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HITTING TIME DISTRIBUTIONS FOR BIRTH–DEATH PROCESSES WITH BILATERAL ABSORBING BOUNDARIES

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For the birth–death process on a finite state space with bilateral boundaries, we give a simpler derivation of the hitting time distributions by h-transform and φ-transform. These transforms can then be used to construct a quick derivation of the hitting time distributions of the minimal birth–death process on a denumerable state space with exit/regular boundaries.

Keyword: stochastic modelling

1. INTRODUCTION

In this paper, we study the hitting time distributions for irreducible continuous time birth–death processes having state space $\{-1, 0, 1, 2, \ldots, N\}$, where both -1 and N are absorbing states. Specifically, we will firstly give by h-transform and φ-transform a simple and interesting proof for the Laplace–Stieltjes transforms of hitting times derived for $N < \infty$ in Gong, Mao, and Zhang [10]. Using these results and the same method, we also derive analogous expressions for the Laplace–Stieltjes transforms of the hitting times when $N = \infty$. These hitting times are associated with the minimal birth–death process with exit or regular boundary.

Very recently, van Doorn [7] studied the hitting time distributions for the birth–death processes on $\mathbb{Z}_+$, by employing Karlin and McGregor's theory of orthogonal-polynomials. (We will remark on the difference between [7] and our paper at the end of this section.)

Consider the continuous time birth–death process on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with q-matrix

$$Q = \begin{pmatrix} -(a_0+b_0) & b_0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & -(a_1+b_1) & b_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} & -(a_{N-1}+b_{N-1}) & b_{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $a_0 \geq 0$, $a_i > 0\,(i \geq 1)$, $b_i > 0\,(i \geq 0)$. If $a_0 = 0$, then 0 is a reflecting state; and if $a_0 > 0$, then -1 can be seen as an absorbing state. Define

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i} \quad (i \geq 1). \tag{1.1}$$

Then (μ_i) is the unique (up to a factor) invariant measure for Q.

Let $T_{i,j}$ be the hitting time of state j starting from state i. Let $Q^{(N)}$ be the submatrix of Q, which serves as a generator of the birth–death process before hitting state N. Namely,

$$Q^{(N)} = \begin{pmatrix} -(a_0+b_0) & b_0 & 0 & \cdots & 0 & 0 \\ a_1 & -(a_1+b_1) & b_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} & -(a_{N-1}+b_{N-1}) \end{pmatrix}. \tag{1.2}$$

Denote by $(0 <)\lambda_1^{(N)} < \cdots < \lambda_{N-1}^{(N)} < \lambda_N^{(N)}$ all eigenvalues of $-Q^{(N)}$. A well-known theorem states that if $a_0 = 0$, the hitting time $T_{0,N}$ has the Laplace–Stieltjes transform

$$\mathbb{E}\, e^{-sT_{0,N}} = \prod_{\nu=1}^{N} \frac{\lambda_\nu^{(N)}}{s + \lambda_\nu^{(N)}}, \quad s \geq 0. \tag{1.3}$$

This theorem can be traced back to Karlin and McGregor [11] and Keilson [13], and it arose many further studies. Fill [9] gave the first probabilistic proof via duality. Diaconis and Miclo [4] presented another probabilistic proof of this theorem.

Gong et al. [10] studied the hitting time distributions of the continuous time birth–death processes on finite and denumerable state spaces when $a_0 \geq 0$. In [10], the case of $a_0 = 0$ was extended to the denumerable state space, and they claimed a similar result holds for the case of $a_0 > 0$ in denumerable space without proof. However, the claim is hurried since the method used in [10, Section 4] needs the eigentime identities. This can be overcome by using h-transform and φ-transform as we do in this paper. Particularly, for $a_0 = 0$, (1.3) was extended to the distributions for all $T_{i,N}$, by observing the independence between $T_{0,i}$ and $T_{i,N}$. That is,

$$\mathbb{E}\, e^{-sT_{i,N}} = \prod_{\nu=1}^{N} \frac{\lambda_\nu^{(N)}}{s + \lambda_\nu^{(N)}} \Big/ \prod_{\nu=1}^{i} \frac{\lambda_\nu^{(i)}}{s + \lambda_\nu^{(i)}}, \quad s \geq 0. \tag{1.4}$$

By (1.3) and (1.4), all the hitting time distributions for an ergodic and finite birth–death process can be deduced.

When $a_0 > 0$, an extra coefficient appears in the Laplace–Stieltjes transform of $T_{i,N}$. This is illustrated in Theorem 1.1, which is given below.

THEOREM 1.1: *Assume* $a_0 > 0$.

(i) *For* $0 \le i < N$*, the Laplace–Stieltjes transform of the hitting time* $T_{i,N}$ *is given by*

$$\mathbb{E}\, e^{-sT_{i,N}} = \frac{h_i}{h_N} \cdot \frac{\prod_{\nu=1}^{N} \lambda_\nu^{(N)}/(s+\lambda_\nu^{(N)})}{\prod_{\nu=1}^{i} \lambda_\nu^{(i)}/(s+\lambda_\nu^{(i)})}, \quad s > 0.$$

Here $\lambda_\nu^{(i)} (1 \le \nu < i)$ *are the eigenvalues of* $Q^{(i)}$ *given in (1.2), where* N *is replaced by* i*, and*

$$h_i = \sum_{j=0}^{i} \frac{1}{\mu_j a_j}, \quad i \ge 0. \tag{1.5}$$

(ii) *For* $0 \le i < N$*, let* $T_{i,-1}^{(N)}$ *be the hitting time of* -1 *for* $Q^{(N)}$*-process starting from* i*. Then the Laplace–Stieltjes transform of* $T_{i,-1}^{(N)}$ *is given by*

$$\mathbb{E}\, e^{-sT_{i,-1}^{(N)}} = \left(1 - \frac{h_i}{h_N}\right) \frac{\prod_{\nu=1}^{N} \lambda_\nu^{(N)}/(s+\lambda_\nu^{(N)})}{\prod_{\nu=1}^{N-1-i} \tilde{\lambda}_\nu^{(i)}/(s+\tilde{\lambda}_\nu^{(i)})}, \quad s > 0,$$

where $(0 <)\tilde{\lambda}_1^{(i)} < \tilde{\lambda}_2^{(i)} < \cdots < \tilde{\lambda}_{N-1-i}^{(i)}$ *are the eigenvalues of* $-\tilde{Q}^{(i)}$*, defined as*

$$\tilde{Q}^{(i)} = \begin{pmatrix} -(a_{i+1}+b_{i+1}) & b_{i+1} & 0 & \cdots & 0 & 0 \\ a_{i+2} & -(a_{i+1}+b_{i+1}) & b_{i+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1} & -(a_{N-1}+b_{N-1}) \end{pmatrix}.$$

Remark 1.2: (1) We remark that for $i = 0$ in (i) and $i = N-1$ in (ii), the denominators of the Laplace–Stieltjes transforms both equal to 1.

(2) Theorem 1.1 implies that for $0 \le i < N$,

$$\mathbb{P}[T_{i,N} < T_{i,-1}] = \frac{h_i}{h_N},$$

which explains why the coefficient does not appear when state 0 is a reflecting boundary.

Theorem 1.1 was proved in [10] by a modified duality method established by Fill in [9]. This duality method depends heavily on the associated spectral polynomials for the generator $Q^{(N)}$. Indeed, the link matrices used in [9,10] are related to the problem of the preserving of the Stieljes matrices. These matrices are of finite dimension. See [15] for more details. It is currently unclear to us if the method used in [9] or [15] can also be applied within an infinite-dimensional setting.

The main purpose of this paper is to give a new and simpler proof for Theorem 1.1, by h-transform and φ-transform. These transforms also help us obtain a quick proof for the hitting time distributions of the birth–death process on $\mathbb{Z}_+$ with bilateral absorbing boundaries.

Actually, h-transform and φ-transform are two kinds of Doob transforms. They can transfer a birth–death process with bilateral absorbing boundaries into two different birth–death processes only with one absorbing boundary. Then the distributions of the absorbing times of the initial process can be obtained from those of the latter processes. A similar spirit is used in two most recent papers [5,6] from Diaconis and Miclo, where they reduce the study of the convergence rate to quasi-stationarity of the absorbing Markov process to the convergence rate to equilibrium of related ergodic processes, via Doob transform.

To deal with the birth–death processes on $\mathbb{Z}_+$, we need to recall Feller's classification of ∞ boundary for Q. Let

$$R=\sum_{i=0}^{\infty}\frac{1}{\mu_i b_i}\sum_{j=0}^{i}\mu_j,\quad S=\sum_{i=0}^{\infty}\frac{1}{\mu_i b_i}\sum_{j=i+1}^{\infty}\mu_j.$$

By [1] or [8], there are four types of ∞ boundaries, which are exit ($R<\infty, S=\infty$), entrance ($R=\infty, S<\infty$), regular ($R<\infty, S<\infty$) and natural ($R=\infty=S$) boundaries. In this paper, we consider the exit and regular boundaries. Namely, we will assume

$$R<\infty. \tag{1.6}$$

This means that the corresponding Q-processes are not unique. Let $(X_t^i, t\ge 0)$ be the corresponding continuous-time Markov chain with $X_0=i$ and the minimal Q-function $P(t)=(p_{ij}(t): i,j\ge 0)$. That is,

$$p_{ij}(t)=\mathbb{P}[X_t^i=j,\ t<\zeta_i]$$

with $\zeta_i=\lim_{n\to\infty}\xi_n$, where ξ_n are the epochs of successive jumps:

$$\xi_0=0,\quad \xi_n=\inf\left\{t: t>\xi_{n-1},\ X_t^i\neq X_{\xi_{n-1}}^i\right\},\quad n\ge 1.$$

So ∞ can be seen as an absorbing state for the minimal process. See also [3]. It is known that $T_{i,\infty}:=\lim_{N\to\infty}T_{i,N}=\zeta_i$.

Denote by $L^2(\mathbb{Z}_+,\mu)$ the Hilbert space $\left\{f:\mathbb{Z}_+\to\mathbb{R} \text{ and } \sum_{i\ge 0}\mu_i f_i^2<\infty\right\}$ with norm $||f||_2=(\sum_{i\ge 0}\mu_i f_i^2)^{1/2}$. Let L be the generator of the minimal process in $L^2(\mathbb{Z}_+,\mu)$ with domain $\mathscr{D}(L)$, and let $\sigma_{\text{ess}}(L)$ be the essential spectrum of L. By [10, Section 4 and Subsection 7.1], we know $\sigma_{\text{ess}}(L)=\emptyset$. Thus denote the whole spectrum of $-L$ by $\{\lambda_\nu,\nu\ge 1\}$, where $\lambda_1<\lambda_2<\cdots$.

Now we can state our main results. Firstly, we will use the h-transform to derive the following expression for the Laplace–Stieltjes transform of $T_{i,\infty}$.

Theorem 1.3: *Assume $a_0>0$ and $R<\infty$. Then the absorbing time $T_{i,\infty}$ $(i\ge 0)$ has the Laplace–Stieltjes transform*

$$\mathbb{E}\,e^{-sT_{i,\infty}}=\frac{h_i}{h_\infty}\cdot\frac{\prod_{\nu=1}^{\infty}\lambda_\nu/(s+\lambda_\nu)}{\prod_{\nu=1}^{i}\lambda_\nu^{(i)}/(s+\lambda_\nu^{(i)})},\quad s>0,$$

where $h_\infty=\sum_{j=0}^{\infty}1/(\mu_j a_j)<\infty$. In particular, the eigentime identity holds:

$$\mathbb{E}\left[T_{i,\infty}\mathbb{1}_{\{T_i<\infty\}}\right]=\frac{h_i}{h_\infty}\sum_{j=0}^{\infty}\mu_j h_j^2\left(\frac{1}{h_{i\vee j}}-\frac{1}{h_\infty}\right).$$

For $i \geq 0$, let

$$\overline{Q}^{(i)} = \begin{pmatrix} a_{i+1} & -(a_{i+1}+b_{i+1}) & b_{i+1} & 0 & \cdots & \cdots \\ 0 & a_{i+2} & -(a_{i+2}+b_{i+2}) & b_{i+2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{1.7}$$

Since $R < \infty$, the corresponding $\overline{Q}^{(i)}$-processes are also not unique, and the minimal process $P^{(i)}(t)$ is the birth–death process on $\{i, i+1, \ldots\}$ before $P(t)$ reaching state i. Let $L^{(i)}$ be its generator in $L^2(\mathbb{Z}_+, \mu)$. Then the essential spectrum $\sigma_{\text{ess}}(L^{(i)}) = \emptyset$ since $\sigma_{\text{ess}}(L) = \emptyset$ and $Q - \overline{Q}^{(i)}$ is of finite rank: see [12, Theorem 5.35].

Denote by $\{\overline{\lambda}_\nu^{(i)}, \nu \geq 1\}$ all the positive eigenvalues of $-L^{(i)}$. We will use the φ-transform to derive the following expression for the Laplace–Stieltjes transform of $T_{i,-1}$.

THEOREM 1.4: *Assume $a_0 > 0$ and $R < \infty$. Then the absorbing time $T_{i,-1}$ $(i \geq 0)$ has the Laplace–Stieltjes transform*

$$\mathbb{E}\, e^{-sT_{i,-1}} = \frac{\varphi_i}{\varphi_{-1}} \cdot \frac{\prod_{\nu=1}^{\infty} \lambda_\nu/(s+\lambda_\nu)}{\prod_{\nu=1}^{\infty} \overline{\lambda}_\nu^{(i)}/(s+\overline{\lambda}_\nu^{(i)})}, \quad s > 0,$$

where

$$\varphi_i = h_\infty - h_i, \quad \varphi_{-1} = h_\infty. \tag{1.8}$$

In particular, the eigentime identity holds:

$$\mathbb{E}\left[T_{i,-1}\mathbb{1}_{\{T_{i,-1}<\infty\}}\right] = \frac{\varphi_i}{\varphi_{-1}} \sum_{j=0}^{\infty} \mu_j \varphi_j^2 \left(\frac{1}{\varphi_{i\wedge j}} - \frac{1}{\varphi_{-1}}\right).$$

Remark 1.5: For the minimal process under the assumption (1.6), the distributions of $T_{i,n}$ are all known. If $0 \leq i < n < \infty$, the distribution of $T_{i,n}$ is given by Theorem 1.1 (i), while if $0 \leq i < n = \infty$, it is given by Theorem 1.3. If $-1 \leq n < i < \infty$, the distribution is given by Theorem 1.4.

After this paper was completed, we discovered that some of our results are also obtained in [7]. By using the classical orthogonal polynomial approach of Karlin and McGregor [11], van Doorn [7] gives a new proof for the hitting time distributions, which have been obtained in [8] via the theory of Dirichlet form and approximation argument. The present thesis is a continuation of [8]. We emphasize the importance of h-transform and φ-transform – which are different from Karlin and McGregor's classical duality and Fill's duality – in dealing with the bilateral absorbing boundaries. Moreover, by using the approximation argument, we can handle both the infinity regular and infinity exit boundaries in the same way (that is $R < \infty$ here, or $C < \infty$ in the notation of [7]).

The rest of this paper is organized as follows. We will give the proof by h-transform and φ-transform for Theorem 1.1 in Section 2, and Theorems 1.3 and 1.4 are proved in Section 3.

2. FINITE STATE SPACE

We first prove Theorem 1.1 (i) by h-transform. Set

$$\widehat{Q} = \begin{pmatrix} Q^{(N)} & \mathbf{b} \\ \mathbf{0} & 0 \end{pmatrix},$$

with $\mathbf{b} = (0, \ldots, 0, b_{N-1})^{\mathrm{T}}$ and $\mathbf{0}$ the zero vector. For h_i given in (1.5), we define

$$H = \mathrm{diag}(h_0, h_1, \ldots, h_N), \quad Q^* = H^{-1}\widehat{Q}H. \tag{2.1}$$

That is,

$$Q^* = \begin{pmatrix} -(a_0+b_0) & \frac{h_1}{h_0}b_0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{h_0}{h_1}a_1 & -(a_1+b_1) & \frac{h_2}{h_1}b_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{h_{N-2}}{h_{N-1}}a_{N-1} & -(a_{N-1}+b_{N-1}) & \frac{h_N}{h_{N-1}}b_{N-1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}. \tag{2.2}$$

Then Q^* is a birth–death q-matrix with state N absorbing and state 0 reflecting. In fact, since $\mu_i b_i = \mu_{i+1}a_{i+1}$ $(0 \le i \le N-2)$, we have

$$-(a_0+b_0) + \frac{h_1}{h_0}b_0 = -(a_0+b_0) + \frac{1/a_0 + 1/b_0}{1/a_0}b_0 = 0,$$

and for $2 \le i < N$,

$$\frac{h_{i-1}}{h_i}a_i - (a_i+b_i) + \frac{h_{i+1}}{h_i}b_i = \frac{h_{i-1}-h_i}{h_i}a_i + \frac{h_{i+1}-h_i}{h_i}b_i = 0.$$

Proof of Theorem 1.1 (i): Let $P^*(t), \widehat{P}(t)$ be the probability transition matrices of Q^* and $\widehat{Q}$, respectively. From (2.1) we have

$$HQ^{*m} = \widehat{Q}^m H, \quad m \in \mathbb{Z}_+$$

and thus

$$HP^*(t) = \widehat{P}(t)H, \quad t \ge 0.$$

Consider the (i, N) entry of each side. Let $T^*_{i,N}$ be the hitting time of N from i for $P^*(t)$. On the left-hand side

$$[HP^*(t)](i,N) = \sum_{k=0}^{N} H(i,k)P^*(t)(k,N) = h_i\mathbb{P}(T^*_{i,N} \le t)$$

and on the right-hand side

$$\left[\widehat{P}(t)H\right](i,N) = \sum_{k=0}^{N} \widehat{P}(t)(i,k)H(k,N) = h_N\mathbb{P}(T_{i,N} \le t).$$

So

$$\mathbb{P}(T_{i,N} \le t) = \frac{h_i}{h_N}\mathbb{P}(T^*_{i,N} \le t), \quad t \ge 0. \tag{2.3}$$

For $1 \le i \le N$, let $Q^{*(i)}$ be the leading principal i-by-i sub-matrix of Q^*, and let $0 < \lambda_1^{*(i)} < \lambda_2^{*(i)} < \cdots < \lambda_i^{*(i)}$ be the eigenvalues of $-Q^{*(i)}$. The relationship between Q^* and $\widehat{Q}$ in (2.1), which remains for $Q^{*(i)}$ and $Q^{(i)}$, implies

$$\lambda_\nu^{*(i)} = \lambda_\nu^{(i)} \ (1 \le \nu \le i \le N). \tag{2.4}$$

Combining (1.3), (1.4), (2.3) and (2.4), we get Theorem 1.1 (i). ∎

Analogously, we can prove Theorem 1.1 (ii) by φ-transform. Let φ_i $(-1 \le i \le N-1)$ be the analogue of (1.8), where ∞ is replaced by N, and

$$\tilde{Q} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{a} & Q^{(N)} \end{pmatrix}$$

with $\mathbf{a} = (a_0, 0, \dots, 0)^T$ and $\mathbf{0}$ the zero vector. Next, define

$$\Phi = \operatorname{diag}(\varphi_{-1}, \varphi_0, \dots, \varphi_{N-1}), \quad Q^{**} = \Phi^{-1}\tilde{Q}\Phi,$$

It is easy to check that Q^{**} is a birth–death q-matrix with state -1 absorbing and state $N-1$ reflecting. Furthermore,

$$\Phi Q^{**m} = \tilde{Q}^m \Phi, \quad m \in \mathbb{Z}_+.$$

Thus the corresponding probability transition matrices $P^{**}(t)$ ad $\tilde{P}(t)$ satisfy

$$\Phi P^{**}(t) = \tilde{P}(t)\Phi, \quad t \ge 0.$$

Following the proof for Theorem 1.1 (i), we derive Theorem 1.1 (ii).

Remark 2.1: As the referees suggested, the proof of Theorem 1.1 (ii) can be reduced to Theorem 1.1 (i) by the involution of the states: $i \to N-1-i$. However, we preserved φ transform on finite state space to support Eq. (3.8) in the proof of Theorem 1.4.

By h-transform and φ-transform, we get the explicit form of the eigentime identities.

Corollary 2.2: *For $0 \le i < N$, the eigentime identify*

$$\mathbb{E}\left[T_{i,N}\mathbb{1}_{\{T_{i,N}<\infty\}}\right] = \frac{h_i}{h_N}\sum_{j=0}^{N-1}\mu_j h_j^2\left(\frac{1}{h_{i\vee j}} - \frac{1}{h_N}\right);$$

$$\mathbb{E}\left[T_{i,-1}\mathbb{1}_{\{T_{i,-1}<\infty\}}\right] = \frac{\varphi_i}{\varphi_{-1}}\sum_{j=0}^{N-1}\mu_j \varphi_j^2\left(\frac{1}{\varphi_{i\wedge j}} - \frac{1}{\varphi_{-1}}\right);$$

$$\mathbb{E}\left[T_{i,N}\wedge T_{i,-1}\right] = \frac{\varphi_i}{\varphi_{-1}}\sum_{j=0}^{i-1}\mu_j h_j + \frac{h_i}{h_N}\sum_{j=i}^{N-1}\mu_j\varphi_j.$$

PROOF: We begin by deriving the first equation. Indeed, by [14, Eq. (3.3)], we can get

$$\mathbb{E}(T^*_{i,N}) = \sum_{j=0}^{N-1} \mu^*_j \sum_{k=i\vee j}^{N-1} \frac{1}{\mu^*_k b^*_k}, \tag{2.5}$$

where b^*_i and a^*_i are the birth and death rates for Q^*, and $(\mu^*_i,\, 0 \le i < N)$ are the invariant measure defined similarly as (1.1). Then Eq. (2.2) gives that

$$\mu^*_0 = 1, \quad \mu^*_i = \frac{b^*_0 b^*_1 \cdots b^*_{i-1}}{a^*_1 a^*_2 \cdots a^*_i} = \frac{\mu_i h_i^2}{h_0^2} \quad (0 \le i < N),$$

and Eq. (2.3) gives that

$$\mathbb{E}(T_{i,N}\mathbb{1}_{\{T_{i,N}<\infty\}}) = \frac{h_i}{h_N}\mathbb{E}(T^*_{i,N}).$$

Then the first equation follows from (2.2) and above three equations.

For the second equation, by (2.5) and the involution $i \mapsto N-1-i\,(-1 \le i \le N)$,

$$\mathbb{E}(T^{**}_{i,-1}) = \sum_{j=0}^{N-1} \mu^{**}_j \sum_{k=0}^{i\wedge j} \frac{1}{\mu^{**}_k a^{**}_k},$$

where b^{**}_i and a^{**}_i are the birth and death rates for Q^{**}, and $(\mu^{**}_i,\, 0 \le i < N)$ are the invariant measure also defined similarly as (1.1). Then the rest of the proof is similar as that of the first equation.

For the third equation, by (1.8) and the fact that $T_{i,N} < \infty$ if and only if $T_{i,-1} = \infty$, we have

$$\mathbb{E}\left[T_{i,N} \wedge T_{i,-1}\right] = \mathbb{E}\left[T_{i,N}\mathbb{1}_{\{T_{i,N}<\infty\}}\right] + \mathbb{E}\left[T_{i,-1}\mathbb{1}_{\{T_{i,-1}<\infty\}}\right],$$

which shows the third equation follows from the first two. ■

3. DENUMERABLE STATE SPACE

Based upon the proofs in Section 2, we can prove Theorems 1.3 and 1.4 for the minimal birth–death process corresponding to Q.

Assume $a_0 > 0$ and $R < \infty$. We first study the distribution of $T_{i,\infty}$ by h-transform. Redefine

$$H = \mathrm{diag}(h_0, h_1, h_2, \ldots), \quad Q^* = H^{-1}QH. \tag{3.1}$$

Then Q^* is a birth–death q-matrix with state 0 reflecting. That is, the birth and death rates

$$b^*_i = \frac{h_{i+1}}{h_i} b_i \;\; (i \ge 0), \quad a^*_i = \frac{h_{i-1}}{h_i} a_i \;\; (i \ge 1), \tag{3.2}$$

and the invariant measure

$$\mu^*_0 = 1, \quad \mu^*_i = \frac{b^*_0 b^*_1 \cdots b^*_{i-1}}{a^*_1 a^*_2 \cdots a^*_i} = \frac{\mu_i h_i^2}{h_0^2} \quad (i \ge 1). \tag{3.3}$$

Thus

$$R^* := \sum_{j=0}^{\infty} \mu^*_j \sum_{i=j}^{\infty} \frac{1}{\mu^*_i b^*_i} = \sum_{j=0}^{\infty} \mu_j h_j^2 \sum_{i=j}^{\infty} \left(\frac{1}{h_i} - \frac{1}{h_{i+1}}\right) < \sum_{j=0}^{\infty} \mu_j \sum_{i=j}^{\infty} \frac{1}{\mu_i b_i} = R < \infty.$$

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This means the corresponding Q^*-processes are not unique. We consider the minimal process $P^*(t)$. Let L^* be its generator with domain $\mathscr{D}(L^*)$ in the space $L^2(\mathbb{Z}_+, \mu^*)$, which is defined similarly as $L^2(\mathbb{Z}_+, \mu)$. By [10, Section 4 and Subsection 7.1], we know that $\sigma_{\text{ess}}(L^*) = \emptyset$.

PROPOSITION 3.1: *Let $\{\lambda_\nu^*, \nu \geq 1\}$ be the eigenvalues of $-L^*$, increasing in ν. Then*

$$\lambda_\nu = \lambda_\nu^*, \quad \nu \geq 1.$$

PROOF: Consider the linear operator $A : \mathscr{D}(L^*) \to \mathscr{D}(L)$, defined by $Af^* = Hf^*/h_1$. According to (3.1) and the Dirichlet form theory about the minimal process in [2, Chapter 6], A is isometric, and $L^* = A^{-1}LA$. Therefore L^* and L have the same spectrum. ∎

Now we can give the proof of Theorem 1.3.

PROOF OF THEOREM 1.3: Let $T_{i,N}^*$ be the hitting time of state N for $P^*(t)$ starting from i. From Theorem 1.1 (i), we have

$$\mathbb{P}(T_{i,N} \leq t) = \frac{h_i}{h_N} \mathbb{P}(T_{i,N}^* \leq t), \quad t \geq 0.$$

So

$$\mathbb{P}(T_{i,\infty} \leq t) = \lim_{N\to\infty} \mathbb{P}(T_{i,N} \leq t) = \lim_{N\to\infty} \frac{h_i}{h_N} \mathbb{P}(T_{i,N}^* \leq t) = \frac{h_i}{h_\infty} \mathbb{P}(T_{i,\infty}^* \leq t), \quad t \geq 0. \quad \textbf{(3.4)}$$

By [10, Theorems 4.6 and 7.1], we have

$$\mathbb{E}\, e^{-sT_{i,\infty}^*} = \frac{\prod_{\nu=1}^{\infty} \lambda_\nu^*/(s + \lambda_\nu^*)}{\prod_{\nu=1}^{i} \lambda_\nu^{*(i)}/(s + \lambda_\nu^{*(i)})}, \quad s \geq 0,$$

where $\left\{\lambda_\nu^{*(i)}, 1 \leq \nu \leq i\right\}$ are the eigenvalues of the leading principal i-by-i submatrix of $-Q^*$. Then the first assertion follows from Proposition 3.1 and (2.4).

For the eigentime identity, by [10, Sections 2 and 4], we get

$$\mathbb{E}T_{i,\infty}^* = \sum_{j=0}^{\infty} \mu_j^* \sum_{m=i\vee j}^{\infty} \frac{1}{\mu_m^* b_m^*}.$$

Then the eigentime identity follows from (3.2) and (3.4). ∎

Next we investigate the distribution of $T_{i,-1}$ by φ-transform. For this, we extend Q on $\mathbb{Z}_+$ to $\overline{Q}$ on $\mathbb{Z}_+ \cup \{-1\}$ as follows:

$$\overline{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_0 & -(a_0 + b_0) & b_0 & 0 & \cdots & 0 & 0 \\ 0 & a_1 & -(a_1 + b_1) & b_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Then $\{\lambda_\nu, \nu \geq 1\} \cup \{0\}$ is the spectrum (all eigenvalues) of $\overline{Q}$. Indeed, assume $\overline{Q}\overline{g} = -\lambda \overline{g}$ for some $\lambda > 0$. Note that $\overline{g}_{-1} = -(1/\lambda)\left[\overline{Q}\overline{g}\right](-1) = 0$. Then $Qg = -\lambda g$ for $g_i = \overline{g}_i (i \geq 0)$.

Conversely, if $Qg = -\lambda g$, then by letting $\overline{g}_{-1} = 0$ and $\overline{g}_i = g_i (i \geq 0)$, we have $\overline{Q}\overline{g} = -\lambda \overline{g}$. Actually, to get the distribution of $T_{i,-1}$, we need only the process before it hits state -1. In other words, only the spectrum (all eigenvalues) of Q will appear in the distribution of $T_{i,-1}$.

Define

$$\Phi = \text{diag}(\varphi_{-1}, \varphi_0, \varphi_1, \ldots), \quad \overline{Q}^{**} = \Phi^{-1}\overline{Q}\Phi.$$

It is easy to check that $\overline{Q}^{**}$ is a birth-death q-matrix with -1 absorbing. Let Q^{**} be the restriction of $\overline{Q}^{**}$ on $\mathbb{Z}_+$. Then the birth and death rates of Q^{**} are

$$b_i^{**} = \frac{\varphi_{i+1}}{\varphi_i} b_i, \quad a_i^{**} = \frac{\varphi_{i-1}}{\varphi_i} a_i \quad (i \geq 0). \tag{3.5}$$

Let

$$\mu_0^{**} = 1, \quad \mu_i^{**} = \frac{b_0^{**} b_1^{**} \cdots b_{i-1}^{**}}{a_1^{**} a_2^{**} \cdots a_i^{**}} = \frac{\mu_i \varphi_i^2}{\varphi_0^2} \quad (i \geq 1). \tag{3.6}$$

Then

$$R^{**} = \sum_{j=0}^{\infty} \mu_j^{**} \sum_{i=j}^{\infty} \frac{1}{\mu_i^{**} b_i^{**}} = \sum_{j=0}^{\infty} \mu_j \varphi_j^2 \sum_{i=j}^{\infty} \left[\frac{1}{\varphi_{i+1}} - \frac{1}{\varphi_i} \right] = \infty, \tag{3.7}$$

and

$$S^{**} = \sum_{j=1}^{\infty} \mu_j^{**} \sum_{i=0}^{j-1} \frac{1}{\mu_i^{**} b_i^{**}} = \sum_{j=1}^{\infty} \mu_j \varphi_j^2 \left[\frac{1}{\varphi_j} - \frac{1}{\varphi_1} \right] < \sum_{j=1}^{\infty} \mu_j \varphi_j < R < \infty.$$

This means the corresponding Q^{**}-process is unique and ∞ is an entrance boundary. Denote this process by $P^{**}(t)$ and let L^{**} be its generator with domain $\mathscr{D}(L^{**}) \subset L^2(\mathbb{Z}_+, \mu^{**})$. Then by [10, Theorem 5.1], $\sigma_{\text{ess}}(L^{**}) = \emptyset$. Similar to the relation between Q and $\overline{Q}$ mentioned in the previous paragraph, the positive eigenvalues of Q^{**} and $\overline{Q}^{**}$ are the same.

For $i \geq 0$, consider the φ-transform on $\{i, i+1, \ldots\}$ with i absorbing. Namely, set

$$\Phi^{(i)} = \text{diag}(\varphi_i, \varphi_{i+1}, \ldots), \quad \overline{Q}^{**(i)} = \Phi^{(i)^{-1}} \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{a} & \overline{Q}^{(i)} \end{pmatrix} \Phi^{(i)},$$

where $\mathbf{a} = (a_{i+1}, 0, \ldots)^T$, $\mathbf{0}$ is the zero vector, and $\overline{Q}^{(i)}$ is defined in (1.7). Then $\overline{Q}^{**(i)}$ is absorbing at state i. Let $Q^{**(i)}$ be the restriction of $\overline{Q}^{**(i)}$ on $\{i+1, i+2, \ldots\}$. By (3.7), there exists a unique $Q^{**(i)}$-process $P^{**(i)}(t)$ with generator $L^{**(i)}$ in $L^2(\mathbb{Z}_+, \mu^{**})$, whose essential spectrum is also empty.

A similar argument as in the proof of Proposition 3.1 gives

Proposition 3.2: *Let $\{\lambda_\nu^{**}, \nu \geq 1\}, \{\lambda_\nu^{**(i)}, \nu \geq 1\}$ be the (positive) eigenvalues for $-L^{**}$ and $-L^{**(i)}$ respectively, all in increasing order with ν. Then*

$$\lambda_\nu = \lambda_\nu^{**}, \quad \lambda_\nu^{(i)} = \lambda_\nu^{**(i)},$$

where $\{\lambda_\nu, \nu \geq 1\}$ and $\{\lambda_\nu^{(i)}, \nu \geq 1\}$ are defined the same as in Theorem 1.4.

Now we give the proof for Theorem 1.4.

Proof of Theorem 1.4: Note that the distribution of $T_{i,-1}$ depends only on Q-process. By the φ-transform on $\{-1, 0, 1, \ldots, N-1\}$ as in Section 2 and the approximating procedure, we have

$$\mathbb{P}(T_{i,-1} \le t) = \frac{\varphi_i}{\varphi_{-1}} \mathbb{P}(T^{**}_{i,-1} \le t), \quad t \ge 0. \tag{3.8}$$

By [10, Theorem 5.5], we have

$$\mathbb{E}\, e^{-sT^{**}_{i,-1}} = \frac{\prod_{\nu=1}^{\infty} \lambda_\nu^{**}/(s+\lambda_\nu^{**})}{\prod_{\nu=1}^{\infty} \lambda_\nu^{**(i)}/(s+\lambda_\nu^{**(i)})}, \quad s \ge 0.$$

Then the first assertion follows from Proposition 3.2.

For the eigentime identity, by [10, Section 5], we get

$$\mathbb{E}T^{**}_{i,-1} = \sum_{j=0}^{\infty} \mu_j^{**} \sum_{m=0}^{j} \frac{1}{\mu_m^{**} a_m^{**}}.$$

Then the eigentime identity follows from (3.5), (3.6) and (3.8). ■

Note that $T_{i,\infty} < \infty$ if and only if $T_{i,-1} = \infty$. We can derive the lifetime distribution from Theorems 1.3 and 1.4.

Corollary 3.3: *Assume $a_0 > 0$ and $R < \infty$. Then the lifetime $T_i = T_{i,-1} \wedge T_{i,\infty}$ $(i \ge 0)$ has the Laplace–Stieltjes transform*

$$\mathbb{E}\, e^{-sT_i} = \left(1 - \frac{h_i}{h_\infty}\right) \cdot \frac{\prod_{\nu=1}^{\infty} \lambda_\nu/(s+\lambda_\nu)}{\prod_{\nu=1}^{\infty} \overline{\lambda}_\nu^{(i)}/(s+\overline{\lambda}_\nu^{(i)})} + \frac{h_i}{h_\infty} \cdot \frac{\prod_{\nu=1}^{\infty} \lambda_\nu/(s+\lambda_\nu)}{\prod_{\nu=1}^{i-1} \lambda_\nu^{(i)}/(s+\lambda_\nu^{(i)})}, \quad s > 0,$$

and the eigentime identity

$$\mathbb{E}T_i = \sum_{\nu=0}^{\infty} \frac{1}{\lambda_\nu} - \frac{\varphi_i}{\varphi_{-1}} \sum_{\nu=0}^{\infty} \frac{1}{\overline{\lambda}_\nu^{(i)}} - \frac{h_i}{h_\infty} \sum_{\nu=0}^{i-1} \frac{1}{\lambda_\nu^{(i)}} = \frac{\varphi_i}{\varphi_{-1}} \sum_{j=0}^{i-1} \mu_j h_j + \frac{h_i}{h_\infty} \sum_{j=i}^{\infty} \mu_j \varphi_j. \tag{3.9}$$

Remark 3.4: Note that Eq. (3.9) can also be derived from Corollary 2.2 by letting N go to infinity, as an anonymous referee suggested.

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