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Indefinite Sturm–Liouville problems

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We study the spectrum of regular and singular Sturm–Liouville problems with real-valued coefficients and a weight function that changes sign. The self-adjoint boundary conditions may be regular or singular, separated or coupled. Sufficient conditions are found for (i) the spectrum to be real and unbounded below as well as above and (ii) the essential spectrum to be empty. Also found is an upper bound for the number of non-real eigenvalues. These results are achieved by studying the interplay between the indefinite problems (with weight function which changes sign) and the corresponding definite problems. Our approach relies heavily on operator theory of Krein space.

1. Introduction

We study the spectrum of Sturm–Liouville problems associated with the differential equation

$$-(py')' + qy = \lambda wy \quad \text{on } J, \tag{1.1}$$

where

$$J = (a, b), \quad -\infty \leqslant a < b \leqslant \infty,$$

and the coefficients satisfy the basic conditions

$$\frac{1}{p}, q, w \in L_{\text{loc}}(J, \mathbb{R}), \quad p > 0, \quad |w| > 0 \text{ a.e. on } J, \quad w \text{ changes sign on } J, \tag{1.2}$$

and suitable boundary conditions. The form of these boundary conditions depends on the classification of the endpoints a, b of the interval J as regular, or singular and, when singular, whether limit-point or limit-circle in the space

$$H = L^2(J, |w|) = \left\{ f : J \to \mathbb{C} : \int_J |f|^2 |w| < \infty \right\},$$

with inner product and norm given by

$$(f, g) = \int_J f\bar{g}|w|, \qquad \|f\|^2 = \int_J |f|^2 |w|.$$

For details, see [22].

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Our approach is based on the well-established theory based on the equation

$$-(py')' + qy = \lambda |w| y \quad \text{on } J. \tag{1.3}$$

This is the so-called right-definite case, which can be studied using operator theory in the Hilbert space H. Let S be a self-adjoint realization of (1.3) in H, i.e.

$$S_{\min} \subseteq S = S^* \subseteq S_{\max},$$

where $S_{\min}$, $S_{\max}$ are the minimal and maximal operators associated with (1.3), respectively. Let $D(S)$ denote the domain of such an operator S. We refer to such domains as self-adjoint domains in H. If both endpoints are in the limit-point case in H, then $S_{\min} = S = S^* = S_{\max}$, and there is no proper self-adjoint restriction of $S_{\max}$ and hence there are no boundary conditions. In all other cases, the operators S are determined by restricting the domain $D(S_{\max})$ of $S_{\max}$ with self-adjoint boundary conditions. The theory of self-adjoint operators in H yields a characterization of all self-adjoint domains. For details, see [16, 22].

To 'transfer' results from the right-definite theory with weight function $|w|$ to corresponding problems for the indefinite weight function w, we use results and methods of operator theory in the Krein space $K = L^2(J, w)$. This is the space of all (equivalence classes) of functions from H, but with the indefinite inner product

$$[f, g] = \int_a^b f\bar{g}w, \quad f, g \in H. \tag{1.4}$$

Thus the Hilbert space H and the Krein space K consist of the same set of elements, but have different inner products. Operators between these spaces are 'connected' by means of the so-called fundamental symmetry operator $\mathcal{J}$, defined by

$$(\mathcal{J}f)(t) = f(t)\,\mathrm{sgn}(w(t)), \quad f \in H. \tag{1.5}$$

We are particularly interested in the self-adjoint realizations T of (1.1) in the Krein space K. These are given by

$$T = \mathcal{J}S, \tag{1.6}$$

where S is a self-adjoint realization of (1.3). Equation (1.6) determines a one-to-one onto correspondence between the self-adjoint realizations of (1.3) in the Hilbert space H and the self-adjoint realizations of (1.1) in the Krein space K. Therefore, the self-adjoint boundary conditions that determine S, obtained from the right-definite theory in H involving $|w|$, are precisely the same boundary conditions that determine T in the Krein space K involving w.

Note that the operators T given by (1.6) map H into H and thus are operators in this space. But, as an operator in H, T is not self-adjoint or even symmetric. However, observe that the spectrum of T considered as an operator in H is the same as the spectrum of T considered as an operator in K; this follows directly from the definition of the spectrum and from the fact that the topology of K is the same as that of H, both being generated by the norm of H. *Thus, by the spectrum of the Sturm–Liouville problem consisting of (1.1) together with the boundary conditions that determine S, and therefore also T, we mean the spectrum of T. Note that this reduces to the right-definite case for positive weight functions.*

How is the spectrum of S related to the spectrum of T? This is the question we study in this paper. There seems to be no simple answer to this question. The spectrum of S is real, the spectrum of T may not be. In 1918, Richardson [19] showed that, even in the regular case, there are such problems with non-real eigenvalues.

The study of the existence and number of non-real eigenvalues is our primary interest in this paper. Our first theorem identifies a class of problems that have only real spectrum.

THEOREM 1.1. *Let the operators S and T be defined as above and assume that $\inf(\sigma(S)) > 0$. Suppose there exist (non-degenerate) subintervals J_+, J_- of J such that w is positive a.e. on J_+ and negative a.e. on J_-. Then $\sigma(T)$ is real and is unbounded above as well as below.*

Proof. To be given in § 5. □

The unboundedness of the spectrum, above and below, when $w > 0$ and p changes sign, was established by Möller in [15].

There is an extensive literature on regular problems satisfying the hypotheses of theorem 1.1; these are called left-definite. For some recent results, see [2, 3, 5, 10, 20, 21]. More information can be obtained from the references of these papers.

Our second main result gives conditions for the essential spectrum to be real and gives an upper bound for the number of non-real eigenvalues. Recall that a closed operator in Banach spaces is called a Fredholm operator if its null space is finite dimensional and its range has finite codimension (this implies that its range is closed). The essential spectrum of a closed operator T in a Banach space, denoted by $\sigma_{\mathrm{e}}(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not a Fredholm operator, where I denotes the identity operator.

THEOREM 1.2. *Let the operators S and T be defined as above and assume that, for some $\varepsilon > 0$, there are exactly m points of $\sigma(S)$, counting multiplicity, to the left of ε, for some $m \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. Then the essential spectrum of T is real and T has at most $2m$ non-real eigenvalues, counting multiplicity.*

Proof. This theorem is a special case of the abstract theorem 3.3 below. □

Since the coefficients p, q, w are real, the non-real eigenvalues occur in conjugate pairs.

We do not know if the essential spectrum of T is always real. However, under an additional condition, we can give an affirmative answer.

THEOREM 1.3. *Let the operators S and T be defined as above. Assume that J is the union of finitely many intervals $J_1, \dots, J_n$ such that w does not change sign on J_k for $k = 1, \dots, n$. Then the essential spectrum of T is real.*

Proof. To be given in § 5. □

The next result shows that there is a close relationship between the discreteness of the spectra of S and T.

THEOREM 1.4. *Let the operators S and T be defined as above. If the essential spectrum of S, $\sigma_{\mathrm{e}}(S)$, is empty, then either $\sigma_{\mathrm{e}}(T)$ is empty or consists of the entire complex plane. In particular, $\sigma_{\mathrm{e}}(S)$ is empty when each endpoint is either regular or limit-circle.*

Proof. The first part follows directly from proposition 3.4 below. The last sentence follows from this proposition and the well-known fact (see [16, § 24]) that $\sigma_e(S)$ is empty, i.e. the spectrum is discrete, when each endpoint is either regular or limit-circle. □

For regular problems with separated boundary conditions, the upper bound of the number of non-real eigenvalues given by theorem 1.2 is well known (see [5, 13]).

Our proofs are heavily dependent on the Krein-space-theory approach. A detailed treatment of such an approach has been undertaken by Čurgus and Langer in the fundamental paper [5]. This is based on Langer's theory of definitizable operators in Krein spaces and a deep understanding of the spectral theory of self-adjoint operators is required. Čurgus and Langer are mainly interested in the spectral resolution and characterizations of singular critical points. As a byproduct, they have some information about non-real eigenvalues.

Here, since we are mainly interested in the non-real spectrum, we give a more direct description of the underlying Krein-space theory. Our restriction to this particular topic leads to a treatment that only uses basic Krein-space theory. Apart from the right Krein space, which is the space $L^2(J, w)$, we also construct 'a left Krein space', which reduces to the Hilbert space $D(S^{1/2})$ for strictly positive-definite S. We show that, under certain assumptions, which correspond to the conditions in [5], that the quadratic form $[Tf, f]$ has only finitely many negative squares, the left Krein space reduces to a Pontryagin space.

For the convenience of the reader, we present the basic definitions and facts on Krein and Pontryagin spaces. For more in-depth results, we refer the interested reader to the monographs of Bognar [4] and Azizov and Iohvidov [1]. Most of the results on Krein spaces established here are known to specialists in this area. However, we feel it is appropriate to present these results in as simple and self-contained a manner as possible to readers mainly interested in differential equations. In this regard, we observe that, in [5], Čurgus and Langer start with a study of minimal operators in Krein space and develop the theory of self-adjoint extensions of these—in parallel with the Hilbert-space theory. However, since boundary conditions are self-adjoint for the weight w if and only if they are self-adjoint for the weight $|w|$, nothing is lost by using the one-to-one onto correspondence given by (1.6). Thus the reader can go directly from the self-adjoint, regular or singular, separated or coupled, boundary conditions obtained from the Hilbert-space theory for $|w|$ to the same boundary conditions for w.

The characterization of $(S^{1/2})$ used below is due to Krein [11, 12], but since this work is not easily accessible, we give a self-contained proof of it here.

This paper is organized as follows. Following this introduction, § 2 gives a short introduction to Krein spaces, and § 3 contains an overview of self-adjoint operators in Krein spaces and their spectra. In § 4, a left Krein space is constructed, which is related to the 'left-definite Hilbert-space theory'. Section 5 contains proofs of theorems 1.1 and 1.3.

2. Krein spaces

In this section, we give basic definitions and results on Krein spaces. For more details, we refer the interested reader to [1, 4, 8].

For a linear operator T, we use $D(T)$, $R(T)$ and $N(T)$ to denote its domain, range and null-space, respectively.

A space $(K, [\cdot,\cdot])$ is called a Krein space if K is a vector space and

$$[\cdot,\cdot] : K \times K \to \mathbb{C} \tag{2.1}$$

is a sesquilinear form such that there are linear submanifolds K_+ and K_- with $K = K_+ \dotplus K_-$ (algebraic direct sum) such that $(K_+, [\cdot,\cdot])$ as well as $(K_-, -[\cdot,\cdot])$ are Hilbert spaces and $[K_+, K_-] = \{0\}$, i.e. $[f, g] = 0$ for all $f \in K_+$ and $g \in K_-$. We write $K = K_+ \oplus K_-$ and call this a fundamental decomposition of the Krein space K. Fundamental decompositions are not unique.

Note that $[f, f] \in \mathbb{R}$ for any $f \in K$. Unless $K_+ = \{0\}$ or $K_- = \{0\}$, there are elements $f \in K \setminus \{0\}$ such that $[f, f] = 0$. Such an element f is called neutral.

If K_- is finite dimensional, then the Krein space K is called a Pontryagin space and κ, the dimension of K_-, is called the index of this Pontryagin space. If K_+ is finite dimensional, then we can consider the Krein space $(K, -[\cdot,\cdot])$ instead, and it is therefore no restriction to assume that K_- is finite dimensional if at least one of K_+, K_- has this property.

The space $H = K$, equipped with the inner product

$$(f, g) = [f_+, g_+] - [f_-, g_-], \tag{2.2}$$

where $f, g \in K$, $f = f_+ + f_-$, $g = g_+ + g_-$, $f_+, g_+ \in K_+$, $f_-, g_- \in K_-$, is a direct sum of Hilbert spaces and therefore a Hilbert space itself, to be called the associated Hilbert space of K (with respect to the decomposition $K = K_+ \dotplus K_-$). The inner product of H is not unique, since it depends on the decomposition of K. The topology of the Krein space K is defined to be the topology generated by the norm of this Hilbert space H. It is this topology that determines the continuity of linear operators in K and their resolvent sets and hence their spectra.

The map $\mathcal{J}$ on K given by

$$\mathcal{J}f = f_+ - f_-, \quad f = f_+ + f_-, \quad f_+ \in K_+, \quad f_- \in K_-, \tag{2.3}$$

is linear and continuous and satisfies $\mathcal{J}^2 = I$, $\mathcal{J}^* = \mathcal{J}$, where $\mathcal{J}^*$ is the adjoint of $\mathcal{J}$ in the associated Hilbert space H. This map $\mathcal{J}$ is called a fundamental symmetry; it connects the Krein-space inner product $[\cdot,\cdot]$ with the Hilbert-space inner product $(\cdot,\cdot)$ by means of the formulae

$$[f, g] = (\mathcal{J}f, g), \quad (f, g) = [\mathcal{J}f, g], \quad f, g \in K. \tag{2.4}$$

Conversely, if $\mathcal{J}$ is a bounded self-adjoint linear operator on a Hilbert space H such that $\mathcal{J}^2 = I$, then, by the spectral theory of self-adjoint operators in Hilbert space, $[f, g] = (\mathcal{J}f, g)$ defines a Krein-space structure on H, $\mathcal{J}$ is a fundamental symmetry and $N(\mathcal{J} - I) \oplus N(\mathcal{J} + I)$ is its fundamental decomposition.

Of particular interest in this paper are the Hilbert and Krein spaces constructed as follows.

Let $J = (a, b)$, $-\infty \leqslant a < b \leqslant \infty$, be an interval and $w \in L_{\text{loc}}(J, \mathbb{R})$, i.e. w is a locally Lebesgue integrable real-valued function, with $w(t) \neq 0$ for almost all $t \in J$. Then it is well known that the set of (equivalence classes of) measurable functions

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f on J such that

$$\int_J |w||f|^2 < \infty$$

is a Hilbert space, denoted $L^2(J, |w|)$, with inner product

$$(f, g) = \int_J |w| f\bar{g}. \tag{2.5}$$

The linear operator $\mathcal{J}$, defined by

$$(\mathcal{J}f)(t) = \operatorname{sgn} w(t) f(t) = \frac{w(t)}{|w(t)|} f(t), \quad t \in J, \quad f \in H, \tag{2.6}$$

is continuous, self-adjoint and satisfies $\mathcal{J}^2 = I$. Hence, with respect to the inner product

$$[f, g] = \int_J w f\bar{g}, \tag{2.7}$$

$L^2(J, |w|)$ becomes a Krein space, which we denote by $L^2(J, w)$. Its fundamental symmetry $\mathcal{J}$ is generated by multiplication by $\operatorname{sgn}(w)$ and its fundamental decomposition is given by

$$L^2(J, w) = L^2(J_-, |w|) \oplus L^2(J_+, |w|), \tag{2.8}$$

where

$$J_\pm = \{t \in J : \pm w(t) > 0\}.$$

Note that $J_\pm$ is not necessarily an interval and need not even contain an interval.

3. Self-adjoint operators in Krein spaces

In this section, we discuss some of the basic theory of self-adjoint operators in a Krein space $(K, [\cdot,\cdot])$ with fundamental symmetry $\mathcal{J}$ and its associated Hilbert space $H = (K, (\cdot,\cdot))$. Let T be a densely defined linear operator from H into H and T^* its Hilbert-space adjoint. The Krein-space adjoint T^+ of T is defined by

$$[Tf, g] = [f, T^+ g] \tag{3.1}$$

for $f \in D(T)$ and $g \in D(T^+)$, where $D(T^+)$ is the set of all $g \in K$ such that the map $f \to [Tf, g]$ is continuous. Note that

$$T^+ = \mathcal{J} T^* \mathcal{J}.$$

A densely defined linear operator T in a Krein space is called Hermitian if $T \subset T^+$. As in the Hilbert-space case, this is equivalent to requiring that $[Tf, f] \in \mathbb{R}$ for all $f \in D(T)$. The operator T is called self-adjoint in the Krein space K if $T = T^+$. Note that T is self-adjoint in K if and only if $\mathcal{J}T$ is self-adjoint in H. Thus if T is self-adjoint in K, then it is closed and its resolvent set is given by

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is one-to-one and onto}\}.$$

This follows from the definition of the resolvent set and the closed graph theorem.

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In contrast to the Hilbert-space case, the spectrum of self-adjoint operators in Krein spaces is, in general, not real. But the points in the spectrum occur in complex conjugate pairs.

PROPOSITION 3.1. *If T is a self-adjoint operator in a Krein space, then its spectrum is symmetric with respect to the real axis.*

Proof. Since a closed linear operator is one-to-one onto if and only if its Hilbert-space adjoint has this property, it is clear that any of $T-\lambda$, $T^*-\bar{\lambda}$, $\mathcal{J}T^*\mathcal{J}-\bar{\lambda}=T^+-\bar{\lambda}=T-\bar{\lambda}$ being one-to-one onto implies that all these operators are one-to-one onto. Thus $\lambda\in\rho(T)$ if and only if $\bar{\lambda}\in\rho(T)$. Therefore, $\rho(T)$ is symmetric with respect to the real axis, and hence so is the spectrum of T, $\sigma(T)=\mathbb{C}\setminus\rho(T)$. □

Let $\mathbb{C}^+=\{\lambda\in\mathbb{C}:\operatorname{Im}\lambda>0\}$, $\mathbb{C}^-=\{\lambda\in\mathbb{C}:\operatorname{Im}\lambda<0\}$.

PROPOSITION 3.2. *Let T be a self-adjoint operator in a Krein space $(K,[\cdot,\cdot])$ and let L be a finite-dimensional subspace of K, which is invariant under the operator T, such that all the eigenvalues of the restriction of T to L are either in $\mathbb{C}^+$ or in $\mathbb{C}^-$. Then $[f,f]=0$ for all $f\in L$. In particular, $[f,f]=0$ for all eigenvectors f corresponding to non-real eigenvalues.*

Proof. We only establish the case for $\mathbb{C}^+$, since the case for $\mathbb{C}^-$ is similar. Let I_L denote the identity map on L and let $Q=T$ restricted to L. For any $r>0$ such that the eigenvalues of Q all are inside the semicircle in the upper half-plane of radius r centred at the origin, we have that

$$I_L=-\frac{1}{2\pi\mathrm{i}}\int_{-r}^{r}(Q-\alpha I)^{-1}\,\mathrm{d}\alpha+\frac{1}{2\pi}\int_0^{\pi}\left(I-\frac{\mathrm{e}^{-\mathrm{i}\varphi}}{r}Q\right)^{-1}\mathrm{d}\varphi.$$

Since

$$\frac{1}{2\pi}\int_0^{\pi}\left(I-\frac{\mathrm{e}^{-\mathrm{i}\varphi}}{r}Q\right)^{-1}\mathrm{d}\varphi\to\tfrac{1}{2}I_L$$

as $r\to\infty$, it follows that

$$\lim_{r\to\infty}-\frac{1}{2\pi\mathrm{i}}\int_{-r}^{r}(Q-\alpha I)^{-1}\,\mathrm{d}\alpha=\tfrac{1}{2}I_L.$$

Thus, for any $f\in L$, we have that

$$[f,f]=\left[\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}(Q-\alpha)^{-1}\,\mathrm{d}\alpha f,f\right]=\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}[(Q-\alpha)^{-1}f,f]\,\mathrm{d}\alpha.$$

Since T, and therefore Q, is symmetric, $[(Q-\alpha)^{-1}f,f]\in\mathbb{R}$ for all real α, the above identity can only hold if both sides are zero. □

THEOREM 3.3. *Let T be a self-adjoint operator in a Krein space with fundamental symmetry $\mathcal{J}$, and let $S=\mathcal{J}T$. Assume the spectrum of S is finite below some positive number ε, i.e. $\sigma(S)\cap(-\infty,\varepsilon)$ consists of at most a finite number of eigenvalues of finite multiplicity. Let m be the total multiplicity of these eigenvalues. (If there are no eigenvalues below ε, then $m=0$.) Then the essential spectrum of T (if any)*

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is real, the non-real part of the spectrum of T (if any) consists of finitely many non-real eigenvalues and the root subspaces of these eigenvalues have finite dimensions and their total dimension is at most $2m$. In particular, the non-real eigenvalues have finite multiplicities and their total multiplicity is at most $2m$.

Proof. First we show that the essential spectrum of T is real. In a manner similar to the proof of proposition 3.1, we establish that $T-\lambda$ is Fredholm if and only if $T-\bar{\lambda}$ is Fredholm and thus if and only if $N(T-\lambda)$ and $N(T-\bar{\lambda})$ are finite dimensional and $R(T-\lambda)$ and $R(T-\bar{\lambda})$ are closed, since the finite codimensionality of the last two spaces immediately follows from, for example,

$$R(T^*-\bar{\lambda}) = \overline{R(T^*-\bar{\lambda})} = N(T-\lambda)^{\perp} \quad \text{and} \quad R(T-\bar{\lambda}) = \mathcal{J}R(T^*-\bar{\lambda})\mathcal{J}.$$

Thus $\lambda \in \sigma_{\mathrm{e}}(T)$ if and only if $N(T-\lambda)$ or $N(T-\bar{\lambda})$ is infinite dimensional or $T-\lambda$ or $T-\bar{\lambda}$ not open, and hence there is an infinite-dimensional submanifold M of $D(T)$ such that either $\|(T-\lambda)f\| \leqslant \varepsilon\,\mathrm{Im}(\lambda)/3|\lambda|$ for all $f \in M$ with $\|f\| = 1$ or $\|(T-\bar{\lambda})f\| \leqslant \varepsilon\,\mathrm{Im}(\lambda)/3|\lambda|$ for all $f \in M$ with $\|f\| = 1$. Here and in the following, the norm is the one associated with the Hilbert-space inner product $(\cdot,\cdot)$.

Assume there is $\lambda \in \sigma_{\mathrm{e}}(T) \setminus \mathbb{R}$. Without loss of generality, we may assume the first case. Then

$$\mathrm{Im}[(T-\lambda)f,f] = -\mathrm{Im}\,\lambda[f,f]$$

and

$$|[(T-\lambda)f,f]| \leqslant \|\mathcal{J}(T-\lambda)f\|\|f\| = \|(T-\lambda)f\|\|f\|$$

yield

$$|[f,f]| \leqslant \frac{\varepsilon}{3|\lambda|} \quad \text{for } f \in M, \quad \|f\| = 1.$$

Then

$$(Sf,f) = [Tf,f] = [(T-\lambda)f,f] + \lambda[f,f] \leqslant \frac{\varepsilon|\,\mathrm{Im}\,\lambda|}{3|\lambda|} + \tfrac{1}{3}\varepsilon < \varepsilon$$

for all $f \in M$, with $\|f\| = 1$. But this contradicts the minimax principle, since $\sigma(S) \cap (-\infty,\varepsilon)$ consists of at most finitely many eigenvalues of finite multiplicity.

By proposition 3.2, the submanifold L spanned by all eigenvectors belonging to eigenvalues in the upper half-plane $\mathbb{C}^+$ consists of neutral elements. By the polarization formula for inner products, the inner product is identically zero on L. From $TL \subset L$, $[Tf,f] = 0$ for all $f \in L$ and the minimax principle, it follows that $\dim L \leqslant m$. Since the total multiplicity of an isolated eigenvalue λ of T coincides with the total multiplicity of the eigenvalue $\bar{\lambda}$ of $T^* = T$, it follows that the total multiplicity of all non-real eigenvalues is at most $2m$.

To show that there is no other non-real spectrum, we note that if $\lambda \in \sigma(T)\backslash\sigma_{\mathrm{e}}(T)$, then the observations at the beginning of this proof imply that $N(T-\lambda) \neq \{0\}$ or $N(T-\bar{\lambda}) \neq \{0\}$. But since there are only finitely many non-real eigenvalues, there is $\mu \in \mathbb{C} \setminus \mathbb{R}$ such that $N(T-\mu) = \{0\}$ and $N(T-\bar{\mu}) = \{0\}$. Thus $T-\mu$ and $T-\bar{\mu}$ have index 0 and, by the index stability theorem (see [9, theorem IV.5.22]), the index must be 0 in $\mathbb{C}^+$ and $\mathbb{C}^-$ and, more generally, on $\mathbb{C} \setminus \sigma_{\mathrm{e}}(T)$. So every point in $\sigma(T) \setminus \sigma_{\mathrm{e}}(T)$ must be an eigenvalue. □

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Proposition 3.4. *Let K be a Krein space, T a self-adjoint operator in K, $\mathcal{J}$ a fundamental symmetry on K and let $S = \mathcal{J}T$. Assume that $\rho(T)$, the resolvent set of T, is not empty. If the essential spectrum of S is empty, then the essential spectrum of T is empty.*

Proof. Since S is self-adjoint in a Hilbert space, $\mathbb{C} \setminus \mathbb{R} \subset \rho(S)$. Hence we can choose $\lambda \in \rho(T)$ and $\mu \in \rho(S)$. Since $\sigma_{\mathrm{e}}(S) = \emptyset$, the spectral theorem for self-adjoint operators shows that

$$S = \sum_{j=1}^{\infty} \lambda_j P_j,$$

with mutually orthogonal finite-rank projections P_j and $|\lambda_j| \to \infty$ as $j \to \infty$, and thus

$$(S - \mu)^{-1} = \sum_{j=1}^{\infty} (\lambda_j - \mu)^{-1} P_j$$

is compact. Hence, in view of $(T - \mu\mathcal{J})^{-1} = (S - \mu)^{-1}\mathcal{J}$,

$$\begin{aligned}(T - \mu\mathcal{J})^{-1} - (T - \lambda)^{-1} &= (T - \mu\mathcal{J})^{-1}(T - \lambda - T + \mu\mathcal{J})(T - \lambda)^{-1} \\ &= (T - \mu\mathcal{J})^{-1}(\mu\mathcal{J} - \lambda)(T - \lambda)^{-1}\end{aligned}$$

is compact, and hence also $(T - \lambda)^{-1}$ is compact. This completes the proof, since an operator with compact resolvent has no essential spectrum (see, for example, [9, theorem III.6.29]). □

The following example shows that $\rho(T) = \emptyset$ can occur even if $\sigma_{\mathrm{e}}(S) = \emptyset$.

Example 3.5. Let $Ly = y'$ on $L^2(0, 1)$, with domain

$$D(L) = H^1(0, 1) = \{y \in L^2(0, 1) : y \in AC_{\mathrm{loc}}(0, 1),\ y' \in L^2(0, 1)\}.$$

Then L is a closed operator with dense domain and

$$D(L^*) = \{y \in H^1(0, 1) : y(0) = 0 = y(1)\}.$$

Let $H = (H^1(0, 1))^2$ and let $\mathcal{J} : H \to H$ be given by

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where I is the identity operator on $L^2(0, 1)$. Then H becomes a Krein space K with fundamental symmetry $\mathcal{J}$. Define $T : K \to K$ by

$$T = \begin{pmatrix} L & 0 \\ 0 & L^* \end{pmatrix}.$$

Since

$$\mathcal{J}T = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}$$

is self-adjoint in H, T is self-adjoint in K. For each $\lambda \in \mathbb{C}$, $y(t) = \mathrm{e}^{\lambda t}$ is in $D(L)$ and satisfies $Ly = \lambda y$. Thus $\rho(T)$ is empty.

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On the other hand, $D(LL^*)$ is the set of all

$$y \in H^2(0,1) = \{y \in H^1(0,1) : y' \in H^1(0,1)\}$$

satisfying $y(0) = 0 = y(1)$, and $LL^*y = y''$. Thus LL^* is a regular self-adjoint Sturm–Liouville operator with Dirichlet boundary conditions and its spectrum is discrete, consisting entirely of simple isolated eigenvalues. By [14, theorem 1.2], $\sigma(\mathcal{J}T)$ is discrete and $\sigma_e(\mathcal{J}T) \subset \{0\}$. But

$$\dim N(\mathcal{J}T) = \dim N(T) = \dim N(L) = 1,$$

and consequently the essential spectrum of $\mathcal{J}T$ is empty.

4. A construction of left-definite Krein spaces

Given an invertible self-adjoint operator T in a Krein space K, we construct a new Krein space associated with T. This new space is called the left-definite Krein space associated with T and is denoted by K_T. We continue to use the notation from the previous sections.

Theorem 4.1. *Let T be an invertible self-adjoint operator in a Krein space with fundamental symmetry $\mathcal{J}$ and let $S = \mathcal{J}T$. Let $U = |S|^{1/2}$. Then there is an inner product $[\cdot,\cdot]_1$ on $D(U)$ such that $K_T = (D(U), [\cdot,\cdot]_1)$ is a Krein space with the following properties.*

(1) *$D(T)$ is dense in $D(U)$ with respect to the Hilbert-space norm generated by the inner product $[\cdot,\cdot]_1$.*

(2) *$[Tf, g] = [f, g]_1$ for all f, g in $D(T)$.*

(3) *$R = T^{-1}|_{D(U)}$ is a continuous self-adjoint operator in the Krein space K_T.*

Proof. It is known (see, for example, [9, theorem VI.6.23]) that $D(U)$ is a Hilbert space with respect to the inner product

$$(f, g)_1 = [\mathcal{J}Uf, Ug] = (Uf, Ug). \tag{4.1}$$

From the functional calculus for self-adjoint operators in Hilbert space, it follows that $\mathcal{J}_1 := \operatorname{sgn}(S)$ is unitary on the Hilbert space $D(U)$ and $\mathcal{J}_1^2 = I$. Thus

$$[f, g]_1 = (\operatorname{sgn}(S)f, g)_1 = (U \operatorname{sgn}(S)f, Ug) \tag{4.2}$$

defines a Krein-space structure on $D(U)$. Part (1) follows from the fact that $D(T)$ is dense on the Hilbert space $D(U)$ (see [9, theorem VI.2.23]).

(2) For $f, g \in D(T)$, from (4.2) it follows that

$$[f, g]_1 = (U \operatorname{sgn}(U)f, g) = [Tf, g]. \tag{4.3}$$

(3) The inclusions

$$T^{-1}D(U) \subset D(T) = D(|S|) \subset D(|S|^{1/2}) \tag{4.4}$$

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show that R maps $D(U)$ into itself. Also, the continuity of the embedding $K_T \hookrightarrow K$ shows that R has a closed graph. Thus R is continuous by the closed graph theorem. Let $f, g \in D(T)$. Then $T^{-1}f, T^{-1}g \in D(T)$ and, by (2),

$$[Rf, g]_1 = [TT^{-1}f, g] = [f, g] = \overline{[g, f]} = \overline{[Rg, f]_1} = [f, Rg]_1. \tag{4.5}$$

Hence R is symmetric on $D(T)$. Since R is continuous on $D(U)$ and $D(T)$ is dense in $D(U)$, R is self-adjoint in $D(U)$. □

In the following, $\|\cdot\|$ and $\|\cdot\|_1$ denote the norms associated with the Hilbert-space inner products on K and on $D(U)$, respectively.

THEOREM 4.2. *Let the notation and hypotheses of theorem 4.1 hold. Then, for each $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda \in \sigma(T)$ if and only if $1/\lambda \in \sigma(R)$.*

Proof. Let $\lambda \neq 0$ be an eigenvalue of T. Then there is a $y \in D(T)$, $y \neq 0$, such that $Ty = \lambda y$. Then $y, Ty \in D(T) \subseteq D(R)$ and $y = \lambda Ry$. Thus $1/\lambda$ is an eigenvalue of R.

Conversely, let $\lambda \neq 0$ be an eigenvalue of R. Then there is $y \in D(U)$, $y \neq 0$, such that $Ry = \lambda y$. Then $y, Ry \in R(R) \subseteq D(T)$ and $y = \lambda Ty$. Thus $1/\lambda$ is an eigenvalue of T.

Now, let $\lambda \in \rho(T) \setminus \{0\}$. By what we have already shown, $R - 1/\lambda$ is injective. Let $f \in D(U)$. Since $T - \lambda$ is surjective, there is a $g \in D(T) \subseteq D(U)$ such that $(T - \lambda)g = -\lambda Tf$, which implies that

$$\left(R - \frac{1}{\lambda}\right)g = \left(T^{-1} - \frac{1}{\lambda}\right)g = f. \tag{4.6}$$

This shows that $R - 1/\lambda$ is also surjective, and hence $1/\lambda \in \rho(R)$.

Finally, let $\lambda \in \rho(R) \setminus \{0\}$. From the first part of the proof, it follows that $T - 1/\lambda$ is injective. Let $g \in \mathcal{K}$ and put $h = -\lambda T^{-1}g \in D(T)$. Since $R - \lambda$ is surjective, there is an $f \in D(R)$ such that $(R - \lambda)f = h$. Since $f = (Rf - h)/\lambda \in D(T)$, we have that

$$Tf = \frac{1}{\lambda}f - \frac{1}{\lambda}Th = \frac{1}{\lambda}f + g. \tag{4.7}$$

This shows that $T - 1/\lambda$ is also surjective and hence $1/\lambda \in \rho(T)$. □

The importance of theorem 4.2 is due to the fact that $K_T = (D(U), [\cdot, \cdot]_1)$ may be a Pontryagin space or a Hilbert space.

COROLLARY 4.3. *Under the assumptions and with the notation of theorem 4.1, suppose that $\sigma(S) \cap (-\infty, 0)$ consists of at most finitely many negative eigenvalues of finite multiplicity. Let $m \geqslant 0$ be the total multiplicity of these negative eigenvalues. Then K_T is a Pontryagin space with Pontryagin index m, and $\sigma(T) \setminus \mathbb{R}$ consists of at most finitely many eigenvalues of total multiplicity not exceeding $2m$.*

Proof. Let $(E_t)_{t \in \mathbb{R}}$ be the spectral family of S. Then

$$(I - E_0)D(U) \oplus E_0 D(U) \tag{4.8}$$

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is a fundamental decomposition of the Krein space K_T, and $E_0D(U) = R(E_0)$ has dimension m. Since $0 \notin \sigma(S)$, there is $\varepsilon > 0$ such that

$$\sigma(S) \cap (-\infty, \varepsilon) = \sigma(S) \cap (-\infty, 0)$$

consists of finitely many eigenvalues of finite multiplicity. By theorem 3.3, $\sigma(T) \setminus \mathbb{R}$ consists of at most finitely many eigenvalues with total multiplicity not exceeding $2m$. A routine modification of the proof of theorem 4.2 yields that the multiplicities of the eigenvalues $1/\lambda \in \sigma(S)$ and $\lambda \in \sigma(T)$ coincide. □

REMARK 4.4. It would be desirable to have an estimate of the magnitude of the non-real eigenvalues of T, or at least of their imaginary parts, in terms of the negative eigenvalues of S. The following example shows that, in general, such an estimate does not exist.

EXAMPLE 4.5. For $c > 0$, $d > 0$, let

$$T = \begin{pmatrix} c & c+d \\ -(c+d) & -c \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on $\mathbb{C}^2$. Then $S = \mathcal{J}T$ is self-adjoint with eigenvalues $2c + d$, and $-d$, whereas the eigenvalues of T are $\pm \mathrm{i}\sqrt{2cd + d^2}$. Note that, for fixed d and c varying in $(0, +\infty)$, the negative eigenvalue of S is bounded, but the imaginary parts of the non-real eigenvalues of T are not bounded.

5. Proof of theorems 1.1 and 1.3

We now proceed with the proof of theorem 1.1, using the notation from previous sections.

Proof of theorem 1.1. In view of corollary 4.3, $\sigma(T)$ is real and the space K_T from theorem 4.1 is a Hilbert space. In terms of the operator R from theorems 4.1 and 4.2, we have to show that $\sigma(R) \cap (-\varepsilon, 0) \neq \emptyset$ and $\sigma(R) \cap (0, \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. Assume there is some $\varepsilon > 0$ such that $\sigma(R) \cap (-\varepsilon, 0) = \emptyset$ or $\sigma(R) \cap (0, \varepsilon) = \emptyset$. This means that, for $\lambda = -2/\varepsilon$ or $\lambda = 2/\varepsilon$, the estimate

$$\left\| \left(R - \frac{1}{\lambda} \right) f \right\|_1 \geqslant \frac{1}{|\lambda|} \|f\|_1$$

holds for all $f \in D(R)$. This is equivalent to

$$\left[\left(R - \frac{1}{\lambda} \right) f, \left(R - \frac{1}{\lambda} \right) f \right]_1 \geqslant \frac{1}{\lambda^2} [f, f]_1.$$

An application of part (2) of theorem 4.1 shows that

$$[(\lambda - T)f, (\lambda R - I)f] \geqslant [Tf, f]$$

for $f \in D(T)$. This leads to

$$\lambda^2 [f, Rf] - 2\lambda [f, f] \geqslant 0$$

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for all $f \in D(T)$. Since R is a restriction of T^{-1} and T^{-1} is invertible in $L^2(J, w)$, it follows, by continuity, that

$$\lambda^2[f, T^{-1}f] - 2\lambda[f, f] \geqslant 0$$

for all $f \in L^2(J, w)$. Putting $g = T^{-1}f$, it follows that

$$\lambda^2[Tg, g] - 2\lambda[Tg, Tg] \geqslant 0 \tag{5.1}$$

for all $g \in D(T)$.

Now let T_+ be the minimal operator associated with (1.1) on $L^2(J_+, w)$. Note that $L^2(J_+, w)$ is a Hilbert space and that T_+ is unbounded above. Hence there is $y \in D(T_+) \subset D(T)$ such that $\|Ty\| = \|T_+y\| > \lambda\|y\|$ for $\lambda = 2/\varepsilon$. Then

$$\begin{aligned} \lambda^2[Ty, y] - 2\lambda[Ty, Ty] &\leqslant \lambda^2\|Ty\|\|y\| - 2\lambda\|Ty\|^2 \\ &= \|Ty\|(\lambda^2\|y\| - 2\lambda\|Ty\|) \\ &< -\lambda^2\|Ty\|\|y\| \\ &< 0, \end{aligned}$$

which contradicts (5.1) for $\lambda = 2/\varepsilon$.

Let T_- be the minimal operator associated with (1.1) on $L^2(J_-, -w)$. Note that $L^2(J_+, -w)$ is a Hilbert space and that T_- is unbounded below. Hence there is $y \in D(T_-) \subset D(T)$ such that $\|Ty\| = \|T_-y\| > -\lambda\|y\|$ for $\lambda = -2/\varepsilon$. Then

$$\begin{aligned} \lambda^2[Ty, y] - 2\lambda[Ty, Ty] &\leqslant \lambda^2\|Ty\|\|y\| + 2\lambda\|Ty\|^2 \\ &= \|Ty\|(\lambda^2\|y\| + 2\lambda\|Ty\|) \\ &< -\lambda^2\|Ty\|\|y\| \\ &< 0, \end{aligned}$$

which contradicts (5.1) for $\lambda = -2/\varepsilon$. □

Proof of theorem 1.3. Let T_k, $k = 1, \dots, n$, be the minimal operators associated with the restriction of (1.1) to J_k. Then the direct sum $T_1 \oplus \cdots \oplus T_n$ on $L^2(J_1, w) \oplus \cdots \oplus L^2(J_n, w)$ can be identified with an operator $\tilde{T} \subset T$. Since each of the T_j is a minimal operator in the Hilbert space $L^2(J_k, |w|)$ (we may replace λ with $-\lambda$, if necessary), $\sigma_{\mathrm{e}}(T_j) \subset \mathbb{R}$. That is, $T_k - \lambda$ is a Fredholm operator for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Thus $(T_1 - \lambda) \oplus \cdots \oplus (T_n - \lambda)$ is also Fredholm, and so is $\tilde{T} - \lambda$. Since $\tilde{T} \subset T$, $T - \lambda$ has a finite-codimensional, and thus closed, range for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (see [7, IV.1.3]). This, together with $N(T - \lambda) = R(T - \bar{\lambda})^{\perp}$, shows that $T - \lambda$ is a Fredholm operator for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. □

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