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Hilbert rings with maximal ideals of different heights and unruly Hilbert rings

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Abstract. Let $f : R \to S$ be a ring homomorphism and J be an ideal of S. Then the subring $R \bowtie^f J := \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\}$ of $R \times S$ is called the amalgamation of R with S along J with respect to f. In this paper, we characterize when $R \bowtie^f J$ is a Hilbert ring. As an application, we provide an example of Hilbert ring with maximal ideals of different heights. We also construct non-Noetherian Hilbert rings whose maximal ideals are all finitely generated (unruly Hilbert rings).

1 Introduction

Throughout, let R and S be two commutative rings with unity, let J be a non-zero proper ideal of S and $f : R \to S$ be a ring homomorphism. D'Anna *et al.* in [9, 10] have introduced the following subring

$$R \bowtie^f J := \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\},$$

of $R \times S$, called the *amalgamated algebra* (or *amalgamation*) of R with S along J with respect to f. This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in [12]). Moreover, several classical constructions such as Nagata's idealization (cf. [18, p. 2]), the $R + XS[X]$ and the $R + XS[\![X]\!]$ constructions can be studied as particular cases of this new construction (see [9, Example 2.5 and Remark 2.8]). Amalgamation, in turn, can be realized as a pullback. The construction has proved its worth providing numerous examples and counterexamples in commutative ring theory [6–8, 11, 13, 20].

Recall that a ring R is *Hilbert* if each proper prime ideal of R is an intersection of maximal ideals of R. Thus, in view of Cohen's theorem, it is natural to ask if a Hilbert ring which all its maximal ideals are finitely generated is necessarily Noetherian. There are counterexamples for this question, called *unruly* Hilbert rings. Gilmer and Heinzer, in 1976, constructed an unruly Hilbert ring [14]. Then, in [4, 17], the authors used the $R + XS[X]$ construction and pullbacks to provide a family of unruly Hilbert rings. In this paper we characterize when $R \bowtie^f J$ is a Hilbert ring (Theorem 3.3) and, as an application, we give examples of unruly Hilbert rings.

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The simplest examples of Hilbert rings include the polynomial rings over a field and the ring of integers. These rings have the property that all the maximal ideals are of the same height. Therefore, it is interesting to find Hilbert rings with maximal ideals of different heights. Roberts in [19] and Heinzer in [15] constructed interesting examples of this kind. In this paper, we provide an easy way to construct such rings.

The outline of the paper is as follows. In Section 2, we fix our notation and give some elementary results on which we base our approach. In Section 3, we prove our main theorem, which provides a characterization of Hilbert property on the amalgamations (Theorem 3.3). Our attempt results in examples of Hilbert rings with maximal ideals of different heights and unruly Hilbert rings. These examples presented in Section 4.

2 Preliminaries

Let us first fix some notation which we shall use. For a commutative ring A, the set of nilpotent elements, prime ideals, and maximal ideals of A will be denoted by $\mathrm{Nil}(A)$, $\mathrm{Spec}(A)$, and $\mathrm{Max}(A)$, respectively. $\mathrm{V}(I)$ denotes the set of prime ideals of A containing I. For a multiplicatively closed subset T of A, we use the notation $T^{-1}A$ to denote the ring of fractions of A with respect to T. In the sequel, we will use the following remark without explicit mention.

Remark 2.1 [10, Proposition 2.6] For $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\mathfrak{q} \in \mathrm{Spec}(S)\setminus \mathrm{V}(J)$, set

$$\mathfrak{p}'^f := \mathfrak{p} \bowtie^f J := \{(p, f(p) + j) \mid p \in \mathfrak{p}, j \in J\},$$

$$\overline{\mathfrak{q}}^f := \{(r, f(r) + j) \mid r \in R, j \in J, f(r) + j \in \mathfrak{q}\}.$$

Then, the following statements hold.

- $\mathrm{Spec}(R \bowtie^f J) = \{\mathfrak{p}'^f \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \cup \{\overline{\mathfrak{q}}^f \mid \mathfrak{q} \in \mathrm{Spec}(S)\setminus V(J)\}$.
- $\mathrm{Max}(R \bowtie^f J) = \{\mathfrak{p}'^f \mid \mathfrak{p} \in \mathrm{Max}(R)\} \cup \{\overline{\mathfrak{q}}^f \mid \mathfrak{q} \in \mathrm{Max}(S)\setminus V(J)\}$.

We need the following lemmas in the proof of our main result.

Lemma 2.2 *Let $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(R)$ and $\mathfrak{q}, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(S)\setminus \mathrm{V}(J)$. Then*

(1) $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ *if and only if* $\mathfrak{p}_1'^f \subseteq \mathfrak{p}_2'^f$.
(2) $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ *if and only if* $\overline{\mathfrak{q}_1}^f \subseteq \overline{\mathfrak{q}_2}^f$.
(3) $\overline{\mathfrak{q}}^f \subseteq \mathfrak{p}'^f$ *if and only if* $f^{-1}(\mathfrak{q} + J) \subseteq \mathfrak{p}$.
(4) $\mathfrak{p}'^f \nsubseteq \overline{\mathfrak{q}}^f$.

Proof (1)–(3) are from [5, Lemmas 2.2 and 2.3]. To see (4), pick $j \in J\setminus\mathfrak{q}$. Then $(0, j) \in \mathfrak{p}'^f\setminus\overline{\mathfrak{q}}^f$. ∎

Lemma 2.3 *Let $\mathfrak{p}, \mathfrak{p}_\alpha \in \mathrm{Spec}(R)$ and $\mathfrak{q}, \mathfrak{q}_\beta \in \mathrm{Spec}(S)\setminus \mathrm{V}(J)$; $\alpha \in \Lambda$, $\beta \in \Delta$. Then*

(1) $\mathfrak{p} = \cap_{\alpha\in\Lambda}\mathfrak{p}_\alpha$ *if and only if* $\mathfrak{p}'^f = \cap_{\alpha\in\Lambda}\mathfrak{p}_\alpha'^f$.
(2) $\mathfrak{q} = \cap_{\beta\in\Delta}\mathfrak{q}_\beta$ *if and only if* $\overline{\mathfrak{q}}^f = \cap_{\beta\in\Delta}\overline{\mathfrak{q}_\beta}^f$.

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Proof The proof of (1) follows immediately from definition of $\mathfrak{p}'^f$ in Remark 2.1. The proof of (2) is similar to the proof of [5, Lemma 2.2], but we include a proof for the convenience of the reader. Note that, by [5, Lemma 2.2], $\mathfrak{q} \subseteq \cap_{\beta\in\Delta}\mathfrak{q}_\beta$ if and only if $\overline{\mathfrak{q}}^f \subseteq \cap_{\beta\in\Delta}\overline{\mathfrak{q}_\beta}^f$. It is also clear that if $\cap_{\beta\in\Delta}\mathfrak{q}_\beta \subseteq \mathfrak{q}$, then $\cap_{\beta\in\Delta}\overline{\mathfrak{q}_\beta}^f \subseteq \overline{\mathfrak{q}}^f$. Now let $\cap_{\beta\in\Delta}\overline{\mathfrak{q}_\beta}^f \subseteq \overline{\mathfrak{q}}^f$, and pick $y \in \cap_{\beta\in\Delta}\mathfrak{q}_\beta$. Let $v \in J\backslash\mathfrak{q}$. Then $(0, yv) \in \cap_{\beta\in\Delta}\overline{\mathfrak{q}_\beta}^f$, which implies $yv \in \mathfrak{q}$ by assumption and so $y \in \mathfrak{q}$. ∎

In order to construct examples of the title, we are required to know when maximal ideals of amalgamations are finitely generated. Here, we collect some elementary properties of this concept.

Lemma 2.4 *Let* $\mathfrak{m} \in \mathrm{Spec}(R)$ *and* $\mathfrak{n} \in \mathrm{Spec}(S)\setminus \mathrm{V}(J)$. *Then*

(1) *If* $\mathfrak{m}$ *and* J *are finitely generated, then* $\mathfrak{m}'^f$ *is finitely generated.*
(2) *If* $\mathfrak{m} = \langle x\rangle$ *is nonzero principal and* J *is divisible by* $f(x)$ *(i.e., for any* $i \in J$*, there exists* $j \in J$ *such that* $i = f(x)j$*), then* $\mathfrak{m}'^f$ *is principal.*
(3) *If* $\mathfrak{m}$ *is finitely generated and* $f(\mathfrak{m})S = S$*, then* $\mathfrak{m}'^f$ *is finitely generated.*
(4) *Assume that* f *is injective,* $f^{-1}(J) = 0$*, and* $\mathfrak{n}$ *is generated by an element of* $f(R) + J = S$*. Then* $\overline{\mathfrak{n}}^f$ *is principal.*

Proof For the proof of (1) see the beginning of the proof of [6, Theorem 4.1]. To prove (2), Let $\mathfrak{m} = \langle x\rangle$ and pick $(r, f(r) + i) \in \mathfrak{m}'^f$. We want to prove that $\mathfrak{m}'^f = \langle(x, f(x))\rangle$. By assumption, $r = xa$ and $i = f(x)j$, for some $a \in R$ and $j \in J$. Then $(r, f(r) + i) = (x, f(x))(a, f(a) + j)$, as claimed. (3) is an immediate consequence of [11, Proposition 3.1(3)]. (4): Let $\mathfrak{n} = \langle f(r) + i\rangle$. We wish to show that $\overline{\mathfrak{n}}^f = \langle(r, f(r) + i)\rangle$. To this end, let $(x, f(x) + j) \in \overline{\mathfrak{n}}^f$. One can write $f(x) + j = (f(r) + i)(f(s) + k)$, for some $f(s) + k \in f(R) + J$. Then $f(x) - f(r)f(s) \in J$, which implies $x = rs$. Hence $(x, f(x) + j) = (r, f(r) + i)(s, f(s) + k)$. ∎

3 Main result

In this section, we investigate when the amalgamated algebra $R \bowtie^f J$ is a Hilbert ring. Then we present several corollaries that recover or generalize previous works. Let us first introduce the concept of Hilbert condition for an arbitrary subset of $\mathrm{Spec}(R)$.

Definition 3.1 Let $X \subseteq \mathrm{Spec}(R)$. We call X Hilbert if each proper prime ideal $P \in X$ is an intersection of elements of $\mathrm{Max}(R) \cap X$. We simply call R Hilbert if $\mathrm{Spec}(R)$ is Hilbert.

Note that R is Hilbert in the usual sense, if it is Hilbert in the sense of above definition. For $X \subseteq \mathrm{Spec}(R)$, when we write that "$\widehat{X}$ is Hilbert," it is to be understood that each $P \in X$ is an intersection of elements of $\mathrm{Max}(R)$ (all maximal ideals of the ring not only those who are in X). We shall need the following well-known fact (see, e.g., [16, Theorem 31]).

Lemma 3.2 *For a ring* R *and indeterminates* $X_1, \ldots, X_n$ *over* R*, the following conditions are equivalent:*

(1) *R is a Hilbert ring.*

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(2) *R/I is a Hilbert ring for each proper ideal I of R.*
(3) *$R[X_1, \dots, X_n]$ is a Hilbert ring.*

The following theorem is the main result of this section.

Theorem 3.3 *The following statements are equivalent*:

(1) *$R \bowtie^f J$ is a Hilbert ring.*
(2) *R and $\mathrm{Spec}(S)\setminus \mathrm{V}(J)$ are Hilbert.*
(3) *R and $\mathrm{Spec}\widehat{(S)\setminus \mathrm{V}}(J)$ are Hilbert.*

Proof (1) $\Rightarrow$ (2) Assume that $R \bowtie^f J$ is a Hilbert ring. Then, by Lemma 3.2, R is also a Hilbert ring since it is a homomorphic image of $R \bowtie^f J$. Next, let $Q \in \mathrm{Spec}(S)\setminus \mathrm{V}(J)$. Then $\overline{Q}^f$ is a prime ideal of the Hilbert ring $R \bowtie^f J$, and so $\overline{Q}^f = (\cap_{\alpha\in\Lambda}\mathfrak{p}_\alpha'^f) \cap (\cap_{\beta\in\Delta}\overline{\mathfrak{q}_\beta}^f)$, for suitable families $\mathfrak{p}_\alpha \in \mathrm{Max}(R)$ and $\mathfrak{q}_\beta \in \mathrm{Max}(S)\setminus \mathrm{V}(J)$. The proof is completed by showing that $Q = \cap_{\beta\in\Delta}\mathfrak{q}_\beta$. From Lemma 2.2, we have $Q \subseteq \cap_{\beta\in\Delta}\mathfrak{q}_\beta$. To see the converse inclusion, let $x \in \cap_{\beta\in\Delta}\mathfrak{q}_\beta$ and pick $v \in J\setminus Q$. Hence, for all $\beta \in \Delta$ we have $xv \in \mathfrak{q}_\beta \cap J$, hence that $(0, xv) \in \overline{\mathfrak{q}_\beta}^f$. Note that for all $\mathfrak{p} \in \mathrm{Spec}(R)$, we always have $(0, xv) \in \mathfrak{p}'^f$. It follows that $(0, xv) \in \overline{Q}^f$, which implies $xv \in Q$ and so $x \in Q$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Assume that R and $\mathrm{Spec}\widehat{(S)\setminus \mathrm{V}}(J)$ are Hilbert, and let $\mathcal{P} \in \mathrm{Spec}(R \bowtie^f J)$. If $\mathcal{P} = \mathfrak{p}'^f$, for some $\mathfrak{p} \in \mathrm{Spec}(R)$, then $\mathfrak{p}$ is an intersection of maximal ideals, and hence the same is true for $\mathfrak{p}'^f$, by Lemma 2.3. Next, let $\mathcal{P} = \overline{\mathfrak{q}}^f$, for some $\mathfrak{q} \in \mathrm{Spec}(S)\setminus \mathrm{V}(J)$. By the Hilbert assumption on $\mathrm{Spec}\widehat{(S)\setminus \mathrm{V}}(J)$, we have $q = (\cap_{\alpha\in\Lambda}\mathfrak{m}_\alpha) \cap (\cap_{\beta\in\Delta}\mathfrak{n}_\beta)$, for suitable families $\mathfrak{m}_\alpha \in \mathrm{Max}(S) \cap \mathrm{V}(J)$ and $\mathfrak{n}_\beta \in \mathrm{Max}(S)\setminus \mathrm{V}(J)$. Since R is a Hilbert ring, for any $\alpha \in \Lambda$ we have $f^{-1}(\mathfrak{m}_\alpha) = \cap_{\gamma\in\Gamma}M_{\alpha\gamma}$ for a suitable families $M_{\alpha\gamma} \in \mathrm{Max}(R)$, which implies $\left(f^{-1}(\mathfrak{m}_\alpha)\right)'^f = \cap_{\gamma\in\Gamma}\left(M_{\alpha\gamma}\right)'^f$ (Lemma 2.3). Thus, if we prove that $\overline{q}^f = (\cap_{\alpha\in\Lambda}\left(f^{-1}(\mathfrak{m}_\alpha)\right)'^f) \cap (\cap_{\beta\in\Delta}\overline{\mathfrak{n}_\beta}^f)$, the assertion follows. This we do.

It follows from Lemma 2.2(3) that $\overline{q}^f \subseteq \cap_{\alpha\in\Lambda}\left(f^{-1}(\mathfrak{m}_\alpha)\right)'^f$ and from Lemma 2.2(2) that $\overline{q}^f \subseteq \cap_{\beta\in\Delta}\overline{\mathfrak{n}_\beta}^f$. To see the converse inclusion, let $(r, f(r) + i) \in (\cap_{\alpha\in\Lambda}\left(f^{-1}(\mathfrak{m}_\alpha)\right)'^f) \cap (\cap_{\beta\in\Delta}\overline{\mathfrak{n}_\beta}^f)$. Then, for all $\alpha \in \Lambda$ and $\beta \in \Delta$, we have $r \in f^{-1}(\mathfrak{m}_\alpha)$ and $f(r) + i \in \mathfrak{n}_\beta$. Hence $f(r) + i \in (\cap_{\alpha\in\Lambda}\mathfrak{m}_\alpha) \cap (\cap_{\beta\in\Delta}\mathfrak{n}_\beta) = \mathfrak{q}$, which implies $(r, f(r) + i) \in \overline{\mathfrak{q}}^f$. ■

Recall that if $f := id_R$ is the identity homomorphism on R, and I is an ideal of R, then $R \bowtie I := R \bowtie^{id_R} I$ is called the amalgamated duplication of R along I.

Corollary 3.4 *Let I be an ideal of R. Then $R \bowtie I$ is a Hilbert ring if and only if so is R.*

In the following, we observe what happens if we let J be *too small* or *too big*.

Corollary 3.5 *The following hold*:

(1) *Let $J \subseteq \mathrm{Nil}(S)$. Then $R \bowtie^f J$ is Hilbert if and only if so is R.*

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(2) *Let J be a maximal ideal of S. Then $R \bowtie^f J$ is a Hilbert ring if and only if so are R and S.*

Proof In the first case $\mathrm{Spec}(S) \setminus \mathrm{V}(J) = \phi$, and in the second case $\mathrm{Spec}(S) \setminus \mathrm{V}(J) = \mathrm{Spec}(S) \setminus \{J\}$. The assertion follows from Theorem 3.3. ■

Let M (respectively, $N = (M_i)_{i=1}^n$) be an R-module (respectively, a family of R-modules). Then $R \ltimes M$ (respectively, $R \ltimes_n N$) denotes the *trivial extension* of R by M (respectively, the n-trivial extension of R by N). It should be noted that both constructions are special cases of amalgamation with $J^n = 0$ (for definition and more details, see [2, 7, 9]). Hence the next result follows from the first part of Corollary 3.5.

Corollary 3.6 *Let M be an R-module and $N = (M_i)_{i=1}^n$ be a family of R-modules. Then the following hold:*

(1) *$R \ltimes M$ is a Hilbert ring if and only if so is R.*
(2) *$R \ltimes_n N$ is a Hilbert ring if and only if so is R.*

Corollary 3.7 [4, Corollary 6] *Let M be a maximal ideal of a ring T and let D be a subring of T such that $M \cap D = (0)$. Then $D + M$ is a Hilbert ring if and only if D and T are Hilbert rings.*

Proof $D + M \cong D \bowtie^{\iota} M$, where $\iota : D \hookrightarrow T$ is the natural embedding. The result now follows from Corollary 3.5(2). ■

Let $\alpha : A \to C, \beta : B \to C$ be ring homomorphisms. The subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the pullback of α and β. Note that the amalgamation can be studied in the framework of pullback constructions. In fact, if $\pi : S \to S/J$ is the canonical projection and $\check{f} = \pi \circ f$, then $R \bowtie^f J = \check{f} \times_{S/J} \pi$.

Assuming A is a subring of C and β is surjective, Anderson *et al.* prove that D and C are Hilbert rings if and only if A and B are Hilbert rings [4, Theorem 3]. The following Proposition removes the assumption that *A is a subring of C*, in the case of amalgamations.

Proposition 3.8 *$R \bowtie^f J$ and S/J are Hilbert rings if and only if so are R and S.*

Proof Assume that $R \bowtie^f J$ and S/J are Hilbert rings. Then, by (the proof of) Theorem 3.3, R is Hilbert and any $Q \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$ is an intersection of maximal ideals of S. On the other hand, if $Q \in \mathrm{Spec}(S) \cap \mathrm{V}(J)$, then Q/J is an intersection of maximal ideals of S/J and so Q is an intersection of maximal ideals of S, as desired. The converse is clear by Theorem 3.3 and Lemma 3.2. ■

In [17, Theorem 5], assuming that R is a Hilbert domain contained in the field S, the authors prove that $R + XS[X]$ is a Hilbert domain. Then [4, Corollary 4] generalizes this result as follows. Let $X_1, \ldots, X_n$ be finitely many indeterminates over a ring E, and let D be a subring of E. Then $D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$ is a Hilbert ring if

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and only if D and E are Hilbert rings. We proceed with a slight generalization of these results.

Corollary 3.9 *Let $\mathbf{X} := \{X_1, \ldots, X_n\}$ be a finite set of indeterminates over S. Then $f(R) + \mathbf{X}S[\mathbf{X}]$ is a Hilbert ring if and only if so are R and S. In particular, if R is a subring of S, then $R + \mathbf{X}S[\mathbf{X}]$ is a Hilbert ring if and only if so are R and S.*

Proof Let φ be the composition homomorphism $\varphi : R \xrightarrow{f} S \hookrightarrow S[\mathbf{X}]$, and let $J := \mathbf{X}S[\mathbf{X}]$. Then, by [9, Proposition 5.1(3)], $R \bowtie^{\varphi} J \cong f(R) + \mathbf{X}S[\mathbf{X}]$. Therefore, by Proposition 3.8 and and Lemma 3.2, $f(R) + \mathbf{X}S[\mathbf{X}]$ and S are Hilbert rings if and only if so are R and S. Observe that S is a homomorphic image of $f(R) + \mathbf{X}S[\mathbf{X}]$ (via evaluation at n-tuple 1) and the result follows. ■

Let R be an integral domain and T be a multiplicatively closed subset of R. In [1, Theorem 4.1], under certain conditions, the authors characterize when $R + XR_T[X]$ is a Hilbert ring and conjectured that the conditions on T are not necessary. This conjecture is proved in [4, Corollary 5] in a more general case; only assuming that each element of T is a nonzerodivisor. As an applications of Corollary 3.9, we now drop this assumption too. In the following, $g : R \to R_T$ stands for the canonical homomorphism.

Corollary 3.10 *Let $\mathbf{X} := \{X_1, \ldots, X_n\}$ be a finite set of indeterminates over R, and T be a multiplicatively closed subset of R. Then $g(R) + \mathbf{X}R_T[\mathbf{X}]$ is a Hilbert ring if and only if R and R_T are Hilbert rings. In particular, if each element of T is a nonzerodivisor, then $R + \mathbf{X}R_T[\mathbf{X}]$ is a Hilbert ring if and only if R and R_T are Hilbert rings.*

4 Examples

In this section, we give examples of the rings of the title. We first present unruly Hilbert rings using previously known rings of this kind [4, 14, 17]. Then we construct new original examples. Finally, we construct a Hilbert ring with maximal ideals of different heights.

Example 4.1 Let R be an unruly Hilbert ring and M be a finitely generated R-module. By Corollary 3.6, $R \ltimes M$ is Hilbert, while [3, Theorem 4.8] shows that it is not Noetherian. It follows from Lemma 2.4(1) that each maximal ideal of $R \ltimes M$ is finitely generated. Therefore $R \ltimes M$ is an unruly Hilbert ring.

Example 4.2 Let R be an unruly Hilbert ring with $0 \neq a \in \mathrm{Nil}(R)$. Then $R \bowtie \langle a \rangle$ is an unruly Hilbert ring, by [9, Proposition 5.6], Theorem 3.3, and Lemma 2.4(1).

To construct unruly Hilbert rings, one should consider the following questions:

- When is $R \bowtie^f J$ a Noetherian ring?
- When is $R \bowtie^f J$ a Hilbert ring?
- When are all maximal ideals of $R \bowtie^f J$ finitely generated?

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The first question is completely answered in [9, Proposition 5.6] and the second one in Theorem 3.3 of this paper. The third question has a partial answer in Lemma 2.4. Before we proceed with new examples, it is worth to notice that the key for further examples via amalgamation is deepening of the answers for third question. In view of Theorem 3.3 and [9, Proposition 5.6], one needs to answer the third question when R is Noetherian and $f(R)+J$ *is not*. The next example is a special case of [17, Corollary 7].

Example 4.3 $\mathbb{Z}+X\mathbb{Q}[X]$ is an unruly Hilbert domain. Indeed, if we set $R=\mathbb{Z}$, $S=\mathbb{Q}[X]$, $J=X\mathbb{Q}[X]$, and f be the natural inclusion, then $R\bowtie^f J\cong\mathbb{Z}+X\mathbb{Q}[X]$. Note that, by Corollary 3.5 (or Corollary 3.10), $\mathbb{Z}+X\mathbb{Q}[X]$ is a Hilbert domain, while it follows from [9, Corollary 5.9] that it is not Noetherian. Finally, Lemma 2.4(2),(4) implies that each maximal ideal of $\mathbb{Z}+X\mathbb{Q}[X]$ is finitely generated (In fact, the maximal ideals are principal since $\mathbb{Z}+X\mathbb{Q}[X]$ is a Bézout domain). Therefore, $R\bowtie^f J$ is an unruly Hilbert domain.

References [15, 19] are devoted to construct examples of Hilbert rings with maximal ideals of different heights (1970s). Theorem 3.3 provides a *very easy* example with this property. The following example is one of the main results of this paper.

Example 4.4 Let k be a field and X, Y be indeterminates over k. Let $R=k[X,Y]$, $S=k[X]$, $J=\langle X^2\rangle$, and f be the natural surjection. Take $\mathfrak{m}=\langle X,Y\rangle$ and $\mathfrak{n}=\langle X-1\rangle$. Then, by Theorem 3.3, $R\bowtie^f J$ is a Hilbert ring (of dimension 2, by [10, Proposition 4.1]). It is easy to see that $\mathrm{ht}(\overline{\mathfrak{n}}^f)=1$. Indeed, by Lemma 2.2, $\mathrm{ht}(\overline{\mathfrak{n}}^f)\leq 1$ and we have the sequence $\overline{0}^f\subsetneq\overline{\mathfrak{n}}^f$ of prime ideals which implies $\mathrm{ht}(\overline{\mathfrak{n}}^f)\geq 1$, as desired. Another use of Lemma 2.2 shows that $\mathrm{ht}(\mathfrak{m}'^f)=2$, since $0'^f\subsetneq\langle X\rangle'^f\subsetneq\mathfrak{m}'^f$ is a sequence of prime ideals of length 2.

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