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THE EMBEDDINGS OF THE HEISENBERG GROUP INTO THE CREMONA GROUP

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Abstract. In this article, we describe the embeddings of the Heisenberg group into the Cremona group.

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1. Introduction. The Heisenberg group is the non-abelian nilpotent group given by:

$$\mathcal{H} = \langle \mathrm{f},\ \mathrm{g} \mid [\mathrm{f}, \mathrm{g}] = \mathrm{h},\ [\mathrm{f}, \mathrm{h}] = [\mathrm{g}, \mathrm{h}] = \mathrm{id}\rangle.$$

It has two generators, f and g, and h is the generator of the center of $\mathcal{H}$.

The Cremona group is the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ of birational maps of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ into itself. Such maps can be written in the form:

$$(x : y : z) \dashrightarrow \big(P_0(x, y, z) : P_1(x, y, z) : P_2(x, y, z)\big),$$

where $P_0, P_1, P_2 \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of the same degree, and this degree is the degree of the map, if the polynomials have no common factor (of positive degree). Recall that if ϕ is a birational self-map of the complex projective plane, then one of the following holds ([**9**, **5**, **8**, **2**]):

- ◇ the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ is bounded, and ϕ is said to be *elliptic*;
- ◇ the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows linearly with n, and ϕ is said to be a *Jonquières twist*;
- ◇ the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows quadratically with n, and ϕ is said to be a *Halphen twist*;
- ◇ $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows exponentially fast with n, and ϕ is said to be *hyperbolic*.

PROPOSITION A. *Let ρ be an embedding of $\mathcal{H}$ into the Cremona group. Then, $\rho(\mathcal{H})$ does not contain hyperbolic birational maps.*

More precisely, $\rho(\mathrm{f})$ and $\rho(\mathrm{g})$ are either elliptic birational maps or Jonquières twists.

We describe the embeddings of $\mathcal{H}$ into the Cremona group (in [**7**], we already looked at such embeddings but with the following assumption: the images of f and g are elliptic birational self maps).

THEOREM B. *Let ρ be an embedding of $\mathcal{H}$ into the Cremona group. Then up to birational conjugacy:*

- ◇ *either $\rho(\mathcal{H})$ is a subgroup of* $\mathrm{PGL}(3, \mathbb{C})$ *and*

$$\rho(\mathrm{f}) = (x + \alpha y, y + \beta) \qquad\qquad \rho(\mathrm{g}) = (x + \gamma y, y + \delta)$$

with α, β, γ, δ in $\mathbb{C}$ such that $\alpha\delta-\beta\gamma=1$;

◇ *or $\rho(\mathcal{H})$ is a subgroup of the group of polynomial automorphisms of $\mathbb{C}^2$ and $(\rho(\mathrm{f}),\,\rho(\mathrm{g}))$ is one of the following pairs:*

$$\begin{gathered}((ax+Q(y),\,y+c),\ (\alpha x+P(y),\,y+\gamma)),\\ \left(\left(ax+Q(y),\,by+\frac{\gamma(b-1)}{\beta-1}\right),\,(\alpha x+P(y),\,\beta y+\gamma)\right)\end{gathered}$$

with a, α, b in $\mathbb{C}^*$, $\beta\in\mathbb{C}^*\smallsetminus\{1\}$, c, γ in $\mathbb{C}$ and P, Q in $\mathbb{C}[y]$;

◇ *or $\rho(\mathrm{f})$ is a Jonquières twist and $(\rho(\mathrm{f}),\,\rho(\mathrm{g}))$ is one of the following pairs:*

$$((x,\,\delta x^{\pm1}y),\ (\gamma x,\,ya(x))),\qquad \left((x,\,\delta x^{\pm2}y),\ (\gamma x,\,ya(x))\right),$$
$$\left((-x,\,\delta x^{\pm1}y),\ (\gamma x,\,yb(x))\right),\qquad \left((\lambda x,\,yc(x)),\,(\delta x,\,yd(x))\right)$$

with δ, $\gamma\in\mathbb{C}^*$, $\lambda\in\mathbb{C}^*\smallsetminus\{1,\,-1\}$ and a, b, c, $d\in\mathbb{C}(x)^*$ such that

$$\frac{b(x)}{b(-x)}\in\mathbb{C}^*,\qquad\qquad \frac{c(\delta x)d(x)}{c(x)d(\lambda x)}\in\mathbb{C}^*.$$

REMARK C. Note that the two last families are not empty. For instance,

$$\left((-x,\,\alpha x^{\pm1}y),\ (\beta x,\,\gamma x^2y)\right),\qquad\qquad \left((\lambda x,\,\alpha x^py),\,(\gamma x,\,\beta x^qy)\right)$$

with α, β, $\gamma\in\mathbb{C}^*$, $\lambda\in\mathbb{C}^*\smallsetminus\{1,\,-1\}$, p, $q\in\mathbb{N}$ are such pairs.

2. Some recalls.

2.1. About birational maps of the complex projective plane. Let ϕ be a birational self-map of the complex projective plane. Then one of the following holds ([9, 5, 8, 2]):

◇ ϕ is *elliptic* if and only if the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ is bounded. In this case, there exist a birational map $\psi\colon S\dashrightarrow\mathbb{P}^2_{\mathbb{C}}$ and an integer $k\geq1$ such that $\psi^{-1}\circ\phi^k\circ\psi$ belongs to the connected component of the identity of the group $\mathrm{Aut}(S)$. Either ϕ is of finite order, or ϕ is conjugate to an automorphism of $\mathbb{P}^2_{\mathbb{C}}$, which restricts to one of the following automorphisms on some open subset isomorphic to $\mathbb{C}^2$:
- $(x,\,y)\mapsto(\alpha x,\,\beta y)$ where the kernel of the group morphism:

$$\mathbb{Z}^2\to\mathbb{C}^2\qquad\qquad (i,j)\mapsto\alpha^i\beta^j$$

  is generated by $(k,\,0)$ for some $k\in\mathbb{Z}$;
- $(x,\,y)\mapsto(\alpha x,\,y+1)$ where $\alpha\in\mathbb{C}^*$.

◇ ϕ is *parabolic* if and only if the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows linearly or quadratically with n. If ϕ is parabolic, there exist a birational map $\psi\colon S\dashrightarrow\mathbb{P}^2_{\mathbb{C}}$ and a fibration $\pi\colon S\to B$ onto a curve B such that $\psi^{-1}\circ\phi\circ\psi$ permutes the fibers of π. If $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows linearly, then the fibration π is rational and ϕ is said to be a *Jonquières twist*. If $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows quadratically, then the fibration π is elliptic and ϕ is said to be a *Halphen twist*.

◇ ϕ is *hyperbolic* if and only if $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows exponentially fast with n: there is a constant $c(\phi)$ such that $\deg(\phi^n)=c(\phi)\lambda(\phi)^n+O(1)$.

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2.2. About distorted elements. If G is a group generated by a finite subset $F\subset \mathrm{G}$ the F-length $|g|_F$ of an element g of G is defined as the least nonnegative integer ℓ such that g admits an expression of the form $g=f_1f_2\dots f_\ell$ where each f_i belongs to $F\cup F^{-1}$. We say that g is *distorted* if $\lim_{k\to+\infty}\frac{|g^k|_F}{k}=0$ (note that the limit $\lim_{k\to+\infty}\frac{|g^k|_F}{k}$ always exists and is a real number since the sequence $k\mapsto |g^k|_F$ is subadditive). This notion actually does not depend on the chosen F, but only on the pair (g, G).

If G is any group, an element $g\in\mathrm{G}$ is *distorted* if it is distorted in some finitely generated subgroup of G.

The element h of

$$\mathcal{H}=\langle \mathrm{f},\ \mathrm{g}\,|\,[\mathrm{f},\mathrm{g}]=\mathrm{h},\ [\mathrm{f},\mathrm{h}]=[\mathrm{g},\mathrm{h}]=\mathrm{id}\rangle$$

satisfies the following property:

$$\forall\, k\in\mathbb{Z}\qquad \mathrm{h}^{k^2}=[\mathrm{f}^k,\mathrm{g}^k]=\mathrm{f}^k\mathrm{g}^k\mathrm{f}^{-k}\mathrm{g}^{-k}$$

so $||\mathrm{h}^{k^2}||\leq 4k$ and $\lim_{k\to+\infty}\frac{||\mathrm{h}^k||}{k^2}=0$. Hence, h is distorted.

An element $\phi\in\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is said to be *algebraic* if it is contained in an algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. By [3, Section 2.6] the map $\phi\in\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is algebraic if and only if the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ is bounded. In other words, elliptic elements and algebraic elements coincide. By [2, Proposition 2.3], this is also equivalent to say that ϕ is of finite order or conjugate to an element of $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$. A straightforward computation shows that every element of $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$ is distorted in $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ (see [4, Lemma 4.40]). As a consequence, every algebraic element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is distorted. The converse statement also holds

THEOREM 2.1 ([4, 6]). *Any distorted element of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ *is elliptic.*

COROLLARY 2.2. *Let ρ be an embedding of $\mathcal{H}$ into* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. *Then* $\rho(\mathrm{h})$ *is elliptic.*

We will use this corollary and the following description of the centralizer of hyperbolic birational maps to prove Proposition A:

PROPOSITION 2.3 ([1]). *Let $\phi\in$* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ *be an hyperbolic map. The infinite cyclic group generated by ϕ is a finite index subgroup of the centralizer of ϕ in* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$.

Proof of the first part of Proposition A. Assume that $\rho(\mathcal{H})$ contains an hyperbolic element ϕ. Since $\rho(\mathrm{h})$ is the generator of the center of $\rho(\mathcal{H})$, $\rho(\mathrm{h})$ commutes with ϕ. Proposition 2.3 implies that either $\rho(\mathrm{h})$ is hyperbolic or $\rho(\mathrm{h})$ is of finite order. But $\rho(\mathrm{h})$ is not hyperbolic (Corollary 2.2) and by definition $\rho(\mathrm{h})$ is of infinite order. As a result, $\rho(\mathcal{H})$ does not contain hyperbolic element. □

2.3. About centralizers of elliptic birational maps. Let us recall the description of the centralizers of the elliptic birational self-maps of infinite order of the complex projective plane obtained in [2].

Consider ϕ an elliptic element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Assume that ϕ is of infinite order. As recalled in Section 2.1, the map ϕ is conjugate to an automorphism of $\mathbb{P}^2_\mathbb{C}$ which restricts to one of the following automorphisms on some open subset isomorphic to $\mathbb{C}^2$:

(1) $(\alpha x,\beta y)$ where α, β belong to $\mathbb{C}^*$;
(2) $(\alpha x, y+1)$ where $\alpha\in\mathbb{C}^*$.

In case (1), the centralizer of ϕ in $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is

$$\big\{(\eta(x), ya(x^k)) \mid a \in \mathbb{C}(x),\ \eta \in \mathrm{PGL}(2, \mathbb{C}),\ \eta(\alpha x) = \alpha\eta(x)\big\};$$

in particular the elements of the centralizer of ϕ are elliptic birational maps or Jonquières twists. In case (2), the centralizer of ϕ in $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is

$$\big\{(\eta(x), y + a(x)) \mid \eta \in \mathrm{PGL}(2, \mathbb{C}),\ \eta(\alpha x) = \alpha\eta(x),\ a \in \mathbb{C}(x),\ a(\alpha x) = a(x)\big\};$$

in particular the elements of the centralizer of ϕ are elliptic birational maps.

Corollary 2.2 and the previous description imply

LEMMA 2.4. *Let ρ be an embedding of $\mathcal{H}$ into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.
Then, $\rho(\mathrm{h})$ is elliptic and up to birational conjugacy:

- *either $\rho(\mathrm{h}) = (\alpha x, \beta y)$, where the kernel of the group morphism:*

$$\mathbb{Z}^2 \to \mathbb{C}^2 \qquad\qquad (i, j) \mapsto \alpha^i\beta^j$$

is generated by $(k, 0)$ for some $k \in \mathbb{Z}$ and both $\rho(\mathrm{f})$, $\rho(\mathrm{g})$ belong to

$$\big\{(\eta(x), ya(x^k)) \mid a \in \mathbb{C}(x),\ \eta \in \mathrm{PGL}(2, \mathbb{C}),\ \eta(\alpha x) = \alpha\eta(x)\big\};$$

- *or $\rho(\mathrm{h}) = (\alpha x, y + 1)$, and both $\rho(\mathrm{f})$, $\rho(\mathrm{g})$ belong to*

$$\big\{(\eta(x), y + a(x)) \mid \eta \in \mathrm{PGL}(2, \mathbb{C}),\ \eta(\alpha x) = \alpha\eta(x),\ a \in \mathbb{C}(x),\ a(\alpha x) = a(x)\big\}.$$

In particular, $\rho(\mathrm{f})$ and $\rho(\mathrm{g})$ are elliptic birational maps or Jonquières twists.

It ends the proof of Proposition A.

3. Proof of Theorem B.

3.1. Assume that all the generators of $\rho(\mathcal{H})$ are elliptic. The group $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of $\mathbb{C}^2$ is a subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. It is generated by the group:

$$\mathrm{A} = \big\{(a_0x + a_1y + a_2, b_0x + b_1y + b_2) \mid a_i,\ b_i \in \mathbb{C},\ a_0b_1 - a_1b_0 \neq 0\big\}$$

and

$$\mathrm{E} = \big\{(\alpha x + P(y), \beta y + \gamma) \mid \alpha,\ \beta \in \mathbb{C}^*,\ \gamma \in \mathbb{C},\ P \in \mathbb{C}[y]\big\}.$$

Let us recall the following result obtained when we study the embeddings of $\mathrm{SL}(n, \mathbb{Z})$ into the Cremona group:

LEMMA 3.1 ([7]). *Let ρ be an embedding of $\mathcal{H}$ into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.
If $\rho(\mathrm{f})$, $\rho(\mathrm{g})$, and $\rho(\mathrm{h})$ are elliptic, then up to birational conjugacy:

- *either $\rho(\mathcal{H})$ is a subgroup of* $\mathrm{PGL}(3, \mathbb{C})$*, and*

$$\rho(\mathrm{f}) = (x + \alpha y, y + \beta) \qquad\qquad \rho(\mathrm{g}) = (x + \gamma y, y + \delta)$$

with α, β, γ, $\delta \in \mathbb{C}$ such that $\alpha\delta - \beta\gamma = 1$;
- *or $\rho(\mathcal{H})$ is a subgroup of* E *and $\rho(h^2) = (x + P(y), y)$ for some $P \in \mathbb{C}[y]$.*

This statement implies the following one:

PROPOSITION 3.2. *Let* ρ *be an embedding from* $\mathcal{H}$ *into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. *Assume that* $\rho(\mathrm{f})$, $\rho(\mathrm{g})$ *and* $\rho(\mathrm{h})$ *are elliptic.*

Then up to birational conjugacy:

- ◇ *either* $\rho(\mathcal{H})$ *is a subgroup of* $\mathrm{PGL}(3,\mathbb{C})$, *more precisely*

$$\rho(\mathrm{f}) = (x+\alpha y, y+\beta) \qquad\qquad \rho(\mathrm{g}) = (x+\gamma y, y+\delta)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *such that* $\alpha\delta - \beta\gamma = 1$;

- ◇ *or* $\rho(\mathcal{H})$ *is a subgroup of* E *and* $(\rho(\mathrm{f}), \rho(\mathrm{g}))$ *is one of the following pairs:*

$$\big((ax+Q(y), y+c),\ (\alpha x+P(y), y+\gamma)\big)$$
$$\left(\left(ax+Q(y), by+\frac{\gamma(b-1)}{\beta-1}\right), \big(\alpha x+P(y), \beta y+\gamma\big)\right)$$

with a, α, b *in* $\mathbb{C}^*$, c, γ *in* $\mathbb{C}$, $\beta \in \mathbb{C}^* \smallsetminus \{1\}$ *and* P, Q *in* $\mathbb{C}[y]$.

Proof. The first assertion follows from Lemma 3.1. Let us focus on the second one.

If $\rho(\mathrm{h})$ belongs to E and $\rho(\mathrm{h}^2) = (x+P(y), y)$, then $\rho(\mathrm{h}) = (\varepsilon x + Q(y), \eta(y))$ with $\varepsilon^2 = 1$, $Q \in \mathbb{C}[y]$ and $\eta(y) \in \big\{-y+\gamma,\ y\big\}$. But $\rho(\mathrm{f})$ and $\rho(\mathrm{g})$ belong to E so $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ implies that $\varepsilon = 1$ and $\eta(y) = y$, that is, $\rho(\mathrm{h}) = (x+Q(y), y)$. Set

$$\rho(\mathrm{f}) = (ax+R(y), by+c), \qquad\qquad \rho(\mathrm{g}) = (\alpha x+P(y), \beta y+\gamma).$$

The second component of $\rho(\mathrm{f})\rho(\mathrm{g})$ has to be equal to the second component of $\rho(\mathrm{h})\rho(\mathrm{g})\rho(\mathrm{f})$, that is,

$$\beta by + b\gamma + c = \beta by + \beta c + \gamma;$$

in other words either $\beta = b = 1$, or $c = \frac{\gamma(b-1)}{\beta-1}$. □

3.2. Assume that $\rho(\mathrm{f})$ is a Jonquières twist with trivial action on the basis of the fibration. Since $\rho(\mathrm{h})$ is elliptic, then up to birational conjugacy either $\rho(\mathrm{h}) = (\alpha x, \beta y)$ or $\rho(\mathrm{h}) = (\alpha x, y+1)$ (see Section 2.1). But $\rho(\mathrm{f})$ belongs to the centralizer of $\rho(\mathrm{h})$ and is a Jonquières twist; therefore, according to Section 2.3, one has $\rho(\mathrm{h}) = (\alpha x, \beta y)$, $\rho(\mathrm{f})$ can be written as $(x, ya(x))$ and $\rho(\mathrm{g})$ as $(\mu(x), yb(x))$ with $\mu \in \mathrm{PGL}(2,\mathbb{C})$ and $a, b \in \mathbb{C}(x)^*$.

Let us remark that if $\mu = \mathrm{id}$, then $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ implies $\alpha = \beta = 1$ so $\mu \neq \mathrm{id}$.

The relation $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ implies that $\alpha = 1$ and $a(\mu(x)) = \beta a(x)$. Let us first look at polynomials P such that $P(\mu(x)) = \beta P(x)$:

CLAIM 3.3. *If* P *is a nonzero polynomial such that* $P(\mu(x)) = \lambda^2 P(x)$, $\lambda^2 \neq 1$, *then one of the following holds*

- ◇ $P(x) = \delta\left(\frac{\gamma}{\lambda^2-1} + x\right)$, $\mu(x) = \gamma + \lambda^2 x$ *with* $a, \delta \in \mathbb{C}$;
- ◇ $P(x) = \delta\left(\frac{\gamma}{\lambda+1} - x\right)^2$, $\mu(x) = \gamma - \lambda x$ *with* $\gamma \in \mathbb{C}$, *and* $\delta \in \mathbb{C}^*$;
- ◇ $P(x) = \delta\left(\frac{\gamma}{\lambda-1} + x\right)^2$, $\mu(x) = \gamma + \lambda x$ *with* $\gamma \in \mathbb{C}$, *and* $\delta \in \mathbb{C}^*$.

Proof. Let us consider the set $Z_P = \big\{z \mid P(z) = 0\big\}$ of roots of P. It is a finite set invariant by μ. As a result, $\mu^n_{|Z_P} = \mathrm{id}$ for some integer n.

If $\#Z_P \geq 3$, then $\mu^n_{|Z_P} = \mathrm{id}$ implies $\mu^n = \mathrm{id}$. Recall that

$$\rho(\mathrm{f}) = (x, ya(x)), \qquad \rho(\mathrm{g}) = (\mu(x), yb(x)), \qquad \rho(\mathrm{h}) = (\alpha x, \beta y)$$

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so

$$\rho(\mathrm{f})^n=(x,yA(x)),\quad \rho(\mathrm{g})^n=(\mu^n(x),yB(x))=(x,yB(x)),\quad \rho(\mathrm{h})^{n^2}=(\alpha^{n^2}x,\beta^{n^2}y).$$

Then, $[\rho(\mathrm{f})^n,\rho(\mathrm{g})^n]=\rho(\mathrm{h})^{n^2}$ implies $\alpha^{n^2}=\beta^{n^2}=1$, that is, $\rho(\mathrm{h})$ is of finite order: contradiction.

Hence, $\#Z_P\leq 2$ so $\deg P\leq 2$. A straightforward computation implies the statement. □

Let us come back to $a(\mu(x))=\beta a(x)$. As β is of infinite order and a belongs to $\mathbb{C}(x)^*$, we can rewrite this equality as follows: $\frac{P(\mu(x))}{Q(\mu(x))}=\frac{\lambda_1^2P(x)}{\lambda_2^2Q(x)}$ where

- ◇ λ_1 and λ_2 are two elements of $\mathbb{C}\smallsetminus\{\pm1\}$ such that $\beta=\frac{\lambda_1^2}{\lambda_2^2}$;
- ◇ P and Q are two polynomials without common factor.

As a result up to birational conjugacy, $(\rho(\mathrm{f}),\rho(\mathrm{g}))$ is one of the following pairs:

$$\Big((x,\delta xy),\ (\lambda x,yb(x))\Big)\qquad \Big((x,\delta x^2y),\ (\lambda x,yb(x))\Big)$$
$$\Big(\big(x,\frac{y}{\delta x}\big),\ (\lambda x,yb(x))\Big)\qquad \Big(\big(x,\frac{y}{\delta x^2}\big),\ (\lambda x,yb(x))\Big)$$

with $\delta\in\mathbb{C}^*$, $\lambda\in\mathbb{C}^*$ of infinite order and $b\in\mathbb{C}(x)$.

We can thus state

PROPOSITION 3.4. *Let ρ be an embedding of $\mathcal{H}$ into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.

If $\rho(\mathrm{f})$ is a Jonquières twist with trivial action on the basis of the fibration, then up to birational conjugacy, $(\rho(\mathrm{f}),\rho(\mathrm{g}))$ is one of the following pairs:

$$\Big((x,\delta xy),\ (\lambda x,yb(x))\Big)\qquad \Big((x,\delta x^2y),\ (\lambda x,yb(x))\Big)$$
$$\Big(\Big(x,\frac{y}{\delta x}\Big),\ (\lambda x,yb(x))\Big)\qquad \Big(\Big(x,\frac{y}{\delta x^2}\Big),\ (\lambda x,yb(x))\Big)$$

with $\delta\in\mathbb{C}^*$, $\lambda\in\mathbb{C}^*$ of infinite order and $b\in\mathbb{C}(x)$.

3.3. Assume that $\rho(\mathrm{f})$ is a Jonquières twist with nontrivial action on the basis of the fibration. Since $\rho(\mathrm{h})$ is elliptic and of infinite order, then up to birational conjugacy either $\rho(\mathrm{h})=(\alpha x,\beta y)$ or $\rho(\mathrm{h})=(\alpha x,y+1)$ (see Section 2.1). But $\rho(\mathrm{f})$ belongs to the centralizer of $\rho(\mathrm{h})$ and is a Jonquières twist; therefore according to Section 2.3, one has $\rho(\mathrm{h})=(\alpha x,\beta y)$, $\rho(\mathrm{f})$ can be written as $(\eta(x),ya(x))$ and $\rho(\mathrm{g})$ as $(\mu(x),yb(x))$ with η, μ in $\mathrm{PGL}(2,\mathbb{C})$ and a, b in $\mathbb{C}(x)$.

Up to conjugacy by an element of $\left\{\left(\frac{ax+b}{cx+d},y\right)\mid\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{PGL}(2,\mathbb{C})\right\}$, one can assume that either $\eta(x)=x+1$ or $\eta(x)=\lambda x$ (remark that this conjugacy does not preserve the first component of $\rho(\mathrm{h})$).

Note that a direct computation implies

$$\big\{\nu\in\mathrm{PGL}(2,\mathbb{C})\mid \nu(\alpha x)=\alpha\nu(x)\big\}=\begin{cases}\mathrm{PGL}(2,\mathbb{C})\ \text{if}\ \alpha=1\\ \big\{\beta x^{\pm1}\mid\beta\in\mathbb{C}^*\big\}\ \text{if}\ \alpha=-1\\ \big\{\beta x\mid\beta\in\mathbb{C}^*\big\}\ \text{if}\ \alpha^2\neq1\end{cases}\qquad(1)$$

so when η is an homothety, we will have to distinguish the cases $\lambda=-1$ and $\lambda\neq-1$.

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3.3.1. Assume that $\eta(x) = x + 1$

Since $\rho(\mathrm{f})$ and $\rho(\mathrm{h})$ commute, $\rho(\mathrm{h})$ can be written as $(x+\gamma, \beta y)$.

If $\gamma \neq 0$, then $[\rho(\mathrm{f}), \rho(\mathrm{h})] = \mathrm{id}$ leads to $a(x+\gamma) = a(x)$, that is, $a(x) = a \in \mathbb{C}$: contradiction with the fact that $\rho(\mathrm{f})$ is a Jonquières twist.

If $\gamma = 0$, then $\rho(\mathrm{h}) = (x, \beta y)$ and $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ leads to $\rho(\mathrm{g}) = (x+\mu, yb(x))$ and $b(x)a(x+\mu) = \beta a(x)b(x+1)$. Let us write a as $\frac{P}{Q}$ and b as $\frac{R}{S}$ with $P, Q, R, S \in \mathbb{C}[y]$ then $b(x)a(x+\mu) = \beta a(x)b(x+1)$ can be rewritten as:

$$P(x+\mu)Q(x)R(x)S(x+1) = \beta P(x)Q(x+\mu)R(x+1)S(x). \qquad (2)$$

Denote by p_i (resp. q_ℓ, resp. r_j, resp. s_k) the coefficient of the highest term of P (resp. Q, resp. R, resp. S). The coefficient of the highest term of the left-hand side of (2) has to be equal to the coefficient of the highest term of the right-hand side of (2), that is, $p_i q_\ell r_j s_k = \beta p_i q_\ell r_j s_k$. So $\beta = 1$, that is, $\rho(\mathrm{h}) = (x, y)$: contradiction.

3.3.2. Suppose that $\eta(x) = -x$, that is, $\rho(\mathrm{f}) = (-x, ya(x))$

REMARK 3.5. The map $\rho(\mathrm{f})^2 = (x, ya(x)a(-x))$ is a Jonquières twist that preserves fiberwise the rational fibration $x = \mathrm{cst}$; consequently, Proposition 3.4 says that $\rho(\mathrm{f})^2$ is one of the following:

$$\big(x, \delta xy\big), \qquad \big(x, \delta x^2 y\big), \qquad \Big(x, \delta \frac{y}{x}\Big), \qquad \Big(x, \delta \frac{y}{x^2}\Big)$$

with $\delta \in \mathbb{C}^*$. Let us try to determine $\rho(\mathrm{f})$. If $\rho(\mathrm{f})^2 = (x, \delta xy)$, then we have to consider the equation $a(x)a(-x) = \delta x$. The right-hand side of this equation is invariant by $x \mapsto -x$, whereas the left-hand side not, so there is no solution. The same holds if $\rho(\mathrm{f})^2 = \big(x, \delta\frac{y}{x}\big)$. Consequently, $\rho(\mathrm{f})^2$ is one of the following:

$$\big(x, \delta x^2 y\big), \qquad \Big(x, \delta \frac{y}{x^2}\Big)$$

with $\delta \in \mathbb{C}^*$ and $\rho(\mathrm{f})$ is thus one of the following:

$$\big(-x, \zeta xy\big), \qquad \Big(-x, \zeta \frac{y}{x}\Big)$$

with $\zeta \in \mathbb{C}^*$.

Since f and h commute, then (1) implies that either $\rho(\mathrm{h}) = \big(\frac{\alpha}{x}, \beta y\big)$ or $\rho(\mathrm{h}) = (\alpha x, \beta y)$. Let us consider these two cases.

- ◇ Assume first that $\rho(\mathrm{h}) = \big(\frac{\alpha}{x}, \beta y\big)$. Note that $\big(\frac{\alpha}{x}, \beta y\big)$ does not commute neither to $(-x, \zeta xy)$ nor to $\big(-x, \zeta\frac{y}{x}\big)$: contradiction with $[\rho(\mathrm{f}), \rho(\mathrm{h})] = \mathrm{id}$.
- ◇ Suppose now that $\rho(\mathrm{h}) = (\alpha x, \beta y)$.
  - If $\alpha^2 \neq 1$, then $[\rho(\mathrm{g}), \rho(\mathrm{h})] = \mathrm{id}$ and (1) imply that $\rho(\mathrm{g}) = (\gamma x, yb(x))$. Then, $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ leads to $\alpha = 1$: contradiction with $\alpha^2 \neq 1$.
  - If $\alpha = -1$, that is $\rho(\mathrm{h}) = (-x, \beta y)$, then according to $[\rho(\mathrm{g}), \rho(\mathrm{h})] = \mathrm{id}$ and (1) we get that either $\rho(\mathrm{g}) = (\gamma x, yb(x))$ or $\rho(\mathrm{g}) = \big(\frac{\gamma}{x}, yb(x)\big)$. In both cases, the relation $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ leads to a contradiction.
  - If $\alpha = 1$, that is, $\rho(\mathrm{h}) = (x, \beta y)$, then $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ implies that either $\rho(\mathrm{g}) = (\gamma x, yb(x))$ or $\rho(\mathrm{g}) = \big(\frac{\gamma}{x}, yb(x)\big)$.

    First, let us assume that $\rho(\mathrm{g}) = (\gamma x, yb(x))$. If $\rho(\mathrm{f}) = (-x, \zeta xy)$, then $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ leads to $\gamma b(x) = \beta b(-x)$, that is $\frac{b(x)}{b(-x)}$ belongs to $\mathbb{C}^*$. If

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$\rho(\mathrm{f})=\left(-x,\,\zeta\frac{y}{x}\right)$, then $[\rho(\mathrm{f}),\,\rho(\mathrm{g})]=\rho(\mathrm{h})$ implies $b(x)=\beta\gamma b(-x)$, that is $\frac{b(x)}{b(-x)}$ belongs to $\mathbb{C}^*$.

Finally, suppose that $\rho(\mathrm{g})=\left(\frac{\gamma}{x},\,yb(x)\right)$. If $\rho(\mathrm{f})=(-x,\,\zeta xy)$, then $[\rho(\mathrm{f}),\,\rho(\mathrm{g})]=\rho(\mathrm{h})$ leads to $\gamma b(x)=\beta x^2b(-x)$. Write b as $\frac{P}{Q}$ with P, Q in $\mathbb{C}[x]$; then $\gamma b(x)=\beta x^2b(-x)$ is equivalent to

$$\gamma P(x)Q(-x)=\beta x^2P(-x)Q(x)$$

and the degree of the left-hand side is $\deg P+\deg Q$, whereas the degree of the right-hand side is $\deg P+\deg Q+2$: contradiction. If $\rho(\mathrm{f})=\left(-x,\,\zeta\frac{y}{x}\right)$, then a straightforward computation implies similarly a contradiction.

PROPOSITION 3.6. *Let ρ be an embedding of $\mathcal{H}$ into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.

If $\rho(\mathrm{f})$ is a Jonquières twist with a order 2 action on the basis of the fibration, then up to birational conjugacy, $(\rho(\mathrm{f}),\,\rho(\mathrm{g}))$ is one of the following pairs:

$$\big((-x,\,\alpha xy),\;(\beta x,\,ya(x))\big),\qquad\qquad \left(\left(-x,\,\alpha\frac{y}{x}\right),\;(\beta x,\,ya(x))\right)$$

with $\alpha,\,\beta\in\mathbb{C}^*$ and $a\in\mathbb{C}(x)^*$ such that $\frac{a(x)}{a(-x)}\in\mathbb{C}^*$.

3.3.3. Assume that $\eta(x)=\lambda x,\ \lambda^2\neq 1$

Recall that

$$\rho(\mathrm{f})=(\lambda x,\,ya(x)),\qquad \rho(\mathrm{g})=(\mu(x),\,yb(x)),\qquad \rho(\mathrm{h})=(\upsilon(x),\,\beta y)$$

with λ in $\mathbb{C}^*\smallsetminus\{1,\,-1\}$, β in $\mathbb{C}^*$, μ and υ in $\mathrm{PGL}(2,\,\mathbb{C})$, and a and b in $\mathbb{C}(x)^*$.

First, note that since $\rho(\mathrm{f})$ and $\rho(\mathrm{h})$ commute, $\upsilon(\lambda x)=\lambda\upsilon(x)$. According to (1), the homography υ is an homothety (recall that $\lambda^2\neq 1$): $\upsilon(x)=\gamma x$ with $\gamma\in\mathbb{C}^*$.

The relations $[\rho(\mathrm{f}),\,\rho(\mathrm{g})]=\rho(\mathrm{h})$, $[\rho(\mathrm{f}),\,\rho(\mathrm{h})]=[\rho(\mathrm{g}),\,\rho(\mathrm{h})]=\mathrm{id}$ imply the following ones:

$$a(x)=a(\gamma x)\tag{3}$$

$$b(x)=b(\gamma x)\tag{4}$$

$$\mu(\gamma x)=\gamma\mu(x)\tag{5}$$

$$\lambda\mu(x)=\gamma\mu(\lambda x)\tag{6}$$

$$b(x)a(\mu(x))=\beta a(x)b(\lambda x).\tag{7}$$

We will distinguish the cases $\gamma=1$, $\gamma=-1$, and $\gamma^2\neq 1$.

- ◇ Assume that $\gamma^2\neq 1$. Then, (1) and (5) lead to $\mu(x)=\mu x$ with $\mu\in\mathbb{C}^*$. Equation (6) can be rewritten $\lambda\mu x=\gamma\mu\lambda x$, that is, $\gamma=1$: contradiction with the assumption $\gamma^2\neq 1$.
- ◇ Suppose that $\gamma=1$. Then, $\lambda^2\neq 1$, (1) and (6) lead to $\mu(x)=\mu x$ with $\mu\in\mathbb{C}^*$. In other words,

$$\rho(\mathrm{f})=(\lambda x,\,ya(x)),\qquad\qquad \rho(\mathrm{g})=(\mu x,\,yb(x))$$

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with λ in $\mathbb{C}^* \smallsetminus \{1, -1\}$, μ in $\mathbb{C}^*$, and a and b in $\mathbb{C}(x)$ such that $\frac{a(\mu x)b(x)}{a(x)b(\lambda x)}$ belongs to $\mathbb{C}^*$.

◇ Assume that $\gamma = -1$. Then, (1) and (5) imply that $\mu(x) = \mu x^{\pm 1}$ with $\mu \in \mathbb{C}^*$. If $\mu(x) = \mu x$, then (6) can be rewritten as $\lambda\mu x = -\lambda\mu x$: contradiction. If $\mu(x) = \frac{\mu}{x}$, then (6) can be rewritten as $\frac{\lambda\mu}{x} = -\frac{\mu}{\lambda x}$; hence, $\lambda^2 = -1$.

- If $\lambda = \mathbf{i}$, then $\rho(\mathrm{f}) = (\mathbf{i}x, ya(x))$ and $\rho(\mathrm{f})^4 = \big(x, ya(x)a(\mathbf{i}x)a(-x)a(-\mathbf{i}x)\big)$ preserves fiberwise the fibration $x =$ cst. According to Proposition 3.4, $\rho(\mathrm{f})^4$ can be written as $(x, \delta xy)$, or $(x, \delta x^2 y)$, or $\big(x, \frac{y}{\delta x}\big)$, or $\big(x, \frac{y}{\delta x^2}\big)$. If $\rho(\mathrm{f})^4 = (x, \delta xy)$, then $\delta x = a(x)a(\mathbf{i}x)a(-x)a(-\mathbf{i}x)$, but the right-hand side of this equality is invariant by $x \mapsto \mathbf{i}x$ whereas the left-hand side is not. As a consequence, $\rho(\mathrm{f})^4$ can not be written $(x, \delta xy)$. Similarly, one sees that $\rho(\mathrm{f})^4$ can not be written as $(x, \delta x^2 y)$, $\big(x, \frac{y}{\delta x}\big)$, and $\big(x, \frac{y}{\delta x^2}\big)$. Thus, $\lambda \neq \mathbf{i}$.
- Similarly, one gets that the case $\lambda = -\mathbf{i}$ does not happen.

PROPOSITION 3.7. *Let ρ be an embedding of $\mathcal{H}$ into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.
If $\rho(\mathrm{f})$ is a Jonquières twist with an action on the basis of the fibration that is neither trivial nor of order 2*, then up to birational conjugacy $(\rho(\mathrm{f}), \rho(\mathrm{g}))$ is one of the following pairs:*

$$((\lambda x, ya(x)), (\mu x, yb(x)))$$

with $\lambda \in \mathbb{C}^* \smallsetminus \{1, -1\}$, $\mu \in \mathbb{C}^*$, and a, $b \in \mathbb{C}(x)^*$ such that $\frac{a(\mu x)b(x)}{a(x)b(\lambda x)} \in \mathbb{C}^*$.

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