Canad. Math. Bull. Vol. 23 (1), 1980

ON SOME GENERALIZATION OF INEQUALITIES OF OPIAL, YANG AND SHUM

BY
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1. **Introduction.** In 1960, Z. Opial [20] proved the following interesting integral inequality:

THEOREM A. *If u is a continuously differentiable function on $[0, b]$, and if $u(0)=u(b)=0$, and $u(x)>0$ for $x \in (0, b)$, then*

$$\int_0^b |u(x)u'(x)|\, dx \le \frac{b}{4}\int_0^b [u'(x)]^2\, dx \tag{1}$$

where the constant $b/4$ is the best possible.

Equality holds in (1) if and only if

$$u(x)=cx, \quad \text{for} \quad 0 \le x \le \frac{b}{2}$$

and

$$u(x)=c(b-x), \quad \text{for} \quad \frac{b}{2} \le x \le b,$$

where c is a constant.

At the same time, C. Olech [19] showed that (1) is valid for any function $u(x)$ which is absolutely continuous on $[0, b]$, and satisfies the boundary conditions $u(0)=u(b)=0$.

C. Olech also pointed out that in order to prove (1) is suffices to prove the following:

THEOREM B *If u is an absolutely continuous function on $[0, b]$, and if $u(0)=0$, then*

$$\int_0^b |u(x)|\,|u'(x)|\, dx \le \frac{b}{2}\int_0^b [u'(x)]^2\, dx \tag{2}$$

where $b/2$ is the best possible constant.

Received by the editors January 16, 1978 and, in revised form, January 19, 1979.

In 1962, P. R. Beesack, [3], gave a different proof of Opial's inequality, and showed that (2) is contained in the following:

THEOREM C *If u is an absolutely continuous function on $[0, b]$, and if $u(0)=0$, then*

$$\int_0^b |u(x)|\,|u'(x)|\,dx+\frac{b}{2}\int_0^b \frac{1}{x^2}\{2\int_0^x |u(t)|\,|u'(t)|\,dt-[u(x)]^2\}\,dx \tag{3}$$
$$\leq \frac{b}{2}\int_0^b [u'(x)]^2\,dx.$$

Equality in (3) *holds if and only if $u(x)=cx$, c being a constant. Since*

$$g_1(x)=2\int_0^x |u(t)|\,|u'(t)|\,dt-[u(x)]^2\geq 0.$$

(3) *gave an improvement of* (2).

In 1966, G. S. Yang, [27], proved the following theorems which are the generalization of Z. Opial's inequality and some extensions of P. R. Beesack's:

THEOREM D-1 ([27]; Theorem 3). *Let $l(x)$ be positive on $a\leq x\leq X$ with $\int_a^X l^{-1}(x)\,dx<\infty$, and let $s(x)$ be bounded, positive and non-increasing on $a\leq x\leq X$; $u(x)$ be any function which is absolutely continuous on $a\leq x\leq X$ with $u(a)=0$. Then*

$$2\int_a^X s(x)\,|u(x)|\,|u'(x)|\,dx\leq \int_a^X l^{-1}(x)\,dx \int_a^X l(x)s(x)u'(x)^2\,dx. \tag{4}$$

There is equality only if $s=\text{constant}$, $u=A\int_a^x l^{-1}(t)\,dt$, A being a constant.

THEOREM D-2 ([27]; Theorem 3′). *Let $l(x)$ be positive on $X\leq x\leq b$ with $\int_X^b l^{-1}(x)\,dx<\infty$, and let $s(x)$ be bounded, positive and non-decreasing on $X\leq x\leq b$; $u(x)$ be any function which is absolutely continuous on $X\leq x\leq b$ with $u(b)=0$. Then*

$$2\int_X^b s(x)\,|u(x)|\,|u'(x)|\,dx\leq \int_X^b l^{-1}(x)\,dx \int_X^b l(x)s(x)u'(x)^2\,dx. \tag{5}$$

Equality in (5) *holds only if $s=constant$, $u=B\int_x^b l^{-1}(t)\,dt$ with constant B.*

THEOREM E ([27]; Theorem 6). *If $u(x)$ is absolutely continuous on $a\leq x\leq b$ with $u(a)=u(b)=0$, then*

$$\int_a^b |u|^p\,|u'|^q\,dx\leq \frac{q}{p+q}\left(\frac{b-a}{2}\right)^p \int_a^b |u'|^{p+q}\,dx, \qquad p, q\geq 1. \tag{6}$$

In a recent paper [24], D. T. Shum gave a generalization of Theorem C as following:

Theorem F ([24]; Theorem 4). *Let u be an absolutely continuous function on $[a, b]$, and $u(a)=0$. If $p>0$, and $\int_a^b |u'|^{p+1}\,dx<\infty$, then*

$$\int_a^b |u|^p\,|u'|\,dx+\frac{p(b-a)^p}{p+1}\int_a^b \frac{g_2(x)}{(x-a)^{p+1}}\,dx\le\frac{(b-a)^p}{p+1}\int_a^b |u'|^{p+1}\,dx \tag{7}$$

where

$$g_2(x)=(p+1)\int_a^x |u|^p\,|u'|\,dt-|u(x)|^{p+1}\,(\ge 0)(a\le x\le b).$$

If either $p<-1$ and both $\int_a^b |u|^p\,|u'|\,dx<\infty$ and $\int_a^b |u'|^{p+1}\,dx<\infty$; or $-1<p<0$ and $\int_a^b |u|^p\,|u'|\,dx<\infty$, the reverse inequality holds.

For $p>0$, equality holds in (7) *if and only if $u(x)=c(x-a)$, for some constant c; for $p<-1$, equality never holds; for $-1<p<0$, equality holds if and only if $u(x)=c(x-a)$, for some constant $c\neq 0$.*

The method of the proof of (7) (Theorem F) was first used by Benson [6], and was modified by Shum [23; 24].

D. T. Shum [24] also stated that the inequality (7) with $p>0$, can further be generalized as following:

Theorem G ([24]; (15)). *With the conditions of Theorem* F (7), *if s is positive and non-increasing on (a, b) with $-\infty<a<b<\infty$ and $\int_a^x s\,|u'|^{p+1}\,dt<\infty$, then*

$$\begin{aligned}\frac{(p+1)}{(b-a)^p}\int_a^b s\,|u|^p\,|u'|\,dx+p\int_a^b \frac{s(x)}{(x-a)^{p+1}}&\\ \times\left\{\frac{p+1}{s(x)}\int_a^x s\,|u|^p\,|u'|\,dt-|u(x)|^{p+1}\right\}dx&\le\int_a^b s\,|u'|^{p+1}\,dx.\end{aligned} \tag{8}$$

The well-known inequality of Z. Opial had led to numerous articles (see, e.g. [3]–[10], [12]–[21] and [24]–[27]). The object of this paper is to give a generalization of Theorem D-1, Theorem D-2, Theorem E, Theorem F and Theorem G in Section 3.

We also note that, in [25], Shum had combined Benson's method ([6]) with that of Bessack's ([2]–[5]) to get some inequalities, which also gave a further generalization of Opial's inequality.

2. **Preliminary lemmas**

Lemma 1. *Let $l(t)$ be positive on $a\le t\le x$ with $\int_a^x l^{-q}(t)\,dt<\infty$. If u is absolutely continuous on $[a, b]$ with $u(a)=0$, and if s is positive and non-increasing*

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on $[a, b]$, *then, for all* $a \le x \le b$, $p > -q$, $q \ge 1$,

$$(9)\quad g_3(x) = \left(\frac{p+q}{q}\right)^q \left(\int_a^x l^{-q}(t)\,dt\right)^{q-1} \cdot \left(\int_a^x l^{q(q-1)}(t)\,|u(t)|^p\,s(t)^q\,|u'(t)|^q\,dt\right) - |u(x)|^{p+q}\,s(x)^q \ge 0.$$

Equality holds in (9) *if and only if* u' *does not change sign on* $[a, b]$, $s = $ *constant and* $u(x) = k(\int_a^x l^{-q}(t)\,dt)^{q/(p+q)}$ *with constant* k.

Proof. Since

$$\left(\frac{q}{p+q}\right)|u(x)|^{(p+q)/q} = \left|\int_a^x u^{p/q}u'\,dt\right| \le \int_a^x |u|^{p/q}\,|u'|\,dt.$$

By Hölder's inequality ([12], p. 81; or [22], p. 113). We have

$$\left(\frac{q}{p+q}\right)|u(x)|^{(p+q)/q} \le \left(\int_a^x l(t)^{-q}\,dt\right)^{(q-1)/q}\left(\int_a^x l(t)^{q(q-1)}\,|u(t)|^p\,|u'(t)|^q\,dt\right)^{1/q}.$$

Now, with the fact that s is positive and non-increasing on $[a, b]$, we obtain

$$\left(\frac{q}{p+q}\right)s(x)\,|u(x)|^{(p+q)/q} \le \left(\int_a^x l(t)^{-q}\,dt\right)^{(q-1)/q}\left(\int_a^x l(t)^{q(q-1)}s(t)^q\,|u(t)|^p\,|u'(t)|^q\,dt\right)^{1/q}$$

that is

$$\left(\frac{q}{p+q}\right)^q s(x)^q\,|u(x)|^{p+q} \le \left(\int_a^x l(t)^{-q}\,dt\right)^{q-1}\left(\int_a^x l(t)^{q(q-1)}s(t)^q\,|u(t)|^p\,|u'(t)|^q\,dt\right)$$

which proves our result.

Before giving to second lemma, we first state two elementary algebraic inequalities [1; 11; 12; or 18, p. 30]. If $q > 0$, then

$$(10)\qquad qs^{p+q} + pt^{p+q} - (p+q)s^q t^p \ge 0, \quad \text{for all} \quad p > 0; \quad \text{or} \quad p < -q.$$

$$(11)\qquad qs^{p+q} + pt^{p+q} - (p+q)s^q t^p \le 0, \quad \text{for all} \quad -q < p < 0.$$

Here, s and t are nonnegative (positive if $p < -q$), and in both cases strict inequality holds unless $s = t$. (We also note that when $p = 0$ or $p = -q$, the left sides of both (10) and (11) become identically zero for all s and t).

LEMMA 2. *Let* $v(x)$ *be absolutely continuous on* $[\alpha, \beta]$ *with* $v'(x) \ge 0$, *a.e. Also, suppose that* $Q(x)$ *is nonnegative a.e. and measurable on* $[\alpha, \beta]$, *and* $G(v, x)$ *is continuously differentiable for* $x \in [\alpha, \beta]$ *and* v *in the range of the* *function $v(x)$ with $G_v(v, x) \geq 0$.* (or $G_v(v, x) > 0$ in case $p < 0$), then, if the integrals exist and $q > 0$, we have*

$$(12)\quad \int_\alpha^\beta [q(v')^{(p+q)/q}Q + p(G_v)^{(p+q)/p}Q^{-q/p} + (p+q)G_x]\,dx \geq (p+q)\{G(v(\beta), \beta) - G(v(\alpha), \alpha)\}, \qquad (p>0; \text{ or } p<-q).$$

$$(13)\quad \int_\alpha^\beta [q(v')^{(p+q)/q}Q + p(G_v)^{(p+q)/p}Q^{-q/p} + (p+q)G_x]\,dx \leq (p+q)\{G(v(\beta), \beta) - G(v(\alpha), \alpha)\}, \qquad (-q<p<0),$$

where

$$G_v = \frac{\partial}{\partial v}(G(v, x)), \qquad G_x = \frac{\partial}{\partial x}(G(v, x)).$$

Equality in both (12) *and* (13) *holds if and only if the differential equation*

$$(14)\qquad v' = (G_v/Q)^{q/p}$$

is satisfied almost everywhere.

Proof. By taking $s = (v')^{1/q}Q^{1/(p+q)}$, $t = (G_v)^{1/p}Q^{-q/[p(p+q)]}$ in (10), we have almost everywhere,

$$q(v')^{(p+q)/q}Q + p(G_v)^{(p+q)/p}Q^{-q/p} \geq (p+q)v'G_v.$$

This implies that

$$q(v')^{(p+q)/q}Q + p(G_v)^{(p+q)/p}Q^{-q/p} + (p+q)G_x \geq (p+q)\left(\frac{d}{dx}(G(v, x))\right).$$

Which proves (12) by integrating both sides of the above inequality from α to β.

The proof of (13) follows from the above argument, by using (11) instead of (10).

The proof of (14) follows at once from the fact that $s = t$.

3. **Main results.**

THEOREM 1-1. *Let $l(x)$ be positive on $a \leq x \leq X$ with $\int_a^X l(x)^{-q}\,dx < \infty$, and let $s(x)$ be positive and nonincreasing on $a \leq x \leq X$; $u(x)$ be any function which is absolutely continuous on $a \leq x \leq X$ with $u(a) = 0$. Then, for all $p \geq 0$, $q \geq 1$.*

$$(15)\quad (p+q)\int_a^X l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dx \leq q\left(\int_a^X l^{-q}\,dx\right)^p\left(\int_a^X l^{q(p+q-1)}s^q\,|u'|^{p+q}\,dx\right).$$

* (We note that in case $p+q=2n$ with n a positive integer, the restriction $v'(x) \geq 0$, a.e. and $G_v(v, x) \geq 0$ may be removed).

There is equality if and only if u' does not change sign on $[a, X]$, s is a positive constant function, $q=1$ and $u(x)=A\int_a^x l(t)^{-q}\,dt$ with constant A.

Proof. Define $z(x)=\int_a^x l(t)^{q(q-1)}s(t)^{q^2/(p+q)}|u'(t)|^q\,dt$, $a\le x\le X$, then $z'(x)=l(x)^{q(q-1)}s(x)^{q^2/(p+q)}|u'(x)|^q$. Since

$$|u(x)|\le\left(\int_a^x l(t)^{-q}\,dt\right)^{(q-1)/q}\left(\int_a^x l(t)^{q(q-1)}|u'(t)|^q\,dt\right)^{1/q}$$

and $s(x)$ is positive and non-increasing on $a\le x\le X$, we have

$$s(x)^{(p\cdot q)/(p+q)}|u(x)|^p\le\left(\int_a^X l(x)^{-q}\,dx\right)^{[p(q-1)]/q} z^{p/q},\qquad a\le x\le X.$$

This implies that

$$\begin{aligned}(p+q)&\int_a^X l^{q(q-1)}s^q|u|^p|u'|^q\,dx\\ &\le(p+q)\left(\int_a^X l^{-q}\,dx\right)^{[p(q-1)]/q}\int_a^X z^{p/q}z'\,dx\\ &=q\left(\int_a^X l^{-q}\,dx\right)^{[p(q-1)]/q} z(X)^{(p+q)/q}\\ &=q\left(\int_a^X l^{-q}\,dx\right)^{[p(q-1)]/q}\left(\int_a^X l^{q(q-1)}s^{q^2/(p+q)}|u'|^q\,dx\right)^{(p+q)/q}.\end{aligned}$$

By Hölder's inequality, the result (15) follows immediately.

We note that (4), (Theorem D-1), is a special case of (15), by taking $p=q=1$. We also note that Theorem 1-1 give an extension of Beesack's Theorem ([3]).

THEOREM 1-2. *Let $l(x)$ be positive on $X\le x\le b$ with $\int_X^b l(x)^{-q}\,dx<\infty$, and let $s(x)$ be positive and non-decreasing on $X\le x\le b$; $u(x)$ be any function which is absolutely continuous on $X\le x\le b$ with $u(b)=0$. Then, for all $p\ge 0$, $q\ge 1$.*

$$\begin{aligned}(16)\quad (p+q)&\int_X^b l(x)^{q(q-1)}s(x)^q|u(x)|^p|u'(x)|^q\,dx\\ &\le q\left(\int_X^b l(x)^{-q}\,dx\right)^p\left(\int_X^b l(x)^{q(p+q-1)}s(x)^q|u'(x)|^{p+q}\,dx\right).\end{aligned}$$

There is equality if and only if u' does not change sign on $[X, b]$, s is a positive constant function, $q=1$ and $u(x)=B\int_x^b l(t)^{-q}\,dt$ with constant B.

Proof. Define $z(x)=-\int_x^b l(t)^{q(q-1)}s(t)^{q^2/(p+q)}|u'(t)|^q\,dt$, $X\le x\le b$, then $z'(x)=l(x)^{q(q-1)}s(x)^{q^2/(p+q)}|u'(x)|^q$, for all $X\le x\le b$. Since

$$|u(x)|\le\left(\int_x^b l(t)^{-q}\,dt\right)^{(q-1)/q}\left(\int_x^b l(t)^{q(q-1)}|u'|^q\,dt\right)^{1/q}$$

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and $s(x)$ is positive and non-decreasing on $X \le x \le b$; we have

$$s(x)^{pq/(p+q)}\,|u(x)|^p \le \left(\int_X^b l(x)^{-q}\,dx\right)^{[p(q-1)]/q}(-z(x))^{p/q}, \qquad X \le x \le b.$$

This implies that

$$\begin{aligned}(p+q)&\int_X^b l(x)^{q(q-1)}s(x)^q\,|u(x)|^p\,|u'(x)|^q\,dx\\ &\le (p+q)\left(\int_X^b l(x)^{-q}\,dx\right)^{[p(q-1)]/q}\left(\int_X^b (-z)^{p/q}z'\,dx\right)\\ &= q\left(\int_X^b l^{-q}\,dx\right)^{[p(q-1)]/q}\left(\int_X^b l^{q(q-1)}s^{q^2/(p+q)}\,|u'|^q\,dt\right)^{(p+q)/q}.\end{aligned}$$

By Hölder's inequality, we get (16) immediately.

We note that (5), (Theorem D-2), is a special case of (16), by taking $p=q=1$. We also note that Theorem 1-2 gives a generalization of Beesack's Theorem ([3]).

THEOREM 2. *Let $l(x)$ be positive on $a \le x \le b$ with $\int_a^b l(x)^{-q}\,dx < \infty$. Let $K = (\int_X^b l(x)^{-q}\,dx)^p = (\int_a^X l(x)^{-q}\,dt)^p$ for some $X \in [a, b]$. If u is an absolutely continuous function on $[a, b]$ with $u(a) = u(b) = 0$, and if $s(x)$ is a positive and non-increasing on $[a, X]$ and nondecreasing on $[X, b]$, then, for all $p \ge 0$, $q \ge 1$,*

$$(p+q)\int_a^b l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dx \le qK\int_a^b l^{q(p+q-1)}s^q\,|u'|^{p+q}\,dx. \tag{17}$$

Equality holds in (17) if and only if u' does not change sign on $[a, X]$ and on $[X, b]$, respectively, s is a positive constant function, $q=1$ and $u = A\int_a^x l(t)^{-q}\,dt$ if $x \in [a, X]$, and $u = B\int_x^b l(t)^{-q}\,dt$ if $x \in [X, b]$. A, B being a constant.

Proof. If we take $X \in [a, b]$ such that $K = (\int_X^b l^{-q}\,dx)^p = (\int_a^X l^{-q}\,dx)^p$ in (15) and (16) of Theorem 1-1 and Theorem 1-2, respectively, (17) follows immediately.

We note that (6), (Theorem E) is a special case of (17), by taking $X = (a+b)/2$, $p \ge 1$, $q \ge 1$, and s and l be a positive constant function.

THEOREM 3. *Let $l(x)$ be positive on $a \leqslant x \leqslant b$ with $\int_a^b l(x)^{-q}\,dx < \infty$. If u is an absolutely continuous function on $[a, b]$ with $u(a) = 0$, and if $s(x)$ is a positive and non-increasing on $[a, b]$, and $\int_a^b l^{q(p+q-1)}s^q\,|u'|^{p+q}\,dt < \infty$, then, for all $p > 0$, $q \ge 1$,*

$$\begin{aligned}&q^{1-p}(p+q)^{[(p+q)(q-1)]/q}\int_a^b l^{q(p+q-1)}s^q\,|u'|^{p+q}\,dx\\ &\qquad \ge \frac{(p+q)^q}{\left(\int_a^b l^{-q}\,dx\right)^p}\int_a^b l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dx + pq^q\int_a^b \frac{l(x)^{-q}g_3(x)}{\left(\int_a^x l^{-q}\,dt\right)^{p+q}}\,dx.\end{aligned} \tag{18}$$

where $g_3(x)$ *is defined in* (9) (*Lemma* 1). *If* $-q<p<0$, $q\geq 1$, *and* $\int_a^b l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dx<\infty$, *the reverse inequality holds.*

For $p>0$, *equality holds in* (18) *if and only if* $s=$ *constant*, $u(x)=c(\int_a^x l^{-q}\,dt)^{q/(p+q)(q-1)/q}$ *for some constant c; for* $-q<p<0$, *equality holds if and only if* $s=$ *constant*, $u(x)=c(\int_a^x l^{-q}\,dt)^{q/(p+q)(q-1)/q}$ *for some constant* $c\neq 0$.

Proof. By (15), Theorem 1-1, we have, for $p>0$, $q\geq 1$,

$$\text{(19)}\qquad \int_a^b l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dx\leq\frac{q\left(\int_a^b l^{-q}\,dx\right)^p}{p+q}\int_a^b l^{q(p+q-1)}\,|u'|^{p+q}\,s^q\,dx<\infty.$$

Now, from (19) with b replaced by x, it follows that

$$\text{(20)}\qquad \lim_{x\to a^+}\frac{p+q}{\left(\int_a^x l^{-q}\,dt\right)^p}\int_a^x l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dt=0,\qquad (p>0,q\geq 1).$$

Now, let $v(x)=\int_a^x l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dt$, $Q^{-1}=|u|^{[p(p+q)]/q}\,s^p(p+q)^{-[p(q-1)]/q}q^pl^{-p}$, and $G=v(\int_a^x l^{-q}\,dt)^{-p}$. Then from (12) with $[\alpha,\beta]$ replaced by $[\alpha,b]$, we obtain, for $p>0$, $q\geq 1$, and $a<\alpha<b$,

$$\text{(21)}\qquad \begin{aligned} &q^{1-p}(p+q)^{[p(q-1)]/q}\int_\alpha^b l^{q(p+q-1)}s^q\,|u'|^{p+q}\,dx\\ &\qquad+\frac{p+q}{\left(\int_a^\alpha l^{-q}\,dx\right)^p}\int_a^\alpha l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dx\\ &\geq\frac{p+q}{\left(\int_a^b l^{-q}\,dx\right)^p}\int_a^b l^{q(q-1)}s^q\,|u|^p\,|u'|^q\,dx+\frac{pq^q}{(p+q)^{q-1}}\int_\alpha^b\frac{l(x)^{-q}g_3(x)}{\left(\int_a^x l^{-q}\,dt\right)^{p+q}}\,dx.\end{aligned}$$

Now, on taking limits as $\alpha\to a^+$ on both sides of the above inequality, one obtains (18) by using (20). On noting that $g_3(x)$ is nonnegative by Lemma 1, so that both integrals on the right side of (18) exist (finite).

The proof of the case $-q<p<0$ is essentially the same as above except that of (12) we now use (13).

The proof of the equality condition begins by employing (14) in (21) and are similar to those used in [23, 24].

We note that by taking $q=1$ and s and l be a positive constant function, parts of Theorem F is a special case of Theorem 3. We also note that if $q=1$, $p>0$, and l is a positive constant function, we may deduce Theorem G.

THEOREM 4. *Let* $l(x)$ *be positive on* $a\leq x\leq b$ *with* $\int_a^b l^{-q}\,dx<\infty$. *If u is an absolutely continuous function on* $[a,b]$ *with* $u(a)=0$, *and if s is a positive and*

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non-increasing on $[a, b]$, *and both* $\int_a^b l^{q(q-1)} s^q |u|^p |u'|^q\,dx < \infty$ *and* $\int_a^b l^{q(p+q-1)} s^q |u'|^{p+q}\,dx < \infty$. *Then, for all* $p < -q$, $q \geq 1$.

$$\int_a^b q l^{q(p+q-1)} s^q |u'|^{p+q}\,dx \geq \frac{(p+q)}{\left(\int_a^b l^{-q}\,dt\right)^p} \int_a^b l^{q(q-1)} s^q |u|^p |u'|^q\,dx + p\int_a^b \frac{l(x)^{-q} h_3(x)}{\left(\int_a^x l^{-q}\,dt\right)^{p+q}}\,dx. \tag{22}$$

where

$$h_3(x) = (p+q)\left(\int_a^x l^{-q}\,dt\right)^{q-1} \int_a^x l^{q(q-1)} s^q |u|^p |u'|^q\,dt - s(x)^q |u(x)|^{p+q}\ (\leq 0).$$

The equality condition never holds.

Proof. Let $v(x) = \int_a^x l^{q(q-1)} s^q |u|^p |u'|^q\,dt$, $Q^{-1} = |u|^{[p(p+q)]/q} s^p l^{-p}$, and $G = v(\int_a^x l^{-q}\,dt)^{-p}$, then, from (12) with $[\alpha, \beta]$ replaced by $[\alpha, b]$, we obtain

$$\begin{aligned}&\frac{-(p+q)}{\left(\int_a^\alpha l^{-q}\,dx\right)^p} \int_a^\alpha l^{q(q-1)} s^q |u|^p |u'|^q\,dx + p\int_\alpha^b \frac{l(x)^{-q} h_3(x)}{\left(\int_a^x l^{-q}\,dt\right)^{p+q}}\,dx \\ &\leq \frac{-(p+q)}{\left(\int_a^b l^{-q}\,dx\right)^p} \int_a^b l^{q(q-1)} s^q |u|^p |u'|^q\,dx + q\int_\alpha^b l^{q(p+q-1)} s^q |u'|^{p+q}\,dx.\end{aligned} \tag{23}$$

Now, since $p < -q$, $q \geq 1$, it follows, from the definition of $h_3(x)$, that $ph_3 \geq 0$, hence both limits on the left side of (23) exist as $\alpha \to a^+$, and the first limit is zero, since $p < 0$, which proves the case $p < -q$ without the equality condition. Thus, we get the result (22).

Now, by combining Theorem 3 and Theorem 4, we may deduce Theorem F on setting $q = 1$, and s and l be a positive constant function on $[a, b]$.

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