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ON THE LINEAR COMBINATION OF LAPLACE RANDOM VARIABLES

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The distribution of the linear combination $\alpha X + \beta Y$ is derived when X and Y are independent Laplace random variables. Extensive tabulations of the associated percentage points are also given. The work is motivated by examples in automation, control, fuzzy sets, neurocomputing, and other areas of informational sciences.

1. INTRODUCTION

For given random variables X and Y, the distribution of linear combinations of the form $\alpha X + \beta Y$ is of interest in problems in automation, control, fuzzy sets, neurocomputing, and other areas of informational sciences. The distribution of $\alpha X + \beta Y$ has been studied by several authors, especially when X and Y are independent random variables and come from the same family. For instance, see Fisher [8] and Chapman [3] for the Student's t family, Christopeit and Helmes [4] for the normal family, Davies [5] and Farebrother [7] for the chi-squared family, Ali [2] for the exponential family, Moschopoulos [12] and Provost [15] for the gamma family, Dobson, Kulasmaa, and Scherer [6] for the Poisson family, Pham-Gia and Turkkan [14] and Pham and Turkkan [13] for the beta family, Kamgar-Parsi, Kamgar-Parsi, and Brosh [11] and Albert [1] for the uniform family, Hitezenko [9] and Hu and Lin [10] for the Rayleigh family, and Witkovský [17] for the inverted gamma family.

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In this article, we study the distribution of $\alpha X + \beta Y$ when X and Y are independent Laplace random variables with probability density functions (p.d.f.s)

$$f(x) = \frac{\lambda}{2} \exp(-\lambda |x - \theta|) \tag{1}$$

and

$$f(y) = \frac{\mu}{2} \exp(-\mu |y - \phi|), \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $\lambda > 0$, $\mu > 0$, $-\infty < \theta < \infty$, and $-\infty < \phi < \infty$. We assume without loss of generality that $\alpha > 0$. Extensive tabulations of the associated percentage points are also provided.

2. CDF

Theorem 1 derives explicit expressions for the cumulative distribution function (c.d.f.) of $\alpha X + \beta Y$ in terms of elementary functions.

Theorem 1: *Suppose X and Y are distributed according to (1) and (2), respectively. The c.d.f. of* $Z = \alpha X + \beta Y$ *can be expressed as one of the following:*

1. *If* $\beta < 0$ *and* $\phi < (z - \theta\alpha)/\beta$, *then*

$$F(z) = \frac{\mu\alpha \exp(p)}{4(\mu\alpha - \lambda\beta)} + \frac{\mu\alpha\{\exp(p) - \exp(r)\}}{4(\mu\alpha + \lambda\beta)} + \frac{\exp(r)}{2} - \frac{\mu\alpha \exp(r)}{4(\mu\alpha - \lambda\beta)}. \tag{3}$$

2. *If* $\beta < 0$ *and* $\phi > (z - \theta\alpha)/\beta$, *then*

$$F(z) = 1 + \frac{\mu\alpha \exp(-r)}{4(\mu\alpha - \lambda\beta)} - \frac{\exp(-r)}{2} - \frac{\mu\alpha\{\exp(-p) - \exp(-r)\}}{4(\mu\alpha + \lambda\beta)} - \frac{\mu\alpha \exp(-p)}{4(\mu\alpha - \lambda\beta)}. \tag{4}$$

3. *If* $\beta > 0$ *and* $\phi < (z - \theta\alpha)/\beta$, *then*

$$F(z) = 1 - \frac{\exp(r)}{2} - \frac{\mu\alpha \exp(-p)}{4(\mu\alpha + \lambda\beta)} - \frac{\mu\alpha\{\exp(r) - \exp(-p)\}}{4(\lambda\beta - \mu\alpha)} + \frac{\mu\alpha \exp(r)}{4(\mu\alpha + \lambda\beta)}. \tag{5}$$

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4. *If $\beta > 0$ and $\phi > (z - \theta\alpha)/\beta$, then*

$$F(z) = \frac{\exp(-r)}{2} - \frac{\mu\alpha \exp(-r)}{4(\mu\alpha + \lambda\beta)} + \frac{\mu\alpha\{\exp(p) - \exp(-r)\}}{4(\mu\alpha - \lambda\beta)} + \frac{\mu\alpha \exp(p)}{4(\mu\alpha + \lambda\beta)}. \quad \textbf{(6)}$$

For (3)–(6), $p = \lambda(z/\alpha - \theta) - \lambda\beta\phi/\alpha$ and $r = \mu\phi - \mu(z - \theta\alpha)/\beta$.

Proof: The c.d.f. $F(z) = \Pr(\alpha X + \beta Y \leq z)$ can be expressed as

$$F(z) = \frac{\mu}{2}\int_{-\infty}^{\infty} F\left(\frac{z - \beta y}{\alpha}\right)\exp(-\mu|y - \phi|)\, dy, \quad \textbf{(7)}$$

where $F(\cdot)$ inside the integral denotes the c.d.f. corresponding to (1) and given by

$$F(x) = \begin{cases} \dfrac{1}{2}\exp\{\lambda(x - \theta)\} & \text{if } x \leq \theta \\ 1 - \dfrac{1}{2}\exp\{\lambda(\theta - x)\} & \text{if } x > \theta. \end{cases}$$

The results in (3)–(6) follow by elementary integration of (7). ■

The following corollaries provide the c.d.f.s for the sum and the difference of Laplace random variables.

Corollary 1: *Suppose X and Y are distributed according to (1) and (2), respectively. Then the c.d.f. of $Z = X + Y$ can be expressed as one of the following:*

1. *If $\phi < z - \theta$, then*

$$F(z) = 1 - \frac{\exp(r)}{2} - \frac{\mu \exp(-p)}{4(\mu + \lambda)} - \frac{\mu\{\exp(r) - \exp(-p)\}}{4(\lambda - \mu)} + \frac{\mu \exp(r)}{4(\mu + \lambda)}.$$

2. *If $\phi > z - \theta$, then*

$$F(z) = \frac{\exp(-r)}{2} - \frac{\mu \exp(-r)}{4(\mu + \lambda)} + \frac{\mu\{\exp(p) - \exp(-r)\}}{4(\mu - \lambda)} + \frac{\mu \exp(p)}{4(\mu + \lambda)}.$$

In the above, $p = \lambda(z - \theta - \phi)$ and $r = \mu(\theta + \phi - z)$.

Proof: Set $\alpha = 1$ and $\beta = 1$ into (5) and (6). ■

Corollary 2: *Suppose X and Y are distributed according to (1) and (2), respectively. Then the c.d.f. of $Z = X - Y$ can be expressed as one of the following:*

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1. *If* $\phi < \theta - z$, *then*

$$F(z) = \frac{\mu \exp(p)}{4(\mu + \lambda)} + \frac{\mu\{\exp(p) - \exp(r)\}}{4(\mu - \lambda)} + \frac{\exp(r)}{2} - \frac{\mu \exp(r)}{4(\mu + \lambda)}.$$

2. *If* $\phi > \theta - z$, *then*

$$F(z) = 1 + \frac{\mu \exp(-r)}{4(\mu + \lambda)} - \frac{\exp(-r)}{2} - \frac{\mu\{\exp(-p) - \exp(-r)\}}{4(\mu - \lambda)} - \frac{\mu \exp(-p)}{4(\mu + \lambda)}.$$

In the above, $p = \lambda(z + \phi - \theta)$ *and* $r = \mu(z + \phi - \theta)$.

PROOF: Set $\alpha = 1$ and $\beta = -1$ into (3) and (4). ■

3. PERCENTILES

In this section, we provide tabulations of percentage points z_p associated with the c.d.f. of $Z = \alpha X + \beta Y$. These values are obtained by numerically solving the equation $F(z_p) = p$, where F is given by one of the eight formulas in the previous section. The computation involved is elementary and was performed using the algebraic manipulation package MAPLE. Table 1 provides the numerical values of z_p for $\beta = -5, -4.9, \ldots, -0.1, 0.1, \ldots, 4.9, 5$, $\alpha = 1$, $\lambda = 1$, $\mu = 1$, $\theta = 0$, and $\phi = 0$. We hope that these values will be of use to the practitioners mentioned in Section 1.

TABLE 1. Percentage Points of Z

β	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
−5	1.269886	2.753305	4.79235	8.303155	11.75257	19.93882
−4.9	1.229933	2.662441	4.673899	8.064235	11.48831	19.09909
−4.8	1.237609	2.664257	4.653566	8.024714	11.31099	19.34929
−4.7	1.183976	2.588971	4.492772	7.711818	11.02441	18.75246
−4.6	1.159243	2.556934	4.42227	7.607005	10.81862	18.22511
−4.5	1.147941	2.489706	4.325202	7.494501	10.57776	17.80752
−4.4	1.118148	2.435873	4.221948	7.2787	10.33217	17.59266
−4.3	1.075941	2.417815	4.212771	7.243095	10.21812	17.26861
−4.2	1.052733	2.326086	4.046438	6.979804	9.864678	16.62053
−4.1	1.077138	2.322918	4.045482	6.933808	9.732728	16.30139
−4	1.044185	2.267041	3.92152	6.701114	9.521412	15.85187
−3.9	0.983324	2.163155	3.750619	6.389278	9.05702	15.33036
−3.8	0.9844203	2.154377	3.73957	6.350874	9.040072	15.28951
−3.7	0.9656131	2.101158	3.652353	6.241897	8.836837	14.71998
−3.6	0.9397277	2.048364	3.559668	6.07805	8.657727	14.51771
−3.5	0.9217346	2.002067	3.475911	5.94196	8.384877	14.06644

(continued)

Table 1. *continued*

β	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
−3.4	0.9141674	1.990318	3.401986	5.793251	8.130967	13.66567
−3.3	0.8958373	1.942612	3.347019	5.637172	7.928744	13.27655
−3.2	0.8620599	1.883366	3.224607	5.485855	7.65268	12.64174
−3.1	0.8673236	1.843531	3.155617	5.34215	7.5823	12.62730
−3	0.8252196	1.775179	3.058439	5.155332	7.207957	11.93970
−2.9	0.8057673	1.73025	2.960222	4.995596	7.021107	11.75484
−2.8	0.7690012	1.681772	2.879660	4.864316	6.779974	11.39554
−2.7	0.725814	1.60682	2.761501	4.674416	6.567154	10.88609
−2.6	0.7243948	1.581168	2.717280	4.566563	6.39454	10.51924
−2.5	0.7098993	1.553990	2.627271	4.405046	6.16451	10.17800
−2.4	0.7039351	1.519689	2.580577	4.292945	5.974966	9.842581
−2.3	0.6973804	1.473137	2.505532	4.154942	5.767642	9.503593
−2.2	0.6485958	1.395287	2.373490	3.946084	5.51701	9.041901
−2.1	0.63266	1.376890	2.339474	3.873955	5.370945	8.830781
−2	0.6315433	1.330765	2.245799	3.698529	5.139233	8.366212
−1.9	0.6074453	1.283457	2.156510	3.566811	4.903106	8.021856
−1.8	0.5683426	1.227039	2.08443	3.444135	4.756674	7.748426
−1.7	0.5466012	1.170776	2.009779	3.318534	4.559533	7.293159
−1.6	0.5347252	1.141689	1.92735	3.169967	4.367155	6.963851
−1.5	0.4944153	1.083331	1.844599	3.043119	4.177776	6.664472
−1.4	0.477608	1.032904	1.755128	2.880615	3.967981	6.285086
−1.3	0.4748285	1.004484	1.689483	2.756714	3.776963	6.013185
−1.2	0.4508426	0.9714642	1.621683	2.640383	3.60959	5.745885
−1.1	0.4213758	0.9088515	1.530096	2.508098	3.427001	5.420583
−1	0.4136206	0.8761135	1.474739	2.406019	3.2821	5.202133
−0.9	0.3912894	0.8362646	1.404177	2.303931	3.167071	4.974678
−0.8	0.3688268	0.7859067	1.326264	2.153312	2.929195	4.641028
−0.7	0.3334323	0.733099	1.251433	2.057209	2.823428	4.522365
−0.6	0.3278965	0.706174	1.183262	1.954140	2.69384	4.313836
−0.5	0.3090532	0.6613967	1.108774	1.846668	2.541977	4.181668
−0.4	0.2895606	0.6241342	1.071112	1.780455	2.472513	4.044924
−0.3	0.2633741	0.5740716	0.9884361	1.691044	2.390136	3.945381
−0.2	0.2430617	0.5386785	0.9469557	1.623032	2.307419	3.895618
−0.1	0.2303104	0.5216299	0.9246481	1.624099	2.318914	3.935529
0	0.2218475	0.5083896	0.913252	1.605813	2.296609	3.929624
0.1	0.2340491	0.5219954	0.9309548	1.629357	2.334492	3.94729
0.2	0.2493614	0.5472577	0.9491748	1.636378	2.337419	3.910305
0.3	0.2689103	0.5838827	0.9986977	1.700900	2.399532	3.958959
0.4	0.2802333	0.6168189	1.062014	1.782387	2.498430	4.093865
0.5	0.3094367	0.6635058	1.130146	1.874722	2.582250	4.203472
0.6	0.3288045	0.6982174	1.185336	1.955054	2.677204	4.330453
0.7	0.3402623	0.7425279	1.256971	2.054224	2.817570	4.489042
0.8	0.3738329	0.7918859	1.323256	2.165221	2.956779	4.698249

(continued)

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TABLE 1. *continued*

β	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.9	0.3940101	0.8353776	1.399344	2.272601	3.102139	4.905491
1	0.4074398	0.8683306	1.463369	2.416161	3.260905	5.201786
1.1	0.4356899	0.9191873	1.533932	2.520633	3.443708	5.461008
1.2	0.4467717	0.9586074	1.614639	2.643252	3.593918	5.732368
1.3	0.4744901	1.013793	1.705164	2.778661	3.797729	6.08622
1.4	0.4862727	1.040886	1.77036	2.904615	3.990043	6.356266
1.5	0.5091952	1.089889	1.839806	3.043095	4.165568	6.684905
1.6	0.5328304	1.136831	1.925434	3.172319	4.34976	6.99339
1.7	0.5485429	1.169568	1.996928	3.280536	4.535883	7.254919
1.8	0.5719334	1.224448	2.069396	3.4072	4.737882	7.631846
1.9	0.5981902	1.277843	2.153720	3.55993	4.933245	7.975023
2	0.6067255	1.318648	2.228955	3.722612	5.17945	8.53912
2.1	0.6346113	1.365696	2.316015	3.844028	5.345759	8.693022
2.2	0.675663	1.427044	2.411504	3.994706	5.560375	8.97436
2.3	0.6679039	1.457223	2.463158	4.154700	5.768735	9.4077
2.4	0.7052633	1.501751	2.563315	4.268658	5.957857	9.796203
2.5	0.7075419	1.557552	2.650183	4.407963	6.121578	10.15914
2.6	0.7451197	1.599973	2.713357	4.564269	6.382591	10.48015
2.7	0.7683837	1.653628	2.793558	4.690857	6.607219	11.03595
2.8	0.7761896	1.679037	2.881245	4.837329	6.749977	11.44059
2.9	0.7748824	1.712727	2.954184	4.986693	6.986463	11.65935
3	0.842106	1.800565	3.075596	5.145849	7.22117	12.02564
3.1	0.842067	1.826327	3.136586	5.346059	7.55189	12.5835
3.2	0.8499198	1.868401	3.250464	5.496027	7.775593	12.84885
3.3	0.8692725	1.892740	3.287322	5.597908	7.87378	13.28508
3.4	0.909583	1.977510	3.402457	5.801697	8.151449	13.48979
3.5	0.9374156	2.017488	3.463582	5.917772	8.30475	14.01053
3.6	0.9588946	2.059392	3.553161	6.105147	8.613357	14.37841
3.7	0.9453604	2.099449	3.619577	6.210849	8.74757	14.57178
3.8	0.9695422	2.147075	3.747447	6.417495	9.086703	15.24841
3.9	1.007353	2.184966	3.773424	6.471695	9.236661	15.39597
4	1.021035	2.246746	3.881664	6.612216	9.406744	15.79030
4.1	1.069859	2.328406	4.03965	6.908588	9.755943	16.36502
4.2	1.069072	2.344893	4.094675	7.035104	9.934385	16.63983
4.3	1.110764	2.404863	4.203883	7.210277	10.21215	16.99265
4.4	1.144612	2.469894	4.267335	7.338549	10.44586	17.69125
4.5	1.141143	2.510093	4.349608	7.54415	10.67173	18.12223
4.6	1.153421	2.543847	4.4377	7.6387	10.87976	18.48087
4.7	1.189554	2.605208	4.530033	7.801218	11.08637	18.63365
4.8	1.221972	2.663384	4.648657	7.9811	11.33371	19.28335
4.9	1.244603	2.72094	4.787079	8.1891	11.58105	19.37992
5	1.24795	2.723921	4.769694	8.270612	11.76398	19.78177

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Similar tabulations could be easily derived for other values of α, β, λ, μ, θ, and ϕ. Sample programs are shown in the Appendix.

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APPENDIX

The following programs in MAPLE can be used to generate tables similar to that presented in Section 3.

```
#this program gives percentiles when beta > 0
tt:=(mu/2)*exp(-mu*abs(y-phi)):
tt1:=(1/2)*exp(lambda*((z-beta*y)/alpha-theta)):
tt2:=1-(1/2)*exp(-lambda*((z-beta*y)/alpha-theta)):
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ff:=int(tt*tt1,y=(z-alpha*theta)/beta..infinity):
ff:=ff+int(tt*tt2,y=-infinity..(z-alpha*theta)/beta):
p1:=fsolve(ff=0.6,z=-10000..10000):
p2:=fsolve(ff=0.7,z=-10000..10000):
p3:=fsolve(ff=0.8,z=-10000..10000):
p4:=fsolve(ff=0.90,z=-10000..10000):
p5:=fsolve(ff=0.95,z=-10000..10000):
p6:=fsolve(ff=0.99,z=-10000..10000):
print(beta,p1,p2,p3,p4,p5,p6);

#this program gives percentiles when beta < 0
tt:=(mu/2)*exp(-mu*abs(y-phi)):
tt1:=(1/2)*exp(lambda*((z-beta*y)/alpha-theta)):
tt2:=1-(1/2)*exp(-lambda*((z-beta*y)/alpha-theta)):
ff:=int(tt*tt2,y=(z-alpha*theta)/beta..infinity):
ff:=ff+int(tt*tt1,y=-infinity..(z-alpha*theta)/beta):
p1:=fsolve(ff=0.6,z=-10000..10000):
p2:=fsolve(ff=0.7,z=-10000..10000):
p3:=fsolve(ff=0.8,z=-10000..10000):
p4:=fsolve(ff=0.90,z=-10000..10000):
p5:=fsolve(ff=0.95,z=-10000..10000):
p6:=fsolve(ff=0.99,z=-10000..10000):
print(beta,p1,p2,p3,p4,p5,p6);
```

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