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BOOK REVIEW

Random Dynamical Systems

By LUDWIG ARNOLD
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Evolution by its nature is time dependent so physical systems can be better described by compositions of different maps rather than by repeated application of exactly the same transformation. It is natural to study such problems for typical, in some sense, sequences of maps which are picked at random in a stationary fashion. This leads directly to the framework of random dynamical systems (RDS). The most versatile definition of RDS which works equally well both in the discrete and continuous time cases employs the notion of cocycles. Namely, given a P-preserving dynamical system ϑ^t on a probability space $(\Omega, \mathcal{F}, P)$ with discrete or continuous time t, a measurable family $\varphi = \varphi(t, \omega)$ of transformations of a space X is called a cocycle if

$$\varphi(t+s, \omega) = \varphi(t, \vartheta^s \omega) \circ \varphi(s, \omega), \tag{1}$$

where $\circ$ denotes a corresponding composition of maps. The probabilistic state of mind requires here to assume as little as possible about the probability space $(\Omega, \mathcal{F}, P)$ but, on the other hand, $\varphi(t, \omega)$ may act on spaces with rich structures as continuous or smooth maps and may be determined by random or stochastic difference or differential equations. As cocycles pass as a guideline through Arnold's entire book it deserves a subtitle: Saga on Cocycles.

Random transformations were discussed already by Ulam and von Neumann [**UN**], but only in the 1980s did this subject receive a real push when stochastic flows, emerging as solutions of stochastic differential equations, provided a rich source of diffeomorphism valued cocycles (see Kunita [**Ku**]). This brought into probabilists minds such smooth dynamical systems notions as Lyapunov exponents, invariant manifolds, bifurcations etc., which had to be adapted to the random framework. It turned out that the relativized entropy, which emerged in ergodic theory earlier, plays an important role here, in particular, enabling one to extend Pesin's theory to this setup (see Liu and Qian [**LQ**]).

Arnold's book is very welcome as a reliable and rather complete source of reference for topics it contains and it lays the foundation for future work and applications in the fast developing field of RDS. The subject has achieved a certain maturity during the last

20 years and a book providing a comprehensive exposition of the accumulated knowledge spread throughout various sources became necessary. The book does not include topics related to the entropy theory of RDS, some parts of which have already appeared in [**Ki**].

In the continuous time case the basic cocycle equation (1) leads to subtleties since, usually, as in the stochastic flows case, (1) holds true for each fixed s and t but only P-almost surely (a.s.) with the null set depending either on both s and t (very crude cocycle) or just on s (crude cocycle). Several results on perfection of cocycles preserving continuity or smoothness appear in Chapter 1. This leads to helices which satisfy (1) identically. Chapter 1 also deals with another basic problem concerning existence of invariant measures of RDS, which are probability measures μ on $\Omega \times X$ with the fixed marginal P on Ω and whose disintegrations (factorizations) μ_ω satisfy

$$\varphi(t,\omega)\mu_\omega = \mu_{\vartheta^t\omega}, \quad P\text{-a.s.} \tag{2}$$

It turns out using a Krylov–Bogolyubov type construction that any continuous RDS acting on a compact metric space X possesses invariant measures and even, so called, Markov ones whose disintegrations depend only either on the past or on the future. The compactness arguments for the space of measures with a given marginal assuming nothing about the probability space $(\Omega, \mathcal{F}, P)$ are not as standard as in the deterministic case where Ω is just a point, but still they go through (see Crauel [**Cr**]).

Chapter 2 describes the ways of generation of RDS. In the discrete time case any RDS φ can be obtained as a composition of random maps $\varphi(n,\omega) = \psi(\vartheta^{n-1}\omega) \circ \cdots \circ \psi(\vartheta\omega) \circ \psi(\omega)$, $n \geq 1$ and, assuming the invertibility, $\varphi(n,\omega) = \psi(\vartheta^n\omega)^{-1} \circ \cdots \circ \psi(\vartheta^{-1}\omega)^{-1}$ for $n \leq -1$. In the continuous time case there are two substantially different sources of RDS: random differential equations $\dot{x}_t = f(\vartheta^t\omega, x_t)$ and stochastic differential equations $dx_t = f_0(x_t)\,dt + \sum_{j=1}^m f_j(x_t) \circ dW_t^j$, where $\circ$ denotes the Stratonovich differential. In both cases a RDS φ is defined by $\varphi(t,\omega)x = x_t$, $x_0 = x$, but in the first case the equation can be solved ω-wise and the cocycle property (1), smoothness and invertibility of solutions are rather standard facts from ordinary differential equations. In the second case, however, this is more subtle and its understanding at the beginning of the 1980s led to an explosion in the interest in RDS.

Part II of the book comprises Chapters 3–6 and deals exclusively with various aspects of the multiplicative ergodic theorem (MET) of Oseledets. This part being about one third of the book is the most comprehensive treatment of MET I known about and it clearly shows the author's fascination of this, one of most fundamental theorems in modern dynamical systems. Chapter 6 discusses the Furstenberg–Khasminskii type representations of Lyapunov exponents via integrals over projective bundles against certain measures which are invariant with respect to the action of corresponding RDS. This chapter also contains the theory developed by Arnold and San Martin of rotation numbers for RDS which complements MET by measuring rotation of matrix cocycles inside moving 2-planes.

Part III consisting of Chapters 7–9 deals with smooth RDS and it is the longest and most technically complicated part of the book. In Chapter 7 the random stable, unstable, and center manifolds theorems are treated with all details both in the discrete and continuous time cases. This chapter also contains a random version of the Hartman–Grobman theorem which provides a random topological conjugation of a smooth RDS to its linear part near a hyperbolic fixed point. This chapter could serve as a starting point for the exposition of random nonuniformly hyperbolic RDS, i.e. of a random version of Pesin's theory, or of random hyperbolic sets. Since the former topic appeared in Liu and Qian [**LQ**], the author decided not to include it here. A brief account on random hyperbolic sets and random Anosov diffeomorphisms can be found in [**GK**]. Considering a random Anosov diffeomorphism $\psi(\omega)$ on a torus it would be natural to extend the random Hartman–Grobman theorem to the random Manning theorem (see, in the deterministic case, Katok and Hasselblatt [**KH**, §18.6]) which would provide a random topological conjugation

$$h(\vartheta\omega) \circ \psi(\omega) = A(\omega) \circ h(\omega), \tag{3}$$

where $A(\omega)$ is a random linear hyperbolic automorphism which belongs to the same homotopy class as $\psi(\omega)$.

Chapter 8 is devoted to the study of smooth linearizations of RDS near a fixed point. As in the deterministic case the only obstruction to a smooth random conjugation is due to resonances which are now defined via Lyapunov exponents of products of random matrices and not just via eigenvalues of a fixed matrix as in the deterministic case. The questions related to Chapters 7 and 8 appeared recently in Nguyen [**Ng**].

Chapter 9 deals with stochastic bifurcation theory 'which is still in its infancy and is not much more than a collection of (numerical) examples' as the author observes. There is a vast literature on the subject but, on the other hand, even basic definitions are not generally accepted. For instance, Definition 9.2.2 of a D-bifurcation point may produce too many of them in situations which do not look like bifurcations since, for example, in many chaotic dynamical systems any weak neighborhood of any invariant measure contains infinitely many ergodic ones. Probably, one should think only of invariant measures which disintegrate into Dirac ones and bifurcate to invariant measures with more complex support. Otherwise, remaining on safe ground, Definition 9.2.1 of an abstract bifurcation point based on random conjugations could be more preferable. Of course, the notion of P-bifurcations describing changes in the graph of a stationary density of the corresponding Markov process makes sense, though it has very little connection with RDS. This chapter contains few results and beautiful pictures and, as a whole, can be viewed as a warm invitation for further research on stochastic bifurcations, especially, taking into account their importance for applications. I have no doubts that this book will play an important role in further development of RDS.

Some of the topics which go beyond the scope of Arnold's book are presented in a volume called *Stochastic Dynamics* edited by Crauel and Gundlach (Springer, 1999, 440 pp., ISBN 0-387-98512-3) which is a Festschrift on the occasion of Arnold's 60th birthday and consists of papers written by invited speakers to the conference on RDS held in Bremen in Spring 1997. It contains papers on stochastic bifurcations, numerical study of RDS, random hyperbolic systems, Lyapunov exponents, stochastic flows, infinite-dimensional stochastic dynamics and others.

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