} + \ell(u_\epsilon, \varphi_\epsilon, \xi) = (q, \xi)_W. \tag{2.47}$$

We have the following results.

Theorem 2.2 *Under the assumptions of Theorem 2.1 with the same value of L^*, the Problem PV_ϵ has a unique solution such that $x_\epsilon = (u_\epsilon, \varphi_\epsilon) \in V \times W$.*

Remark 1 Assume that (2.22)–(2.26) and (2.29)–(2.31) hold. Then, the Problem PV_ϵ has at least one solution $x_\epsilon = (u_\epsilon, \varphi_\epsilon) \in V \times W$.

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We have the following convergence result.

Theorem 2.3 *Under the assumptions of Theorem 2.2 the solutions* $(u_\epsilon, \varphi_\epsilon)$ *of penalised Problem* PV_ϵ *converge to a solution* (u, φ) *of Problem* PV. *i.e.,*

$$u_\epsilon \rightharpoonup u \quad \text{weakly in } V, \quad \varphi_\epsilon \rightharpoonup \varphi \quad \text{weakly in } W \quad \text{as} \quad \epsilon \to 0.$$

3 Proof of results

We consider the product spaces $X = V \times W$ and $Y = L^2(\Gamma_3) \times L^2(\Gamma_3)$ together with the inner products

$$(x, y)_X = (u, v)_V + (\varphi, \xi)_W, \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{3.1}$$

$$(\eta, \theta)_Y = (g, \lambda)_{L^2(\Gamma_3)} + (z, \zeta)_{L^2(\Gamma_3)}, \forall \eta = (g, z),\ \theta = (\lambda, \zeta) \in Y, \tag{3.2}$$

and the associated norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. Let $U = K \times W$ be a non-empty closed convex subset of X. We define the operator $A : X \to X$, the functions $\widetilde{j}$, $\widetilde{\ell}$ on $X \times X$ and the element $f_3 \in X$ by equalities:

$$(Ax, y)_X = (\mathfrak{F}\varepsilon(u), \varepsilon(v))_{\mathscr{H}} + (\beta\nabla\varphi, \nabla\xi)_{L^2(\Omega)^d} + (\mathscr{E}^*\nabla\varphi, \varepsilon(v))_{L^2(\Omega)^d}$$
$$-(\mathscr{E}\varepsilon(u), \nabla\xi)_{L^2(\Omega)^d}, \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{3.3}$$

$$\widetilde{j}(x, y) = j(u, v), \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{3.4}$$

$$\widetilde{\ell}(x, y) = \int_{\Gamma_3} \psi(u_\nu)\phi_L(\varphi - \varphi_0)\xi \, da, \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{3.5}$$

$$f_3 = (f, q) \in X. \tag{3.6}$$

We introduce the operator $A_\epsilon : X \to X$ defined by

$$(A_\epsilon x, y)_X = (Ax, y)_X + \frac{1}{\epsilon}\Phi(u, v), \tag{3.7}$$

for all $x = (u, \varphi)$, $y = (v, \xi) \in X$, where A given by (3.3), and we extend the functional j_ϵ defined by (2.43) to functional $\widetilde{j}_\epsilon$ defined on $X \times X$, that is

$$\widetilde{j}_\epsilon(x, y) = j_\epsilon(u, v), \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{3.8}$$

with the notations above, and according (3.5)–(3.6), we have the following result.

Lemma 1 *The couple* $x_\epsilon = (u_\epsilon, \varphi_\epsilon)$ *is a solution to Problem* PV_ϵ *if and only if:*

$$(A_\epsilon x_\epsilon, y)_X + \langle \widetilde{j}'_\epsilon(x_\epsilon, x_\epsilon), y\rangle + \widetilde{\ell}(x_\epsilon, y) = (f_3, y)_X, \forall y = (v, \xi) \in V \times W. \tag{3.9}$$

Proof Let $x_\epsilon = (u_\epsilon, \varphi_\epsilon) \in X$ be a solution to Problem PV_ϵ and let $y = (v, \xi) \in X$. We add the equalities (2.46), (2.47) and use (3.1), (3.7) and (3.6) to obtain (3.9). Conversely, let $x_\epsilon = (u_\epsilon, \varphi_\epsilon) \in X$ be a solution to the elliptic variational equalities (3.9). We take $y = (v, 0)$

in (3.9) where v is an arbitrary element of V and obtain (2.46). Then for any $\xi \in W$, we take $y = (0, \xi)$ in (3.9) to obtain (2.47), which concludes the proof of Lemma 1. □

3.1 Proof of Theorem 2.2

The proof of Theorem 2.2 will be carried out in several steps, based on a fixed point argument. For this purpose, let $\eta = (g, z) \in L^2(\Gamma_3) \times L^2(\Gamma_3)$ be given, and we define

$$\ell_z(\xi) = \int_{\Gamma_3} z\ \xi da, \quad \forall \xi \in W, \tag{3.10}$$

$$j_g(v) = \int_{\Gamma_3} g\, v_\tau\, da, \quad \forall v \in V. \tag{3.11}$$

We construct the following intermediate problem.

Problem PV_ϵ^η. *Let $\eta \in L^2(\Gamma_3) \times L^2(\Gamma_3)$ be given, find $u_{\epsilon\eta} \in V$ and $\varphi_{\epsilon\eta} \in W$ such that*

$$(\mathfrak{F}\varepsilon(u_{\epsilon\eta}), \varepsilon(v))_{\mathscr{H}} - (\mathscr{E}^*\nabla\varphi_{\epsilon\eta}, \varepsilon(v))_{L^2(\Omega)^d} + \frac{1}{\epsilon}\Phi(u_{\epsilon\eta}, v) = (f, v)_V - j_g(v), \tag{3.12}$$

$$(\beta\nabla\varphi_{\epsilon\eta}, \nabla\xi)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u_{\epsilon\eta}), \nabla\xi)_{L^2(\Omega)^d} = (q, \xi)_W - \ell_z(\xi), \tag{3.13}$$

for all $v \in V$ and $\xi \in W$.

We consider the element $f_\eta = (f_1, q_1) \in X$ such that

$$(f_1, v)_V = (f, v)_V - j_g(v), \forall v \in V, \tag{3.14}$$

$$(q_1, \xi)_W = (q, \xi)_W - \ell_z(\xi), \forall \xi \in W. \tag{3.15}$$

It is easy to see that $x_{\epsilon\eta} = (u_{\epsilon\eta}, \varphi_{\epsilon\eta})$ is a solution to Problem PV_ϵ^η if and only if

$$(A_\epsilon x_{\epsilon\eta}, y)_X = (f_\eta, y)_X, \quad \forall y = (v, \xi) \in X. \tag{3.16}$$

We now use (3.16) to obtain the following existence and uniqueness result.

Lemma 2 *For any $\eta \in L^2(\Gamma_3) \times L^2(\Gamma_3)$, assume that (2.22)–(2.24) hold. Then*

(i) *The Problem PV_ϵ^η has a unique solution $x_{\epsilon\eta} = (u_{\epsilon\eta}, \varphi_{\epsilon\eta}) \in X$ which depends Lipschitz continuously on $\eta \in L^2(\Gamma_3) \times L^2(\Gamma_3)$.*

(ii) *There exists a constant $c_2 > 0$ such that the solution to problem (3.16) satisfies*

$$\|x_{\epsilon\eta}\|_X \leqslant c_2 \|f_\eta\|_X. \tag{3.17}$$

Proof Consider two elements $x_1 = (u_1, \varphi_1),\ x_2 = (u_2, \varphi_2) \in X$. It follows from (3.7), (3.3) and (2.45) that

$$\begin{aligned}&(A_\epsilon x_1 - A_\epsilon x_2, x_1 - x_2)_X \\ &\quad = (Ax_1 - Ax_2, x_1 - x_2)_X + \frac{1}{\epsilon}\big(\Phi(u_1, u_1 - u_2) - \Phi(u_2, u_1 - u_2)\big).\end{aligned} \tag{3.18}$$

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We use now (2.21), (2.22), (2.24), (2.18) and (3.1), there exists $c_3 > 0$ which depends only on $\mathfrak{F}$, β, Ω and Γ_a such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geqslant c_3 \|x_1 - x_2\|_X^2, \tag{3.19}$$

and observe that

$$\Phi(u_1, u_1 - u_2) - \Phi(u_2, u_1 - u_2) \geqslant 0, \quad \forall u_1, u_2 \in V. \tag{3.20}$$

We find

$$(A_\epsilon x_1 - A_\epsilon x_2, x_1 - x_2)_X \geqslant c_3 \|x_1 - x_2\|_X^2. \tag{3.21}$$

In the same way, using (2.22)–(2.24), (3.1) and (2.17), after some calculations it follows that there exists $c_4 > 0$ which depends only on $\mathfrak{F}$, β and $\mathscr{E}$ such that

$$(A_\epsilon x_1 - A_\epsilon x_2, y)_X \leqslant 4c_4 \|x_1 - x_2\|_X \ \|y\|_X + \frac{1}{\epsilon} c_0^2 \|x_1 - x_2\|_X \ \|y\|_X, \quad \forall y \in X,$$

and, taking $y = A_\epsilon x_1 - A_\epsilon x_2 \in X$, we find

$$\|A_\epsilon x_1 - A_\epsilon x_2\|_X \leqslant (4c_4 + \frac{1}{\epsilon} c_0^2) \|x_1 - x_2\|_X. \tag{3.22}$$

Now for every fixed $\epsilon > 0$, it follows from (3.21), (3.22), $f_\eta \in X$ and a standard result on nonlinear variational equations that there exists a unique element $x_{\epsilon\eta} = (u_{\epsilon\eta}, \varphi_{\epsilon\eta}) \in X$ which satisfies (3.16).

We show next that this solution depends Lipschitz continuously on $\eta \in L^2(\Gamma_3) \times L^2(\Gamma_3)$. Let $\eta_1 = (g_1, z_1), \eta_2 = (g_2, z_2) \in L^2(\Gamma_3) \times L^2(\Gamma_3)$ be given and denote the corresponding solution of the problem (3.16) by $x_{\epsilon\eta_1} = (u_{\epsilon\eta_1}, \varphi_{\epsilon\eta_1})$ and $x_{\epsilon\eta_2} = (u_{\epsilon\eta_2}, \varphi_{\epsilon\eta_2})$. Then, we have

$$(A_\epsilon x_{\epsilon\eta_1}, y_{\eta_1} - x_{\epsilon\eta_1})_X = (f_{\eta_1}, y_{\eta_1} - x_{\epsilon\eta_1})_X, \quad \forall y_{\eta_1} \in U,$$
$$(A_\epsilon x_{\epsilon\eta_2}, y_{\eta_2} - x_{\epsilon\eta_2})_X = (f_{\eta_2}, y_{\eta_2} - x_{\epsilon\eta_2})_X, \quad \forall y_{\eta_2} \in U.$$

We take $y_{\eta_1} = x_{\epsilon\eta_2}$, $y_{\eta_2} = x_{\epsilon\eta_1}$ and add the two equalities to obtain

$$(A_\epsilon x_{\epsilon\eta_1} - A_\epsilon x_{\epsilon\eta_2}, x_{\epsilon\eta_1} - x_{\epsilon\eta_2}) = (f_{\eta_1} - f_{\eta_2}, x_{\epsilon\eta_1} - x_{\epsilon\eta_2})_X,$$

from (3.14)–(3.15) and (3.10)–(3.11), we have

$$\begin{aligned}
&(A_\epsilon x_{\epsilon\eta_1} - A_\epsilon x_{\epsilon\eta_2}, x_{\epsilon\eta_1} - x_{\epsilon\eta_2}) \\
&\quad = \int_{\Gamma_3} (g_1 - g_2)(u_{\epsilon\eta_1\tau} - u_{\epsilon\eta_2\tau}) da + \int_{\Gamma_3} (z_1 - z_2)(\varphi_{\epsilon\eta_1} - \varphi_{\epsilon\eta_2}) da \\
&\quad\quad \leqslant \|g_1 - g_2\|_{L^2(\Gamma_3)} \|u_{\epsilon\eta_1} - u_{\epsilon\eta_2}\|_{L^2(\Gamma_3)^d} + \|z_1 - z_2\|_{L^2(\Gamma_3)} \|\varphi_{\epsilon\eta_1} - \varphi_{\epsilon\eta_2}\|_{L^2(\Gamma_3)}.
\end{aligned}$$

Thus, using (2.17) and (2.19) we deduce

$$\begin{aligned}
&(A_\epsilon x_{\epsilon\eta_1} - A_\epsilon x_{\epsilon\eta_2}, x_{\epsilon\eta_1} - x_{\epsilon\eta_2})_X \\
&\quad \leqslant c_0 \|g_1 - g_2\|_{L^2(\Gamma_3)} \|u_{\epsilon\eta_1} - u_{\epsilon\eta_2}\|_V + c_1 \|z_1 - z_2\|_{L^2(\Gamma_3)} \|\varphi_{\epsilon\eta_1} - \varphi_{\epsilon\eta_2}\|_W,
\end{aligned}$$

and, using (3.1), (3.21) and (3.2)

$$\|x_{\epsilon\eta_1} - x_{\epsilon\eta_2}\|_X \leqslant \frac{\max(c_0, c_1)}{c_3} \sqrt{2} \|(g_1, z_1) - (g_2, z_2)\|_{L^2(\Gamma_3) \times L^2(\Gamma_3)}.$$

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Thus, there exists a positive constant $c_5 = \frac{\max(c_0,c_1)}{c_3}\sqrt{2}$ such that

$$\|x_{\epsilon\eta_1} - x_{\epsilon\eta_2}\|_X \leqslant c_5\|\eta_1 - \eta_2\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)}, \tag{3.23}$$

hence (i) follows. We turn now to the proof of (ii). Let $\eta = (g,z) \in L^2(\Gamma_3) \times L^2(\Gamma_3)$, we take $y = x_{\epsilon\eta}$ in the equality (3.16), we have

$$(A_\epsilon x_{\epsilon\eta}, x_{\epsilon\eta})_X = (f_\eta, x_{\epsilon\eta})_X, \quad \forall x_{\epsilon\eta} \in X.$$

Using (3.21), we deduce

$$\|x_{\epsilon\eta}\|_X \leqslant \frac{1}{c_3}\|f_\eta\|_X.$$

□

We now consider the operator $\Lambda : L^2(\Gamma_3) \times L^2(\Gamma_3) \to L^2(\Gamma_3) \times L^2(\Gamma_3)$ such that for all $\eta \in L^2(\Gamma_3) \times L^2(\Gamma_3)$, we have

$$\Lambda\eta = \left(\mu(\|u_{\epsilon\eta\tau}\|)|R\sigma_\nu(u_{\epsilon\eta}, \varphi_{\epsilon\eta})|\frac{u_{\epsilon\eta\tau}}{\sqrt{\epsilon^2 + \|u_{\epsilon\eta\tau}\|^2}}, \psi(u_{\epsilon\eta\nu})\phi_L(\varphi_{\epsilon\eta} - \varphi_0)\right), \tag{3.24}$$

The operator Λ depends on ϵ, but in order to simplify the notation we will not make this dependence explicit in the following.
It follows from assumptions (2.25)–(2.26) that the operator Λ is well defined. We have the following result.

Lemma 3 *There exists $L^* > 0$ such that if $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$, then Λ has a unique fixed point η^*.*

Proof Since for $g \in L^2(\Gamma_3)$, $\sigma_\nu(u_g, \varphi_g)$ is defined on Γ_3 and belongs to the dual space H'_{Γ_3} and let $\eta_1 = (g_1, z_1)$, $\eta_2 = (g_2, z_2) \in L^2(\Gamma_3) \times L^2(\Gamma_3)$, we have

$$\begin{aligned}
\|\Lambda\eta_1 - \Lambda\eta_2\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)} \leqslant &\left\|\mu(\|u_{\epsilon\eta_1\tau}\|)|R\sigma_\nu(u_{\epsilon\eta_1}, \varphi_{\epsilon\eta_1})|\frac{u_{\epsilon\eta_1\tau}}{\sqrt{\epsilon^2 + \|u_{\epsilon\eta_1\tau}\|^2}}\right.\\
&\left.-\mu(\|u_{\epsilon\eta_2\tau}\|)|R\sigma_\nu(u_{\epsilon\eta_2}, \varphi_{\epsilon\eta_2})|\frac{u_{\epsilon\eta_2\tau}}{\sqrt{\epsilon^2 + \|u_{\epsilon\eta_2\tau}\|^2}}\right\|_{L^2(\Gamma_3)}\\
&+\left\|\psi(u_{\epsilon\eta_1\nu})\phi_L(\varphi_{\epsilon\eta_1} - \varphi_0) - \psi(u_{\epsilon\eta_2\nu})\phi_L(\varphi_{\epsilon\eta_2} - \varphi_0)\right\|_{L^2(\Gamma_3)},\\
\|\Lambda\eta_1 - \Lambda\eta_2\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)} \leqslant &\, J + G,
\end{aligned}$$

where

$$\begin{aligned}
J = &\left\|\big(\mu(\|u_{\epsilon\eta_1\tau}\|) - \mu(\|u_{\epsilon\eta_2\tau}\|)\big)|R\sigma_\nu(u_{\epsilon\eta_1}, \varphi_{\epsilon\eta_1})|\Pi(u_{\epsilon\eta_1})\right\|_{L^2(\Gamma_3)}\\
&+\left\|\mu(\|u_{\epsilon\eta_2\tau}\|)\big(|R\sigma_\nu(u_{\epsilon\eta_1}, \varphi_{\epsilon\eta_1})| - |R\sigma_\nu(u_{\epsilon\eta_2}, \varphi_{\epsilon\eta_2})|\big)\Pi(u_{\epsilon\eta_1})\right\|_{L^2(\Gamma_3)}\\
&+\left\|\mu(\|u_{\epsilon\eta_2\tau}\|)|R\sigma_\nu(u_{\epsilon\eta_2}, \varphi_{\epsilon\eta_2})|\big(\Pi(u_{\epsilon\eta_1}) - \Pi(u_{\epsilon\eta_2})\big)\right\|_{L^2(\Gamma_3)},
\end{aligned} \tag{3.25}$$

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$$G = \left\| \big(\psi(u_{\epsilon\eta_1 v}) - \psi(u_{\epsilon\eta_2 v})\big)\phi_L(\varphi_{\epsilon\eta_1} - \varphi_0) \right\|_{L^2(\Gamma_3)} + \left\| \psi(u_{\epsilon\eta_2 v})\big(\phi_L(\varphi_{\epsilon\eta_1} - \varphi_0) - \phi_L(\varphi_{\epsilon\eta_2} - \varphi_0)\big) \right\|_{L^2(\Gamma_3)}, \tag{3.26}$$

$$\Pi(u_{\epsilon\eta}) = \frac{u_{\epsilon\eta\tau}}{\sqrt{\epsilon^2 + \|u_{\epsilon\eta\tau}\|^2}}. \tag{3.27}$$

Using (2.28), (2.26), (2.17), (3.1), the properties of R and the Lipschitz continuity of the function Π, after some algebra we obtain

$$J \leqslant \big(L_\mu\, c_0^2\, \|R\sigma_\nu(u_{\epsilon\eta_1}, \varphi_{\epsilon\eta_1})\|_{L^\infty(\Gamma_3)} + \mu^*\, c_*\, c_0^2 + \mu^*\, \|R\sigma_\nu(u_{\epsilon\eta_2}, \varphi_{\epsilon\eta_2})\|_{L^\infty(\Gamma_3)}\, L_\Pi\, c_0^2\big)\, \|x_{\epsilon 1} - x_{\epsilon 2}\|_X^2, \tag{3.28}$$

thus, by using (2.27), (2.25), the bounds $|\phi_L(\varphi - \varphi_0)| \leqslant L$, the Lipschitz continuity of the function ϕ_L, (2.17), (2.19) and (3.1) we deduce

$$G \leqslant (M_\psi\, c_1^2 + L\, L_\psi\, c_0\, c_1)\|x_{\epsilon 1} - x_{\epsilon 2}\|_X^2. \tag{3.29}$$

Hence, there exists a constant $c_6 > 0$ such that

$$\|x_{\epsilon 1} - x_{\epsilon 2}\|_X^2 \leqslant c_6(L_\mu + \mu^* + L_\psi L + M_\psi)\, \|x_{\epsilon\eta_1} - x_{\epsilon\eta_2}\|_X^2.$$

and using (3.23), we have

$$\|\Lambda\eta_1 - \Lambda\eta_2\|_{L^2(\Gamma_3)\times L^2(\Gamma_3)} \quad \leqslant \quad c_5 c_6(L_\mu + \mu^* + L_\psi L + M_\psi)\ \|\eta_1 - \eta_2\|_{L^2(\Gamma_3)\times W}.$$

Let $L^* = \frac{1}{c_5 c_6}$, then if $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$ the mapping Λ is contraction of $L^2(\Gamma_3) \times L^2(\Gamma_3)$. By the Banach fixed point theorem, the mapping Λ has a unique fixed point η_ϵ^* on $L^2(\Gamma_3) \times L^2(\Gamma_3)$. □

Let $L_\mu + \mu^* + L_\psi L + M_\psi < L^*$ and let η_ϵ^* be the fixed point of operator Λ. We denote by $(u^*,\ \varphi^*)$ the solution of the variational Problem PV_ϵ^η for $\eta = \eta_\epsilon^*$. Using (3.12)–(3.13) and (3.24), it is easy to see that $(u^*,\ \varphi^*)$ is a solution of PV_ϵ. This proves the existence part of Theorem 2.2. The uniqueness of the solution results from the uniqueness of the fixed point of the operator Λ.

3.2 Convergence result

Taking $v = u_\epsilon$ in (3.12) and $\xi = \varphi_\epsilon$ in (3.13), we have

$$(\mathfrak{F}\varepsilon(u_\epsilon), \varepsilon(u_\epsilon))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_\epsilon, \varepsilon(u_\epsilon))_{L^2(\Omega)^d} + \frac{1}{\epsilon}\Phi(u_\epsilon, u_\epsilon) + \langle j_\epsilon'(u_\epsilon, u_\epsilon), u_\epsilon\rangle = (f, u_\epsilon)_V, \tag{3.30}$$

$$(\beta\nabla\varphi_\epsilon, \nabla\varphi_\epsilon)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u_\epsilon), \nabla\varphi_\epsilon)_{L^2(\Omega)^d} + \ell(u_\epsilon, \varphi_\epsilon, \varphi_\epsilon) = (q, \varphi_\epsilon)_W, \tag{3.31}$$

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we add equalities (3.30), (3.31) and use (2.21), to obtain

$$\begin{aligned}&(\mathfrak{F}\varepsilon(u_\epsilon),\varepsilon(u_\epsilon))_{\mathscr{H}}+(\beta\nabla\varphi_\epsilon,\nabla\varphi_\epsilon)_{L^2(\Omega)^d}+\tfrac{1}{\epsilon}\Phi(u_\epsilon,u_\epsilon)+\langle j_\epsilon'(u_\epsilon,u_\epsilon),u_\epsilon\rangle\\&\quad+\ell(u_\epsilon,\varphi_\epsilon,\varphi_\epsilon)=(f,u_\epsilon)_V+(q,\varphi_\epsilon)_W.\end{aligned}\tag{3.32}$$

Since $\Phi(u_\epsilon,u_\epsilon)\geqslant 0$ and $\langle j_\epsilon'(u_\epsilon,u_\epsilon),u_\epsilon\rangle)\geqslant 0$, then

$$(\mathfrak{F}\varepsilon(u_\epsilon),\varepsilon(u_\epsilon))_{\mathscr{H}}+(\beta\nabla\varphi_\epsilon,\nabla\varphi_\epsilon)_{L^2(\Omega)^d}\leqslant(f,u_\epsilon)_V+(q,\varphi_\epsilon)_W-\ell(u_\epsilon,\varphi_\epsilon,\varphi_\epsilon).\tag{3.33}$$

Now, we define the operator $F:V\to V$ by

$$(Fu,v)_V=(\mathfrak{F}\varepsilon(u),\varepsilon(v))_{\mathscr{H}}\quad\forall u,v\in V,\tag{3.34}$$

by (3.34) and (2.22)(c), we find

$$(Fu-Fv,u-v)_V\geqslant m_{\mathfrak{F}}\|u-v\|_V^2\quad\forall u,v\in V,\tag{3.35}$$

i.e, that $F:V\to V$ is a monotone operator. Choosing $v=0$ in (3.35), we obtain

$$(Fu,u)_V\geqslant m_{\mathfrak{F}}\|u\|_V^2-\|F0_V\|_V\|u\|_V\quad\forall u\in V.\tag{3.36}$$

Using (3.34), (3.36), (2.24), (2.25)(c), the bounds $\phi_L(\varphi_\epsilon-\varphi_0)\leqslant L$ and (2.19) in (3.33), we obtain

$$\begin{aligned}&m_{\mathfrak{F}}\|u_\epsilon\|_V^2+m_\beta\|\varphi_\epsilon\|_W^2\\&\leqslant(\|f\|_V+\|F0_V\|_V)\|u_\epsilon\|_V+\big(\|q\|_W+M_\psi\,L\,c_1\,meas(\Gamma_3)^{\frac12}\big)\|\varphi_\epsilon\|_W,\end{aligned}$$

therefore

$$\|u_\epsilon\|_V+\|\varphi_\epsilon\|_W\leqslant c\big(\|f\|_V+\|F0_V\|_V+\|q\|_W+M_\psi\,L\,c_1\,meas(\Gamma_3)^{\frac12}\big)=C.\tag{3.37}$$

Using again (3.32), (2.22), (2.24) and (3.37), we find that

$$\frac{1}{\epsilon}\int_{\Gamma_3}u_{\epsilon\nu}^+u_{\epsilon\nu}=\frac{1}{\epsilon}\int_{\Gamma_3}u_{\epsilon\nu}^{+2}=\frac{1}{\epsilon}\|u_{\epsilon\nu}^+\|_{L^2(\Gamma_3)}^2\leqslant 2\,C,$$

thus

$$\|u_{\epsilon\nu}^+\|_{L^2(\Gamma_3)}\leqslant\sqrt{2\,C}\,\sqrt{\epsilon},\tag{3.38}$$

From (3.37), we deduce that there exist $\widetilde{u}\in V$, $\widetilde{\varphi}\in W$ and subsequences (u_ϵ), (φ_ϵ), denoted again by (u_ϵ), (φ_ϵ), such that

$$u_\epsilon\rightharpoonup\widetilde{u}\quad\text{weakly in }V,\quad\varphi_\epsilon\rightharpoonup\widetilde{\varphi}\quad\text{weakly in }W.\tag{3.39}$$

Since the trace map $\gamma:V\times W\to L^2(\Gamma_3)^d\times L^2(\Gamma_3)$ is a compact operator, we deduce that

$$u_\epsilon\to\widetilde{u}\quad\text{strongly in}\quad L^2(\Gamma_3)^d,\quad\varphi_\epsilon\to\widetilde{\varphi}\quad\text{strongly in}\quad L^2(\Gamma_3),\tag{3.40}$$

and we have

$$\lim_{\epsilon\to 0}\|u_{\epsilon\nu}^+\|_{L^2(\Gamma_3)}=\|\widetilde{u}_\nu^+\|_{L^2(\Gamma_3)}.$$

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From (3.38), we deduce that

$$\lim_{\epsilon\to 0}\|u^+_{\epsilon\nu}\|_{L^2(\Gamma_3)} = 0,$$

we find that $\widetilde{u}^+_\nu = 0$ a.e. on Γ_3; it follows that $\widetilde{u}_\nu \leqslant 0$ a.e. on Γ_3, which shows that $\widetilde{u}\in K$. Since $\widetilde{\Phi}(x_\epsilon, y-x_\epsilon) \leqslant 0$ for all $y=(v,\xi)\in U = K\times W$, and (3.7), (3.9) and use the inequality

$$\widetilde{j}_\epsilon(x,y) - \widetilde{j}_\epsilon(x,x) \geqslant \langle \widetilde{j}'_\epsilon(x,x), y-x\rangle \quad \forall x,y\in X, \tag{3.41}$$

for all $y=(v,\xi)\in U$, we obtain

$$(Ax_\epsilon, y-x_\epsilon)_X + \widetilde{j}_\epsilon(x_\epsilon,y) - \widetilde{j}_\epsilon(x_\epsilon,x_\epsilon) + \widetilde{\ell}(x_\epsilon, y-x_\epsilon) \geqslant (f_3, y-x_\epsilon)_X, \tag{3.42}$$

then from (3.40) and the properties of R, σ, ψ et ϕ_L we have

$$\widetilde{j}(x_\epsilon,y) - \widetilde{j}(x_\epsilon,x_\epsilon) \to \widetilde{j}(x,y) - \widetilde{j}(x,x), \quad \widetilde{\ell}(x_\epsilon, y-x_\epsilon) \to \widetilde{\ell}(x,y-x). \tag{3.43}$$

From (2.42)–(2.43) and (3.4), we find

$$0 < \widetilde{j}_\epsilon(x_\epsilon,y) - \widetilde{j}(x_\epsilon,y) \leqslant c\,\epsilon, \quad \forall y\in X. \tag{3.44}$$

Let $\widetilde{x} = (\widetilde{u},\widetilde{\varphi})$. Using (3.39), (3.42), (3.43), (3.44) and a lower semicontinuity argument we find that $\epsilon\to 0$

$$(A\widetilde{x}, y-\widetilde{x})_X + \widetilde{j}(\widetilde{x},y) - \widetilde{j}(\widetilde{x},\widetilde{x}) + \widetilde{\ell}(\widetilde{x}, y-\widetilde{x}) \geqslant (f_3, y-\widetilde{x})_X, \tag{3.45}$$

for any $y=(v,\xi)\in U$. We now use the Lemma 4.1 (p. 363) of [7] to show that $\widetilde{x} = x$.

We conclude that $x=(u,\varphi)$ is the unique weak limit in $X = V\times W$ of any subsequence of the sequence $x_\epsilon = (u_\epsilon,\varphi_\epsilon)$ and therefore, we find that the whole sequence $x_\epsilon = (u_\epsilon,\varphi_\epsilon)$ converges weakly to the element $x=(u,\varphi)\in U = K\times W$.

4 Finite element setting and discrete penalty problem

In this section, we introduce and study the finite element approximation of the variational Problem PV_ϵ. Assume Ω is a polygonal domain, let τ^h be a regular family of triangular finite element partitions of $\overline{\Omega}$ that are compatible with the partition of the boundary decompositions $\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3$ and $\Gamma = \Gamma_a\cup\Gamma_b\cup\Gamma_3$, that is, any point when the boundary condition type changes is a vertex of the partitions, then the side lies entirely in $\overline{\Gamma}_1\cup\overline{\Gamma}_2\cup\overline{\Gamma}_3$, and $\overline{\Gamma}_a\cup\overline{\Gamma}_b\cup\overline{\Gamma}_3$. Corresponding to each partition τ^h. We denote by $\mathbb{P}_1(\Omega^e)$ the space of polynomials of global degree less or equal to one in Ω^e. Let us consider two finite-dimensional spaces $V^h\subset V$ and $W^h\subset W$, approximating the spaces V and W, respectively, that is

$$V^h = \{v^h \in C(\overline{\Omega})^d,\ v^h_{/\Omega^e}\in\mathbb{P}_1(\Omega^e)^d,\ \Omega^e\in\tau^h,\ v^h = 0 \ on\ \overline{\Gamma}_1\},$$
$$W^h = \{\psi^h \in C(\overline{\Omega}),\ \psi^h_{/\Omega^e}\in\mathbb{P}_1(\Omega^e),\ \Omega^e\in\tau^h,\ \psi^h = 0 \ on\ \overline{\Gamma}_a\}.$$

Here $h>0$ is a discretisation parameter. Moreover, c denotes a positive constant which depends on the problem data, but is independent of the discretisation parameter h. We consider the following discrete approximation of Problem PV^h_ϵ:

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Problem PV_ϵ^h Find $u_\epsilon^h \in V^h$ and $\varphi_\epsilon^h \in W^h$ such that

$$(\mathfrak{F}\varepsilon(u_\epsilon^h), \varepsilon(v^h))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_\epsilon^h, \varepsilon(v^h))_{L^2(\Omega)^d} + \frac{1}{\epsilon}\Phi(u_\epsilon^h, v^h) + \langle j_\epsilon'(u_\epsilon^h, u_\epsilon^h), v^h\rangle = (f, v^h)_V, \quad \forall v^h \in V^h, \tag{4.1}$$

$$(\beta\nabla\varphi_\epsilon^h, \nabla\xi^h)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u_\epsilon^h), \nabla\xi^h)_{L^2(\Omega)^d} + \ell(u_\epsilon^h, \varphi_\epsilon^h, \xi^h) = (q, \xi^h)_W, \ \forall \xi^h \in W^h. \tag{4.2}$$

Applying Theorem 2.2, for the case when V and W are replaced by V^h and W^h, respectively, we find that the Problem PV_ϵ^h has a unique solution $(u_\epsilon^h, \varphi_\epsilon^h) \in V^h \times W^h$. We have the following convergence result.

Theorem 4.1 *Let us denote by $(u_\epsilon, \varphi_\epsilon)$ and $(u_\epsilon^h, \varphi_\epsilon^h)$ the respective solutions to Problem PV_ϵ and PV_ϵ^h. Under the assumptions of Theorem 2.2 with the same value of L^*, we have*

$$\|u_\epsilon^h - u_\epsilon\|_V \to 0, \quad \|\varphi_\epsilon^h - \varphi_\epsilon\|_W \to 0, \quad \textit{as} \quad h \to 0. \tag{4.3}$$

Proof We consider

$$\mathfrak{U}_1 = \{v \in C^\infty(\overline{\Omega})^d / \ v_i = 0 \text{ in a neighbourhood of } \Gamma_1\},$$

$$\mathfrak{U}_2 = \{\xi \in C^\infty(\overline{\Omega}) / \ \xi_i = 0 \text{ in a neighbourhood of } \Gamma_a\},$$

then $\overline{\mathfrak{U}_1} = V$ and $\overline{\mathfrak{U}_2} = W$ (see [19]). We define $r_1^h : \mathfrak{U}_1 \to V^h$ and $r_2^h : \mathfrak{U}_2 \to W^h$ by

$$\begin{cases} r_1^h v \in V^h, & \forall v \in \mathfrak{U}_1, \\ (r_1^h v)(P) = v(P), & P \text{ is a vertex of triangulation,} \end{cases}$$

$$\begin{cases} r_2^h \xi \in W^h, & \forall \xi \in \mathfrak{U}_2, \\ (r_2^h \xi)(P) = \xi(P), & P \text{ is a vertex of triangulation.} \end{cases}$$

Then since $r_1^h v$ (resp. $r_2^h \xi$) is the "linear" interpolate of v (resp. ξ) on τ^h, under the assumptions made on τ^h, we have (see [4, 18])

$$\|r_1^h v - v\|_V \leqslant c\, h\, |v|_{H^2(\Omega)^d}, \quad \forall v \in \mathfrak{U}_1, \tag{4.4}$$

$$\|r_2^h \xi - \xi\|_W \leqslant c\, h\, |\xi|_{H^2(\Omega)}, \quad \forall \xi \in \mathfrak{U}_2, \tag{4.5}$$

with c independent of h, v and ξ. This implies

$$r_1^h v \to v \ \text{ strongly in } V, \quad r_2^h \xi \to \xi \ \text{ strongly in } W, \quad \text{as} \quad h \to 0. \tag{4.6}$$

We now prove the boundedness of the sequence $\{u_\epsilon^h\}_h$ in V and the sequence $\{\varphi_\epsilon^h\}_h$ in W.

Taking $v^h = u_\epsilon^h$ in (4.1) and $\xi^h = \varphi_\epsilon^h$ in (4.2), we have

$$(\mathfrak{F}\varepsilon(u_\epsilon^h), \varepsilon(u_\epsilon^h))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_\epsilon^h, \varepsilon(u_\epsilon^h))_{L^2(\Omega)^d} + \frac{1}{\epsilon}\Phi(u_\epsilon^h, u_\epsilon^h) + \langle j_\epsilon'(u_\epsilon^h, u_\epsilon^h), u_\epsilon^h\rangle = (f, u_\epsilon^h)_V,$$
$$(\beta\nabla\varphi_\epsilon^h, \nabla\varphi_\epsilon^h)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u_\epsilon^h), \nabla\varphi_\epsilon^h)_{L^2(\Omega)^d} + \ell(u_\epsilon^h, \varphi_\epsilon^h, \varphi_\epsilon^h) = (q, \varphi_\epsilon^h)_W,$$

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as $\Phi(u_\epsilon^h, u_\epsilon^h) \geqslant 0$ and $\langle j_\epsilon'(u_\epsilon^h, u_\epsilon^h), u_\epsilon^h\rangle \geqslant 0$, then

$$(\mathfrak{F}\varepsilon(u_\epsilon^h), \varepsilon(u_\epsilon^h))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_\epsilon^h, \varepsilon(u_\epsilon^h))_{L^2(\Omega)^d} \leqslant (f, u_\epsilon^h)_V, \tag{4.7}$$

$$(\beta\nabla\varphi_\epsilon^h, \nabla\varphi_\epsilon^h)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u_\epsilon^h), \nabla\varphi_\epsilon^h)_{L^2(\Omega)^d} + \ell(u_\epsilon^h, \varphi_\epsilon^h, \varphi_\epsilon^h) = (q, \varphi_\epsilon^h)_W. \tag{4.8}$$

Adding (4.7) and (4.8), from the bounds $\phi_L(\varphi_{\eta\epsilon}^h - \varphi_0) \leqslant L$ and using (2.21)–(2.22), (2.24), (2.25)(c), (3.34), (3.36) and (2.19), we obtain

$$\begin{aligned} &m_{\mathfrak{F}}\|u_\epsilon^h\|_V^2 + m_\beta\|\varphi_\epsilon^h\|_W^2 \\ &\quad\leqslant (\|f\|_V + \|F0_V\|_V)\|u_\epsilon^h\|_V + \big(\|q\|_W + M_\psi\, L\, c_1\, meas(\Gamma_3)^{\frac{1}{2}}\big)\|\varphi_\epsilon^h\|_W, \end{aligned}$$

therefore

$$\|u_\epsilon^h\|_V + \|\varphi_\epsilon^h\|_W \leqslant c\big(\|f\|_V + \|F0_V\|_V + \|q\|_W + M_\psi\, L\, c_1\, meas(\Gamma_3)^{\frac{1}{2}}\big), \tag{4.9}$$

the constant c is independent of u_ϵ^h, φ_ϵ^h and h, thus, there exist $u^* \in V$, $\varphi^* \in W$ and subsequences of the sequences $\{u_\epsilon^h\}_h$, $\{\varphi_\epsilon^h\}_h$ denoted again by $\{u_\epsilon^h\}_h$, $\{\varphi_\epsilon^h\}_h$ such that

$$u_\epsilon^h \rightharpoonup u^* \quad \text{weakly in } V, \quad \varphi_\epsilon^h \rightharpoonup \varphi^* \quad \text{weakly in } W, \quad \text{as} \quad h \to 0. \tag{4.10}$$

Since the trace map $\gamma : V \times W \to L^2(\Gamma_3)^d \times L^2(\Gamma_3)$ is compact operator, it follows from (4.10) that

$$u_\epsilon^h \to u^* \ \text{ strongly in } L^2(\Gamma_3)^d,\ \varphi_\epsilon^h \to \varphi^* \ \text{ strongly in } L^2(\Gamma_3), \text{ as } h \to 0. \tag{4.11}$$

In the next, we prove that (u^*, φ^*) is a solution to Problem PV_ϵ. Since $(u_\epsilon^h, \varphi_\epsilon^h)$ is a solution to Problem PV_ϵ^h and $r_1^h v \in V^h$, $r_2^h\xi \in W^h$, for all h, $v \in \mathfrak{U}_1$ and $\xi \in \mathfrak{U}_2$, we have

$$\begin{aligned} &(\mathfrak{F}\varepsilon(u_\epsilon^h), \varepsilon(u_\epsilon^h - r_1^h v))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_\epsilon^h, \varepsilon(u_\epsilon^h - r_1^h v))_{L^2(\Omega)^d} + \frac{1}{\epsilon}\Phi(u_\epsilon^h, u_\epsilon^h - r_1^h v) \\ &+\langle j_\epsilon'(u_\epsilon^h, u_\epsilon^h), u_\epsilon^h - r_1^h v\rangle = (f, u_\epsilon^h - r_1^h v)_V, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &(\beta\nabla\varphi_\epsilon^h, \nabla(\varphi_\epsilon^h - r_2^h\xi))_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u_\epsilon^h), \nabla(\varphi_\epsilon^h - r_2^h\xi))_{L^2(\Omega)^d} + \ell(u_\epsilon^h, \varphi_\epsilon^h, \varphi_\epsilon^h - r_2^h\xi) \\ &= (q, \varphi_\epsilon^h - r_2^h\xi)_W. \end{aligned} \tag{4.13}$$

From (4.6), (4.11) and the properties of R, σ, ψ and ϕ_L, we have

$$\lim_{h\to 0}\langle j_\epsilon'(u_\epsilon^h, u_\epsilon^h), u_\epsilon^h - r_1^h v\rangle = \langle j_\epsilon'(u^*, u^*), u^* - v\rangle, \forall v \in \mathfrak{U}_1, \tag{4.14}$$

$$\lim_{h\to 0}\Phi(u_\epsilon^h, u_\epsilon^h - r_1^h v) = \Phi(u^*, u^* - v), \forall v \in \mathfrak{U}_1, \tag{4.15}$$

$$\lim_{h\to 0}\ell(u_\epsilon^h, \varphi_\epsilon^h, \varphi_\epsilon^h - r_2^h\xi) = \ell(u^*, \varphi^*, \varphi^* - \xi),\ \forall(\xi, v) \in \mathfrak{U}_2 \times \mathfrak{U}_1. \tag{4.16}$$

Therefore, by (4.10), (4.13), (4.16) and a lower semicontinuity argument we find that

$$\begin{aligned} &(\beta\nabla\varphi^*, \nabla(\varphi^* - \xi))_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u^*), \nabla(\varphi^* - \xi))_{L^2(\Omega)^d} + \ell(u^*, \varphi^*, \varphi^* - \xi) \\ &\leqslant (q, \varphi^* - \xi)_W, \quad \forall\xi \in \mathfrak{U}_2, \quad \forall(u^*, \varphi^*) \in V \times W. \end{aligned} \tag{4.17}$$

Since $\mathfrak{U}_2$ is dense in W and β, $\mathscr{E}$, ψ, ϕ_L are continuous, from (4.17) we obtain

$$\begin{aligned}&(\beta\nabla\varphi^*,\nabla(\varphi^*-\xi))_{L^2(\Omega)^d}-(\mathscr{E}\varepsilon(u^*),\nabla(\varphi^*-\xi))_{L^2(\Omega)^d}+\ell(u^*,\varphi^*,\varphi^*-\xi)\\&\quad\leqslant(q,\varphi^*-\xi)_W,\quad\forall\xi\in W,\quad\forall(u^*,\varphi^*)\in V\times W,\end{aligned}\tag{4.18}$$

by setting $\xi=\varphi^*\pm\xi^*$ in (4.18), where ξ^* is an arbitrary element of W, we find

$$(\beta\nabla\varphi^*,\nabla\xi^*)_{L^2(\Omega)^d}-(\mathscr{E}\varepsilon(u^*),\nabla\xi^*)_{L^2(\Omega)^d}+\ell(u^*,\varphi^*,\xi^*)=(q,\xi^*)_W,\tag{4.19}$$

for all $\xi^*\in W$, $(u^*,\varphi^*)\in V\times W$. From (4.12), we have

$$\begin{aligned}&(\mathfrak{F}\varepsilon(u^h_\epsilon),\varepsilon(u^h_\epsilon-u^*))_{\mathscr{H}}=(\mathfrak{F}\varepsilon(u^h_\epsilon),\varepsilon(u^h_\epsilon-r^h_1v))_{\mathscr{H}}+(\mathfrak{F}\varepsilon(u^h_\epsilon),\varepsilon(r^h_1v-u^*))_{\mathscr{H}}\\&\quad\leqslant(\mathscr{E}^*\nabla\varphi^h_\epsilon,\varepsilon(r^h_1v-u^h_\epsilon))_{L^2(\Omega)^d}+\tfrac{1}{\epsilon}\Phi(u^h_\epsilon,r^h_1v-u^h_\epsilon)+\langle j'_\epsilon(u^h_\epsilon,u^h_\epsilon),r^h_1v-u^h_\epsilon\rangle\\&\quad\quad+(f,u^h_\epsilon-r^h_1v)_V+\|\mathfrak{F}\varepsilon(u^h_\epsilon)\|_{\mathscr{H}}\|\varepsilon(r^h_1v-u^*)\|_{\mathscr{H}},\end{aligned}\tag{4.20}$$

therefore, by (4.6), (4.10), (4.13) and (4.16), we find that

$$\begin{aligned}&\limsup_{h\to0}(\mathfrak{F}\varepsilon(u^h_\epsilon),\varepsilon(u^h_\epsilon-u^*))_{\mathscr{H}}\\&\quad\leqslant(\mathscr{E}^*\nabla\varphi^*,\varepsilon(v-u^*))_{L^2(\Omega)^d}+\frac{1}{\epsilon}\Phi(u^*,v-u^*)+\langle j'_\epsilon(u^*,u^*),v-u^*\rangle\\&\quad\quad+(f,u^*-v)_V+\limsup_{h\to0}\|\mathfrak{F}\varepsilon(u^h_\epsilon)\|_{\mathscr{H}}\|v-u^*\|_V,\end{aligned}$$

for all $v\in\mathfrak{U}_1$. Note that $\|\mathfrak{F}\varepsilon(u^h_\epsilon)\|_{\mathscr{H}}$ is bounded (according to (3.9)), we obtain

$$\begin{aligned}&\limsup_{h\to0}(\mathfrak{F}\varepsilon(u^h_\epsilon),\varepsilon(u^h_\epsilon-u^*))_{\mathscr{H}}\\&\leqslant(\mathscr{E}^*\nabla\varphi^*,\varepsilon(v-u^*))_{L^2(\Omega)^d}+\frac{1}{\epsilon}\Phi(u^*,v-u^*)+\langle j'_\epsilon(u^*,u^*),v-u^*\rangle\\&\quad+(f,u^*-v)_V+c\|v-u^*\|_V,\end{aligned}$$

for all $v\in\mathfrak{U}_1$, we may then substitute $v=u^*$ into the previous inequality to obtain

$$\limsup_{h\to0}(\mathfrak{F}\varepsilon(u^h_\epsilon),\varepsilon(u^h_\epsilon-u^*))_{\mathscr{H}}\leqslant0.$$

Therefore, by pseudo monotonicity of $\mathfrak{F}$, we get

$$(\mathfrak{F}\varepsilon(u^*),\varepsilon(u^*-v))_{\mathscr{H}}\leqslant\liminf_{h\to0}(\mathfrak{F}\varepsilon(u^h_\epsilon),\varepsilon(u^h_\epsilon-r^h_1v))_{\mathscr{H}}.\tag{4.21}$$

Combining (4.12), (4.14), (4.15) and (4.21), one gets

$$\begin{aligned}&(\mathfrak{F}\varepsilon(u^*),\varepsilon(u^*-v))_{\mathscr{H}}+(\mathscr{E}^*\nabla\varphi^*,\varepsilon(u^*-v))_{L^2(\Omega)^d}+\tfrac{1}{\epsilon}\Phi(u^*,u^*-v)\\&\quad+\langle j'_\epsilon(u^*,u^*),u^*-v\rangle=(f,u^*-v)_V,\quad\forall v\in\mathfrak{U}_1,\quad\forall(u^*,\varphi^*)\in V\times W.\end{aligned}\tag{4.22}$$

Since $\mathfrak{U}_1$ is dense in V and $\mathfrak{F}$, $\mathscr{E}$, R, σ are continuous, from (4.22) we obtain

$$\begin{aligned}&(\mathfrak{F}\varepsilon(u^*),\varepsilon(u^*-v))_{\mathscr{H}}+(\mathscr{E}^*\nabla\varphi^*,\varepsilon(u^*-v))_{L^2(\Omega)^d}+\tfrac{1}{\epsilon}\Phi(u^*,u^*-v)\\&\quad+\langle j'_\epsilon(u^*,u^*),u^*-v\rangle\leqslant(f,u^*-v)_V,\quad\forall v\in V,\quad\forall(u^*,\varphi^*)\in V\times W,\end{aligned}\tag{4.23}$$

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by setting $v = u^* \pm v^*$ in (4.23) where v^* is an arbitrary element in V, we find

$$\begin{aligned}(\mathfrak{F}\varepsilon(u^*), \varepsilon(v^*))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi^*, \varepsilon(v^*))_{L^2(\Omega)^d} + \tfrac{1}{\epsilon}\Phi(u^*, v^*) + \langle j_\epsilon'(u^*, u^*), v^*\rangle \\ = (f, v^*)_V, \quad \forall v^* \in V, \quad \forall (u^*, \varphi^*) \in V \times W.\end{aligned} \tag{4.24}$$

By (4.19) and (4.24), we conclude that (u^*, φ^*) is a solution to Problem PV_ϵ. By Theorem 2.2, the solution of PV_ϵ is unique and hence $(u^*, \varphi^*) = (u_\epsilon, \varphi_\epsilon)$. Then, u_ϵ (resp. φ_ϵ) is the only cluster point of $\{u_\epsilon^h\}_h$ (resp. $\{\varphi_\epsilon^h\}_h$) in the weak topology of V (resp. W). Hence, the whole $\{u_\epsilon^h\}_h$ (resp. $\{\varphi_\epsilon^h\}_h$) converge to u_ϵ weakly (resp. φ_ϵ weakly).

Using now the assumptions (2.22), (2.24) on $\mathfrak{F}$, β and (4.12)–(4.13), we deduce that

$$\begin{aligned}
& m_{\mathfrak{F}}\|u_\epsilon^h - u_\epsilon\|_V + m_\beta\|\varphi_\epsilon^h - \varphi_\epsilon\|_W \\
&\quad \leqslant (\mathfrak{F}\varepsilon(u_\epsilon^h) - \mathfrak{F}\varepsilon(u_\epsilon), \varepsilon(u_\epsilon^h - u_\epsilon))_{\mathscr{H}} + (\beta\nabla(\varphi_\epsilon^h - \varphi_\epsilon), \nabla(\varphi_\epsilon^h - \varphi_\epsilon))_{L^2(\Omega)^d} \\
&\quad \leqslant (\mathfrak{F}\varepsilon(u_\epsilon^h), \varepsilon(r_1^h v - u_\epsilon))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_\epsilon^h, \varepsilon(u_\epsilon^h - r_1^h v))_{L^2(\Omega)^d} - \tfrac{1}{\epsilon}\Phi(u_\epsilon^h, u_\epsilon^h - r_1^h v) \\
&\qquad - \langle j_\epsilon'(u_\epsilon^h, u_\epsilon^h), u_\epsilon^h - r_1^h v\rangle + (f, u_\epsilon^h - r_1^h v)_V - (\mathfrak{F}\varepsilon(u_\epsilon), \varepsilon(u_\epsilon^h - u_\epsilon))_{\mathscr{H}} \\
&\qquad + (\beta\nabla\varphi_\epsilon^h, \nabla(r_2^h\xi - \varphi_\epsilon))_{L^2(\Omega)^d} + (\mathscr{E}\varepsilon(u_\epsilon^h), \nabla(\varphi_\epsilon^h - r_2^h\xi))_{L^2(\Omega)^d} \\
&\qquad - \ell(u_\epsilon^h, \varphi_\epsilon^h, \varphi_\epsilon^h - r_2^h\xi) + (q, \varphi_\epsilon^h - r_2^h\xi)_W - (\beta\nabla\varphi_\epsilon, \nabla(\varphi_\epsilon^h - \varphi_\epsilon))_{L^2(\Omega)^d},
\end{aligned}$$

for all $(v, \xi) \in \mathfrak{U}_1 \times \mathfrak{U}_2$. Taking into account the bounds $\|\mathfrak{F}\|_{\mathscr{H}} \leqslant c$, (4.6), (4.14)–(4.16) and the weak convergence of $\{u_\epsilon^h\}_h$ to u_ϵ and $\{\varphi_\epsilon^h\}_h$ to φ_ϵ, we obtain from the previous inequality that

$$\begin{aligned}
& \lim_{h\to 0}(m_{\mathfrak{F}}\|u_\epsilon^h - u_\epsilon\|_V^2 + m_\beta\|\varphi_\epsilon^h - \varphi_\epsilon\|_W^2) \\
&\quad \leqslant c\|v - u_\epsilon\|_V + (\mathscr{E}^*\nabla\varphi_\epsilon, \varepsilon(u_\epsilon - v))_{L^2(\Omega)^d} - \frac{1}{\epsilon}\Phi(u_\epsilon, u_\epsilon - v) \\
&\qquad - \langle j_\epsilon'(u_\epsilon, u_\epsilon), u_\epsilon - v\rangle + (f, u_\epsilon - v)_V - (\mathfrak{F}\varepsilon(u_\epsilon), \varepsilon(u_\epsilon - u_\epsilon))_{\mathscr{H}} \\
&\qquad + (\beta\nabla\varphi_\epsilon, \nabla(\xi - \varphi_\epsilon))_{L^2(\Omega)^d} + (\mathscr{E}\varepsilon(u_\epsilon), \nabla(\varphi_\epsilon - \xi))_{L^2(\Omega)^d} \\
&\qquad - \ell(u_\epsilon, \varphi_\epsilon, \varphi_\epsilon - \xi) + (q, \varphi_\epsilon - \xi)_W - (\beta\nabla\varphi_\epsilon, \nabla(\varphi_\epsilon - \varphi_\epsilon))_{L^2(\Omega)^d},
\end{aligned} \tag{4.25}$$

for all $(v, \xi) \in \mathfrak{U}_1 \times \mathfrak{U}_2$. By the density of $\mathfrak{U}_1$ and $\mathfrak{U}_2$, (4.25) holds, $(v, \xi) \in V \times W$. Replacing (v, ξ) by $(u_\epsilon, \varphi_\epsilon)$ in (4.25), we obtain

$$\lim_{h\to 0}(\|u_\epsilon^h - u_\epsilon\|_V + \|\varphi_\epsilon^h - \varphi_\epsilon\|_W) = 0,$$

this proves the theorem. □

The finite element system (4.1)–(4.2) can be approximated by a fixed point iteration method. This follows from a discrete analogue of the proof of Theorem 2.2. Given an initial guess $(u_{\epsilon,0}^h,\ \varphi_{\epsilon,0}^h)$, we define a sequence $(u_{\epsilon,n}^h,\ \varphi_{\epsilon,n}^h) \in V \times W$ for all $n \in \mathbb{N}$ recursively by

$$\begin{aligned}(\mathfrak{F}\varepsilon(u_{\epsilon,(n+1)}^h), \varepsilon(v^h))_{\mathscr{H}} + (\mathscr{E}^*\nabla\varphi_{\epsilon,(n+1)}^h, \varepsilon(v^h))_{L^2(\Omega)^d} + \frac{1}{\epsilon}\Phi(u_{\epsilon,(n+1)}^h, v^h) \\ + \langle j_\epsilon'(u_{\epsilon,n}^h, u_{\epsilon,n}^h), v^h\rangle = (f, v^h)_V, \quad \forall v^h \in V^h,\end{aligned} \tag{4.26}$$

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$$(\beta\nabla\varphi^h_{\epsilon,(n+1)}, \nabla\xi^h)_{L^2(\Omega)^d} - (\mathscr{E}\varepsilon(u^h_{\epsilon,(n+1)}), \nabla\xi^h)_{L^2(\Omega)^d} + \ell(u^h_{\epsilon,n}, \varphi^h_{\epsilon,n}, \xi^h)$$
$$= (q, \xi^h)_W, \ \forall \xi^h \in W^h. \tag{4.27}$$

We have the following convergence result.

Theorem 4.2 *Under the assumptions of Theorem 2.2 with the same value of L^*, the iteration method (4.26)–(4.27) converges:*

$$\|u^h_{\epsilon,n} - u^h_\epsilon\|_V \to 0 \quad as \quad n \to \infty, \qquad \|\varphi^h_{\epsilon,n} - \varphi^h_\epsilon\|_W \to 0 \quad as \quad n \to \infty.$$

Furthermore, for some constant $0 < k < 1$ (which depends on data and ϵ), we have the estimate

$$\|u^h_{\epsilon,n} - u^h_\epsilon\|_V \leqslant ck^n, \qquad \|\varphi^h_{\epsilon,n} - \varphi^h_\epsilon\|_W \leqslant ck^n. \tag{4.28}$$

Proof Using Lemma 1, it is easy to see that

(i) The couple $x^h = (u^h_\epsilon, \varphi^h_\epsilon)$ is a solution to Problem PV^h_ϵ if and only if:

$$(A_\epsilon x^h_\epsilon, y^h)_X + \langle \widetilde{j}'_\epsilon(x^h_\epsilon, x^h_\epsilon), y^h\rangle + \widetilde{\ell}(x^h_\epsilon, y^h) = (f_3, y^h)_X, \forall y^h = (v^h, \xi^h) \in V^h \times W^h. \tag{4.29}$$

(ii) The couple $x^h_n = (u^h_n, \varphi^h_n)$ is a solution to Problem (4.26)–(4.27) if and only if

$$\begin{aligned}&(A_\epsilon x^h_{\epsilon,(n+1)}, y^h)_X + \langle \widetilde{j}'_\epsilon(x^h_{\epsilon,n}, x^h_{\epsilon,n}), y^h\rangle + \widetilde{\ell}(x^h_{\epsilon,n}, y^h) = (f_3, y^h)_X,\\ &\quad \forall y^h = (v^h, \xi^h) \in V^h \times W^h.\end{aligned} \tag{4.30}$$

We subtract (4.29) from (4.30) and taking $y = x^h_\epsilon - x^h_{\epsilon,(n+1)}$ in the resulting equalities, we have

$$(A_\epsilon x^h_\epsilon - A_\epsilon x^h_{\epsilon,(n+1)}, x^h - x^h_{n+1})_X$$
$$= \langle \widetilde{j}'_\epsilon(x^h_{\epsilon,n}, x^h_{\epsilon,n}), y^h\rangle - \langle \widetilde{j}'_\epsilon(x^h_\epsilon, x^h_\epsilon), y^h\rangle + \widetilde{\ell}(x^h_{\epsilon,n}, y^h) - \widetilde{\ell}(x^h_\epsilon, y^h),$$

and using the inequality (3.41), we find

$$\begin{aligned}&(A_\epsilon x^h_\epsilon - A_\epsilon x^h_{\epsilon,(n+1)}, x^h_\epsilon - x^h_{\epsilon,(n+1)})_X\\ &\quad \leqslant \widetilde{j}_\epsilon(x^h_\epsilon, x^h_{\epsilon,(n+1)}) - \widetilde{j}_\epsilon(x^h_\epsilon, x^h_\epsilon) + \widetilde{j}_\epsilon(x^h_{\epsilon,n}, x^h_\epsilon) - \widetilde{j}_\epsilon(x^h_{\epsilon,n}, x^h_{\epsilon,(n+1)})\\ &\quad + \widetilde{\ell}(x^h_{\epsilon,n}, x^h_\epsilon - x^h_{\epsilon,(n+1)})_X - \widetilde{\ell}(x^h_\epsilon, x^h_\epsilon - x^h_{\epsilon,(n+1)})_X.\end{aligned}$$

Then, as in the proof of the uniqueness of Theorem 2.2, we can derive the estimate

$$\|x^h_\epsilon - x^h_{\epsilon,(n+1)}\|_X \leqslant c_6(L_\mu + \mu^* + L_\psi L + M_\psi)\,\|x^h_\epsilon - x^h_{\epsilon,n}\|_X,$$

thus

$$\|x^h_\epsilon - x^h_{\epsilon,(n+1)}\|_X \leqslant \frac{(L_\mu + \mu^* + L_\psi L + M_\psi)}{L^*}\ \|x^h_\epsilon - x^h_{\epsilon,n}\|_X.$$

Under the stated assumption, $k \equiv \frac{(L_\mu + \mu^* + L_\psi L + M_\psi)}{L^*} < 1$, and we have the estimate (4.28). □

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5 Conclusion

In this work, we presented a penalisation method to solve the frictional contact problem between a piezoelectric body and an electrically conductive foundation. The constitutive relation of the material is assumed to be electro-elastic and involves a nonlinear elasticity operator. The contact is described by the Signorini's conditions and a version of Coulomb's law of dry friction in which the coefficient of friction depends on the slip, including the electrical conductivity conditions. The existence and uniqueness of the solution for the penalised problem as well as its convergence to the solution of the original problem were established. The proofs were based on arguments for elliptic variational inequalities followed by applying Banach's fixed point theorem. Then, we study the discrete problem and prove the convergence of its solution towards the solution of the penalised problem. Moreover, we describe an iterative method for the numerical solutions and obtain its convergence. A numerical validation of the convergence result included in this method will be provided in a forthcoming paper.

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