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Dynamical Diophantine approximation and shrinking targets for C^1 weakly conformal IFSs with overlaps

EDOUARD DAVIAUD

Université Paris-Est, LAMA (UMR 8050) UPEMLV, UPEC, CNRS, F-94010 Créteil, France

(e-mail: edouardaviaud@gmail.com)

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Abstract. In this article, we extend, with a great deal of generality, many results regarding the Hausdorff dimension of certain dynamical Diophantine coverings and shrinking target sets associated with a conformal iterated function system (IFS) previously established under the so-called open set condition. The novelty of the result we present is that it holds regardless of any separation assumption on the underlying IFS and thus extends to a large class of IFSs the previous results obtained by Beresnevitch and Velani [A mass transference principle and the Duffin–Schaeffer conjecture for Hausdorff measures. *Ann. of Math. (2)* **164**(3) (2006), 971–992] and by Barral and Seuret [The multifractal nature of heterogeneous sums of Dirac masses. *Math. Proc. Cambridge Philos. Soc.* **144**(3) (2008), 707–727]. Moreover, it will be established that if S is conformal and satisfies mild separation assumptions (which are, for instance, satisfied for any self-similar IFS on $\mathbb{R}$ with algebraic parameters, no exact overlaps and similarity dimension smaller than 1), then the classical result of Hill–Velani regarding the shrinking target problem associated with a conformal IFS satisfying the open set condition (and for which the Hausdorff measure was later computed by Allen and Barany [On the Hausdorff measure of shrinking target sets on self-conformal sets. *Mathematika* **67** (2021), 807–839]) can be extended.

Key words: Diophantine approximation approximations, fractal geometry, dynamical coverings

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1. *Introduction*

Estimating the Hausdorff dimension of points falling infinitely often in sets U_n having some algebraic or dynamical meaning is a question which arises naturally in Diophantine approximation as well as in dynamical systems. Given a metric space (X, d), a measurable

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mapping $T : X \to X$, and an ergodic probability measure μ, a classical question consists in estimating, for μ-typical points x, the Hausdorff dimension of points falling infinitely often in balls $B(T^n(x), \phi(n))$, centered in $T^n(x)$ and with radius $\phi(n)$. Such problems have been studied for instance in [1, 2, 10, 17, 23, 27, 30] and are called 'dynamical Diophantine approximation problems'.

Estimating these dimensions often relies on establishing mass transference principles for the ergodic probability measure μ. Given a sequence of balls $(B_n := B(x_n, r_n))_{n\in\mathbb{N}}$, these theorems usually aim at giving lower bounds for the dimension of sets of points of the form $\limsup_{n\to+\infty} U_n$, where $U_n \subset B_n$ (typically, $U_n = B_n^\delta = B(x_n, r_n^\delta)$), provided that the sequence of balls $(B_n)_{n\in\mathbb{N}}$ satisfies $\mu(\limsup_{n\to+\infty} B_n) = 1$.

Let $m \geq 2$ be an integer and $S = \{f_1, \ldots, f_m\}$ be a weakly conformal family of mC^1 contracting maps from $\mathbb{R}^d \to \mathbb{R}^d$ (see Definition 2.4). Denote by K the attractor of S, that is, the unique non-empty compact set satisfying $K = \bigcup_{i=1}^m f_i(K)$, $\Lambda = \{1, \ldots, m\}$, $\Lambda^* = \bigcup_{k\geq 0} \Lambda^k$, and, for $k \in \mathbb{N}$, $\underline{i} = (i_1, \ldots, i_k) \in \Lambda^k$, write $f_{\underline{i}} = f_{i_1} \circ \cdots \circ f_{i_k}$.

In this article, we prove that if $\dim_H(K) = \dim(S)$, where $\dim(S)$ is the conformality dimension, defined by Definition 2.8, then for any $x_0 \in K$, for any $\delta \geq 1$,

$$\dim_H \Big(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta)\Big) = \frac{\dim_H(K)}{\delta}. \tag{1}$$

In other words, the set of points x for which the orbit of x_0, $(f_{\underline{i}}(x_0))_{\underline{i}\in\Lambda^*}$, satisfies infinitely many often that $d(x, f_{\underline{i}}(x_0)) \leq |f_{\underline{i}}(K)|^\delta$, has dimension $\dim_H(K)/\delta$. We mention that this dimension result regarding this dynamical Diophantine approximation problem can be deduced from the mass transference principle [7] in the case where the iterated function system (IFS) is conformal and satisfies the open set condition. One emphasizes that the condition $\dim_H(K) = \dim(S)$ is much weaker than the open set condition. For instance, this condition is satisfied for self-similar systems in $\mathbb{R}$, as soon as Hochman's exponential separation condition (given by [24, Theorem 1.4]) is verified, which provides a large number of examples.

An other classical problem in Diophantine approximation on fractals is the so-called shrinking target problem, which was originally introduced by Hill and Velani in [23]. Consider $S = \{f_1, \ldots, f_m\}$ as a self-similar IFS satisfying the strong separation condition and let K be its attractor. It is classical that K can be viewed as the attractor of an expending map. This is done by defining $F : K \to K$ by setting

$$F(x) = f_i^{-1}(x) \quad \text{if } x \in f_i(K).$$

Let $\phi : \mathbb{N} \to \mathbb{R}^+$ be a mapping such that $\phi(n) \to 0$, then the shrinking target set associated with $x_0 \in K$, F, and ϕ is defined as

$$W(x_0, \phi) = \{z \in K : \ F^n(z) \in B(x_0, \phi(n)) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Writing $0 < c_i < 1$ as the contraction ratio of f_i, one can write

$$W(x_0, \phi) = \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), c_{\underline{i}}\phi(|\underline{i}|)).$$

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This naturally leads to further investigate the dimension of limsup sets generated by balls of the form $B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|\phi(|\underline{i}|))$. In the case where the IFS satisfies the open set condition, the dimension of such sets was computed by Hill and Velani in [23], and the Hausdorff measure by Allen and Barany [1]. In this article, we extend the dimension result established in [23] to a large class of overlapping conformal IFSs.

As an application of our approach, a complement of some results established in [2] are also given (see Theorem 3.3).

An important tool to establish equation (1) is Theorem 5.11, which is a mass transference principle for projection of quasi-Bernoulli measures (without assuming any separation condition on the underlying IFS) and strongly relies on the techniques developed in [10].

Before stating our main results, we make some general recalls about contracting iterated function systems. We also recall some known results in the case were the IFS is weakly conformal.

2. *Recalls on geometric measure theory and definition of weakly conformal IFSs*

Let us start with some notation.

Let $d \in \mathbb{N}$. For $x \in \mathbb{R}^d$, $r > 0$, $B(x, r)$ stands for the closed ball of $(\mathbb{R}^d, \| \ \|_\infty)$ of center x and radius r. Given a ball B, $|B|$ stands for the diameter of B. For $t \geq 0$, $\delta \in \mathbb{R}$, and $B = B(x, r)$, tB stands for $B(x, tr)$, that is, the ball with same center as B and radius multiplied by t, and the δ-contracted ball B^δ is defined by $B^\delta = B(x, r^\delta)$.

Given a set $E \subset \mathbb{R}^d$, E stands for the interior of the E, $\overline{E}$ its closure, and $\partial E = \overline{E} \setminus E$ its boundary. If E is a Borel subset of $\mathbb{R}^d$, its Borel σ-algebra is denoted by $\mathcal{B}(E)$.

Given a topological space X, the Borel σ-algebra of X is denoted $\mathcal{B}(X)$ and the space of probability measure on $\mathcal{B}(X)$ is denoted $\mathcal{M}(X)$.

The d-dimensional Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is denoted by $\mathcal{L}^d$.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\text{supp}(\mu) = \{x \in [0, 1] : \ \text{for all } r > 0, \ \mu(B(x, r)) > 0\}$ is the topological support of μ.

Given $E \subset \mathbb{R}^d$, $\dim_H(E)$ and $\dim_P(E)$ denote respectively the Hausdorff and the packing dimension of E.

2.1. *Dimension of measures and Hausdorff content.*

Definition 2.1. Let $\zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+$. Suppose that ζ is increasing in a neighborhood of 0 and $\zeta(0) = 0$. The Hausdorff outer measure at scale $t \in (0, +\infty]$ associated with the gauge ζ of a set E is defined by

$$\mathcal{H}^\zeta_t(E) = \inf\left\{\sum_{n\in\mathbb{N}} \zeta(|B_n|) : \ |B_n| \leq t, \ B_n \text{ closed ball and } E \subset \bigcup_{n\in\mathbb{N}} B_n\right\}. \qquad (2)$$

The Hausdorff measure associated with ζ of a set E is defined by

$$\mathcal{H}^\zeta(E) = \lim_{t\to 0^+} \mathcal{H}^\zeta_t(E). \qquad (3)$$

For $t \in (0, +\infty]$, $s \geq 0$, and $\zeta : x \mapsto x^s$, one simply uses the usual notation $\mathcal{H}^\zeta_t(E) = \mathcal{H}^s_t(E)$ and $\mathcal{H}^\zeta(E) = \mathcal{H}^s(E)$, and these measures are called s-dimensional Hausdorff outer measure at scale $t \in (0, +\infty]$ and s-dimensional Hausdorff measure, respectively. Thus,

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$$\mathcal{H}^s_t(E) = \inf\left\{\sum_{n\in\mathbb{N}} |B_n|^s : |B_n| \leq t, \ B_n \text{ closed ball and } E \subset \bigcup_{n\in\mathbb{N}} B_n\right\}. \tag{4}$$

The quantity $\mathcal{H}^s_\infty(E)$ (obtained for $t = +\infty$) is called the s-dimensional Hausdorff content of the set E.

Definition 2.2. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. For $x \in \text{supp}(\mu)$, the lower and upper local dimensions of μ at x are defined as

$$\underline{\dim}_{\text{loc}}(\mu, x) = \liminf_{r\to 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)} \quad \text{and} \quad \overline{\dim}_{\text{loc}}(\mu, x) = \limsup_{r\to 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)}.$$

Then, the lower and upper Hausdorff dimensions of μ are defined by

$$\underline{\dim}_H(\mu) = \text{ess inf}_\mu(\underline{\dim}_{\text{loc}}(\mu, x)) \quad \text{and} \quad \overline{\dim}_P(\mu) = \text{ess sup}_\mu(\overline{\dim}_{\text{loc}}(\mu, x)), \tag{5}$$

respectively.

It is known (for more details, see [16]) that

$$\underline{\dim}_H(\mu) = \inf\{\dim_H(E) : E \in \mathcal{B}(\mathbb{R}^d), \ \mu(E) > 0\},$$
$$\overline{\dim}_P(\mu) = \inf\{\dim_P(E) : E \in \mathcal{B}(\mathbb{R}^d), \ \mu(E) = 1\}.$$

When $\underline{\dim}_H(\mu) = \overline{\dim}_P(\mu)$, this common value is simply denoted by $\dim(\mu)$ and μ is said to be *exact dimensional*.

2.2. *Weakly conformal IFS.*

2.2.1. *Generalities about contracting IFS.* Let $m \geq 2$ be an integer. An IFS is a set $S = \{f_1, \ldots, f_m\}$ of mappings $f_i : X \to X$, where $X \subset \mathbb{R}^d$ is a closed set. Moreover, one says that f is differentiable on X if there exists an open set $U \supset X$ on which f is differentiable.

Given an open set $U \subset \mathbb{R}^d$ and $f : U \to \mathbb{R}^d$ a differentiable map, for any $x \in U$:

- $f'(x)$ is the differential of f at x;
- let $k \in \mathbb{N}$ be an integer, we write $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^d)$ the set of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^d$. For $\ell \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^d)$, one denotes

$$\|\ell\| = \max_{x\in\mathbb{R}^k\neq 0} \frac{\|\ell(x)\|_\infty}{\|x\|_\infty} \quad \text{and} \quad \underline{\|\ell\|} = \min_{x\in\mathbb{R}^k\neq 0} \frac{\|\ell(x)\|_\infty}{\|x\|_\infty}. \tag{6}$$

Let us recall the following result.

PROPOSITION 2.3. (Hutchinson [25]) *Let $m \geq 2$ be an integer, $X \subset \mathbb{R}^d$ a closed set, and $S = \{f_1, \ldots, f_m\}$ a system of C^1 maps from X to X. Assume that S is uniformly contracting, i.e.,*

$$\max_{1\leq i\leq m} \sup_{x\in X} \|f_i'(x)\| < 1.$$

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Then, there exists a unique non-empty compact set K satisfying

$$K = \bigcup_{1\leq i\leq m} f_i(K).$$

Moreover, for any $(p_1, \ldots, p_m) \in (0, 1)^m$*, there exists a unique measure* $\mu \in \mathcal{M}(\mathbb{R}^d)$ *supported on K satisfying*

$$\mu = \sum_{1\leq i\leq m} p_i \mu(f_i^{-1}(\cdot)). \tag{7}$$

From now on, an IFS designates a uniformly contracting system of C^1 *maps.*

The following notation is used throughout the manuscript.

- $\Lambda(S) = \{1, \ldots, m\}$ and $\Lambda(S)^* = \bigcup_{k\geq 0} \Lambda(S)^k$. When there is no ambiguity on the system S involved, one simply writes $\Lambda(S) = \Lambda$.
- K_S denotes the attractor of S (or simply K when the context is clear).
- For $\underline{i} = (i_1, \ldots, i_k) \in \Lambda^k$, the cylinder $[\underline{i}]$ is defined by

  $$[\underline{i}] = \{(i_1, \ldots, i_k, x_1, x_2, \ldots) : (x_1, x_2, \ldots) \in \Lambda^{\mathbb{N}}\}.$$

  Moreover, if $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of real numbers, one sets

  $$\alpha_{\underline{i}} = \alpha_{i_1} \times \cdots \times \alpha_{i_k}$$

  and

  $$f_{\underline{i}} = f_{i_1} \circ \cdots \circ f_{i_k}.$$

  For example, given the probability vector $(p_1, \ldots, p_m)$, $p_{\underline{i}} = p_{i_1} \times \cdots \times p_{i_k}$.
- The set $\Lambda^{\mathbb{N}}$ is endowed with the topology generated by the cylinders. The set of probability measures on the Borel sets with respect to this topology is denoted $\mathcal{M}(\Lambda^{\mathbb{N}})$.
- The shift operator $\sigma : \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ is defined for any $(i_1, i_2, \ldots) \in \Lambda^{\mathbb{N}}$ by

  $$\sigma((i_1, i_2, \ldots)) = (i_2, i_3, \ldots). \tag{8}$$

- The canonical projection of $\Lambda^{\mathbb{N}}$ on K will be denoted π_Λ (or simply π when there is no ambiguity) and, fixing any $x \in K$, is defined, for any $(i_1, i_2, \ldots .) \in \Lambda^{\mathbb{N}}$, by

  $$K \ni \pi((i_1, \ldots)) = \lim_{k\to+\infty} f_{i_1} \circ \cdots \circ f_{i_k}(x). \tag{9}$$

  It is easily verified that π is independent of the choice of x.

2.2.2. *Weakly conformal IFS and pressure function associated with weakly conformal IFSs.* Let us recall the definition of a weakly conformal map.

Definition 2.4. [19] Let $m \geq 2$ be an integer, $U \subset \mathbb{R}^d$ an open set, $S = \{f_i\}_{i=1}^m$, where $f_i : U \to U$ is a C^1 contraction and K the attractor of S.

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One says that S is weakly conformal when, recalling equation (6),

$$\lim_{k\to+\infty} \frac{\sup_{(x_i)_{i\in\mathbb{N}}\in\{1,\ldots,m\}^{\mathbb{N}}}(\log \|f'_{(x_1,\ldots,x_k)}(\pi(\sigma^k(x)))\| - \log \|f'_{(x_1,\ldots,x_k)}(\pi(\sigma^k(x)))\|)}{k} = 0. \tag{10}$$

In this case, a measure defined by equation (7) is called a weakly conformal measure.

Example 2.5.

- If the maps $f_1, \ldots, f_m$ are affine similarities or conformal maps (i.e., verify $\|f'(x)(y)\| = \|f'(x)\| \cdot \|y\|$ for every $x \in U$, $y \in \mathbb{R}^d$), the system $S = \{f_1, \ldots, f_m\}$ is weakly conformal. In this case, the IFS is called self-similar or self-conformal and the measures satisfying equation (7) are respectively called self-similar and self-conformal measures. Note that this class of IFSs contains, for instance, every system of holomorphic contracting mappings.
- Assume that for any $1 \leq i \leq m$, $f_i : \mathbb{R}^d \to \mathbb{R}^d$ is defined by $f_i(x) = A_i x + b_i$, where for any $1 \leq i \leq m$, $b_i \in \mathbb{R}^d$ and $A_i \in GL_d(\mathbb{R})$ has its eigenvalues equal in modulus to $0 < r_i < 1$, and for any $1 \leq i, j \leq m$, $A_i A_j = A_j A_i$. Then, $S = \{f_1, \ldots, f_m\}$ is weakly conformal.

The pressure function associated with a weakly conformal IFS is naturally related to the dimension of the attractor. It is defined by the following proposition.

PROPOSITION 2.6. *Let $m \geq 2$ be an integer, $S = \{f_1, \ldots, f_m\}$ be a C^1 weakly conformal IFS, and let K be its attractor.*

Let us fix $s \geq 0$ and $z \in K$. The following quantity is well defined and independent of the choice of z:

$$P_z(s) = \lim_{k\to+\infty} \frac{1}{k} \log \sum_{\underline{i}\in\Lambda^k} \|f'_{\underline{i}}(z)\|^s. \tag{11}$$

Remark 2.7. Let us mention that the proof of Proposition 2.6 is very standard and does not diverge much from the standard proofs made in [**8**, **34**], but strictly speaking, due to the weakly conformal settings, unfortunately, one cannot recover the result from these references. So, for the sake of completeness, a proof is given in §4.

Since $P_z(s)$ does not depend on z, one writes

$$P_z(s) = P(s) = \lim_{k\to+\infty} \frac{1}{k} \log \bigg(\sum_{\underline{i}\in\Lambda^k} |f_{\underline{i}}(K)|^s\bigg).$$

As said above, the pressure function is naturally connected to the dimension of the attractor K associated with the underlying IFS. More precisely, the following quantity is a natural candidate to be the Hausdorff dimension of K.

Definition 2.8. Let $m \geq 2$ be an integer. Let $S = \{f_1, \ldots, f_m\}$ be a C^1 weakly conformal IFS and K its attractor.

The unique real number $\dim(S)$ satisfying $P(\dim(S)) = 0$ is called the conformality dimension of S.

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Remark 2.9. If the mappings $f_1, \ldots, f_m$ are affine similarities, then the conformality dimension is called the similarity dimension. It is the real number solution to

$$\sum_{i=1}^{m} c_i^s = 1, \tag{12}$$

where c_i is the contraction ratio of f_i (that is, $c_i = \|f_i'\|$).

3. *Statement of the main results*

3.1. *Dimension of weakly conformal dynamical Diophantine set.*

Our main result is the following.

THEOREM 3.1. *Let $m \geq 2$ be an integer. Let U be an open set and let $S = \{f_1 : U \to U, \ldots, f_m : U \to U\}$ be a C^1 weakly conformal IFS with attractor K. For every $\delta > 0$, set*

$$W(x_0, \delta) = \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta}). \tag{13}$$

Then, we have the following.

(1) *For any $x_0 \in U$, for any $\delta < 1$,*

$$W(x_0, \delta) = K.$$

(2) *Assume in addition that $\dim_H(K) = \dim(S)$, then for any $x_0 \in K$, for any $\delta \geq 1$,*

$$\dim_H W(x_0, \delta) = \frac{\dim_H(K)}{\delta}.$$

Remark 3.2. In [**10**, Theorem 2.14], the Hausdorff dimension of sets defined as in equation (13) is estimated for a self-similar IFS satisfying the so-called dimension regularity assumption, meaning that the dimension of every self-similar measure does not drop by projecting it on the attractor (see Definition 4.17 below). In the self-similar case, Theorem 3.1 extends this result to any IFS satisfying the weaker assumption that the similarity dimension and the dimension of the attractor agree. We would like to further mention that, even in the self-similar case, Theorem 3.1 currently applies to strictly more cases than the result in [**10**, Theorem 2.14]. Let $0 < \lambda \leq \frac{1}{3}$, $\tau \in \mathbb{R}$, and set $S = \{g_1, g_2, g_3\}$, where $g_1(x) = \lambda x$, $g_2(x) = \lambda x + 1$ and $g_3(x) = \lambda x + \tau$. Due to a result of Rapaport and Varju established in [**32**], outside a set of parameters $(\lambda, \tau) \in (0, \frac{1}{3}) \times \mathbb{R}$ of Hausdorff dimension 0, the IFS S satisfies that $\dim_H K = \dim(S)$, so Theorem 3.1 applies to S. Note also that every IFS $S = \{f_1, f_2, f_3\}$ on $\mathbb{R}$ is affinely conjugated to the IFS S for some $\lambda, \tau \in (0, 1) \times \mathbb{R}$. It is worth mentioning that for the IFS S corresponding to good parameters, the dimensions of all self-similar measures associated with S are not known, in general. In particular, such cases are not covered by [**10**].

Let us provide concrete examples of IFSs for which Theorem 3.1 applies and examples of IFS to which the conclusion of Theorem 3.1 does not hold.

- Theorem 3.1 applies to any self-similar IFS S on $\mathbb{R}$ satisfying the exponential separation condition [**24**, Theorem 1.4] or having algebraic contraction ratio and no

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exact overlaps (i.e., for every $k \in \mathbb{N}$, every $\underline{i} \neq \underline{j} \in \Lambda^k$, $f_{\underline{i}} \neq f_{\underline{j}}$) with $\dim(S) \leq 1$ (due to a result of Rapaport [31]). For instance, define g_1, g_2, and g_3 as three mappings $\mathbb{R} \to \mathbb{R}$ by setting for every $x \in \mathbb{R}$, $g_1(x) = \frac{1}{4}x$, $g_2(x) = \frac{1}{4}(x+1)$, and $g_3(x) = \frac{1}{4}(x+t)$, where $t \in \mathbb{R} \setminus \mathbb{Q}$. Then, Theorem 3.1 applies to $S = \{g_1, g_2, g_3\}$.

- Let $m \geq 2$ be an integer and $0 < c_1, \ldots, c_m < 1$ be m real numbers satisfying

$$c_1 + \cdots + c_m \leq 1.$$

Then (due to a result of Hochman, [24]), for Lebesgue-almost every choice of $a_1, \ldots, a_m \in \mathbb{R}$, Theorem 3.1 applies to the IFS $S = \{f_1, \ldots, f_m\}$ where, for every $1 \leq i \leq m$, $f_i(x) = c_i x + a_i$.

- It is relatively easy to see that Theorem 3.1 cannot hold when $\dim(S) > d$. One could take, for instance, so many similarities so that for every $1 \leq \delta \leq 2$,

$$\dim_H \Big(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta})\Big) = d.$$

To be more explicit, fix t a badly approximable (by rationals) number and define $\phi_1(x) = x/2$, $\phi_2(x) = (x+1)/2$, $\phi_3(x) = (x+t)/2$, $\phi_4(x) = (x+1+t)/2$, and $\Lambda = \{1, \ldots, 4\}$. Then, it is proved in [2, Theorem 2.10] that for every $x_0 \in [0, 1+t]$,

$$\dim_H \Bigg(\limsup_{\underline{i} \in \Lambda^*} B\bigg(\phi_{\underline{i}}(x_0), \frac{1}{4^{|\underline{i}|}}\bigg)\Bigg) = 1.$$

In §3.1, we establish that the result applies to a large class of weakly conformal IFSs, namely the weakly conformal IFS satisfying the asymptotically weak separation condition (AWSC).

3.2. *An application in the case of homogeneous self-similar IFS.* In Theorem 3.1, the choice of the radii of the balls plays an important role on the hypothesis that one needs to assume to be able to estimate $\dim_H W(x, \delta)$. In the case of homogeneous IFSs, we will be able to treat the general case where the radii are simply given by a mapping $\phi : \mathbb{N} \to \mathbb{R}^+$. In our settings, we will call a system of affine similarities $S = \{f_i\}_{1 \leq i \leq m}$ homogeneous if for every $1 \leq i, j \leq m$,

$$\|f_i'\| = \|f_j'\|.$$

THEOREM 3.3. *Let S be an homogeneous self-similar IFS of common contraction ratio $0 < c < 1$. Let $\phi : \mathbb{N} \to \mathbb{R}^+$ be such that $\lim_{n \to +\infty} \phi(n) = 0$ and set*

$$s_\phi = \inf \Big\{ s \geq 0 : \sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^k} \phi(k)^{s \dim(S)} < +\infty \Big\}. \tag{14}$$

Assume that $\dim_H K = \dim(S)$. Then,

$$\dim_H W(x_0, \phi) := \dim_H \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), \phi(|\underline{i}|)) = \min\{1, s_\phi\} \dim(S).$$

Remark 3.4. If $\lim_{n \to +\infty} \phi(n) \neq 0$, since K is compact, $W(x_0, \phi)$ contains a ball, and hence $\dim_H K = d$.

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Let us provide a short application to Theorem 3.3. Let $t \in (0, 1)$ be a transcendental number and $\mathcal{A} = \{q_1 = 0, \ldots, q_N\} \subset \mathbb{Q}$, where $N \geq 2$ satisfies $\log N \leq -\log t$. Define $S = \{f_1(x) = tx + q_1, \ldots, f_N(x) = tx + q_N\}$. It is established in [32] that, denoting K as the attractor of S, one has

$$\dim_H K = \dim S = \frac{\log N}{-\log t}.$$

In addition, one has

$$\mathcal{P}_{\mathcal{A},t,n} := \left\{ P(t) = \sum_{k=0}^{n} a_i t^i \,:\, a_0, \ldots, a_n \in \mathcal{A} \right\} = \{ f_{\underline{i}}(0), \underline{i} \in \{1, \ldots, m\}^{n+1} \}.$$

So, given $\phi : \mathbb{N} \to \mathbb{R}^+$, writing

$$W_{\mathcal{A},t}(\phi) = \left\{ x \in \mathbb{R}^d : |x - P(t)| \leq \phi(\deg P) \text{ for infinitely many } P \in \bigcup_{n \geq 0} \mathcal{P}_{\mathcal{A},n,t} \right\},$$

a direct application of Theorem 3.3 yields that

$$\dim_H W_{\mathcal{A},t}(\phi) = \min\{1, s_\phi\} \frac{\log N}{-\log t},$$

where s_ϕ is defined as in equation (14). Finally, we mention that, for the sake of clarity, the present article deals with the case of IFSs without exact overlaps, but one easily could extend the results to the case where the IFS has exact overlaps under suitable assumption (related to the Garcia entropy of the IFS). In particular, $\dim_H W_{\mathcal{A},t}(\phi)$ can also be estimated for t algebraic. The short note [11] has been made available to explain how one should proceed in this case.

3.3. *The classical shrinking target problem.* As mentioned in §1, thanks to the techniques developed in this article, we are able to extend the dimension results established by Hill and Velani in [23] with a great deal of generality. Let us explain first why one needs a different hypothesis to extend the result of [23] than in Theorem 3.1. Reading the rest of the article, the reader will find that the proof of Theorem 3.1 is almost completely geometrical (that is, the proof is essentially non-symbolic). The combination of this fact and the fact that establishing dimension results in the context of self-similar IFSs is, in general, difficult and relies on deep ideas (see [19, 24, 31, 32, 36]), which makes it somewhat surprising that Theorem 3.1 holds under the rather weak hypothesis that the similarity dimension and the dimension of the attractor agree. This is completely due to the specific choice of the radii. More precisely, the choice of the necessary hypotheses in Theorem 3.1 are made with regard to the analyzing measures one will need to consider to solve this specific Diophantine problem. In the case of the classical shrinking target problem, these analyzing measures will be Gibbs measures, so the hypothesis one will require to deal with it will be made to ensure that the projections of Gibbs measure have the expected dimension on K. For the sake of simplicity, we will place ourselves in the case where the IFS is conformal and satisfies the so-called bounded distortion property, which we recall.

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Definition 3.5. Let $S = \{f_1, \ldots, f_m\}$ be a conformal IFS. We say that S satisfies the bounded distortion property if there exists $\kappa > 0$ such that for every $n \in \mathbb{N}$, for every $\underline{i} \in \{1, \ldots, m\}^n$ and every $x, y \in K$, one has

$$\kappa^{-1} \leq \frac{\| f'_{\underline{i}}(x)\|}{\| f'_{\underline{i}}(y)\|} \leq \kappa. \tag{15}$$

Before stating our main theorem, we recall the following result (see for instance [1, equation (2.6), p. 7]).

THEOREM 3.6. *Let $s > 0$ be a real number. Assume that $S = \{f_1, \ldots, f_m\}$ is conformal and satisfies the bounded distortion property, then there exists a unique ergodic measure $\nu_s \in \mathcal{M}(\Lambda^{\mathbb{N}})$ satisfying that for every $n \in \mathbb{N}$ and for every $\underline{i} \in \Lambda^n$, one has*

$$C^{-1} \leq \frac{\nu_s([\underline{i}])}{|f_{\underline{i}}(K)|^s e^{-nP(s)}} \leq C, \tag{16}$$

where $C \geq 1$ is a constant independent of $\underline{i}$. We will write $\mu_s = \nu_s \circ \pi^{-1}$ as the projection of ν_s on the attractor of S.

Remark 3.7. By ergodicity, one easily proves that there exists $h_s \geq 0$ such that for ν_s-almost every $\underline{i} = (i_n)_{n\in\mathbb{N}} \in \{1, \ldots, m\}^{\mathbb{N}}$, one has

$$\lim_{n\to+\infty} \frac{\log \nu_s([i_1, \ldots, i_n])}{\log |f_{(i_1,\ldots,i_n)}(K)|} = h_s.$$

Our main result regarding the Hausdorff dimension of shrinking targets set for overlapping conformal IFSs is the following.

THEOREM 3.8. *Let S be a conformal IFS satisfying the bounded distortion property, given by equation (15), and let $\phi : \mathbb{N} \to \mathbb{R}^+$ be a mapping satisfying $\phi(n) \to 0$. Set*

$$\alpha = \liminf_{n\to 0} \frac{-\log \phi(n)}{n}$$

and denote s the solution to $P(s) = s\alpha/\delta$. Assume that $\dim_H y\mu_s = h_s$ and $0 < \alpha < +\infty$, then for every $\delta \geq 1$, one has

$$\dim_H \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta \phi(|\underline{i}|)) = \frac{s}{\delta}.$$

Remark 3.9.

- If $\alpha = 0$, then for every $\varepsilon > 0$, there exists an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, for every $\underline{i} \in \Lambda^{n_k}$, one has

  $$|f_{\underline{i}}(K)|^{(1+\varepsilon)\delta} \leq |f_{\underline{i}}(K)|^\delta \phi(|\underline{i}|) \leq |f_{\underline{i}}(K)|^\delta.$$

  In that case, using the same argument as in the proof of Theorem 3.1, we conclude that

  $$\dim_H \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta \phi(|\underline{i}|)) = \frac{\dim S}{\delta}.$$

- If $\alpha = +\infty$, then for every $\delta' > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and for every $\underline{i} \in \Lambda^n$, we have

$$\phi(n) \leq |f_{\underline{i}}(K)|^{\delta'},$$

and hence, $|f_{\underline{i}}(K)|^{\delta}\phi(n) \leq |f_{\underline{i}}(K)|^{\delta+\delta'}$. By Theorem 3.1, this implies that

$$\dim_H \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta}\phi(|\underline{i}|)) \leq \frac{\dim S}{\delta+\delta'}.$$

Letting $\delta' \to +\infty$ yields

$$\dim_H \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta}\phi(|\underline{i}|)) = 0.$$

- In the case where S is a self-similar IFS on $\mathbb{R}$ satisfying the exponential separation condition due to a result established by Jordan and Rapaport in [33], for every $s > 0$, one has

$$\dim_H \mu_s = \min\{1, h_s\}.$$

As a consequence, Theorem 3.8 applies to any ϕ satisfying that the root s of $P(s) - s(\alpha/\delta)$ is such that $h_s \leq 1$.

To provide a large class of IFSs to which Theorem 3.8 applies, we start by recalling the definition of the so-called multifractal scaling function of a measure. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and define

$$\Theta_\mu(q, r) = \inf\left\{\sum_{i\in I} \mu(B(x_i, r))^q\right\},$$

where the infimum is taken over all countable collections of balls $\{B(x_i, r)\}_{i\in I}$ satisfying the following two properties:

(1) $x_i \in \mathrm{supp}(\mu)$ for every $i \in I$; and
(2) $B(x_i, r) \cap B(x_j, r) = \emptyset$ for $i \neq j$.

Then, the multifractal scaling function of μ is defined as

$$\tau_\mu(q) = \liminf_{r\to 0} \frac{\log(\Theta_\mu(q, r))}{\log r}. \tag{17}$$

Such mappings were first considered in the context of turbulence by Frisch and Parisi in [21]. The mapping τ_μ is often connected to the so-called multifractal spectrum of μ. More precisely, for $h \geq 0$, denote by $E_{\mu,h}$ the set of points of lower μ-local dimension h, that is,

$$E_{\mu,h} = \{x : \underline{\dim}(\mu, x) = h\}.$$

Many natural measures satisfy for every $h \geq 0$ that

$$\dim_H E_{\mu,h} = \tau_\mu^*(h),$$

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where $\tau_\mu^*(h) = \inf_{q\in\mathbb{R}}\{qh - \tau_\mu(q)\}$. We refer, for instance, to [9] for a complete mathematical foundation and to [18, 29, 36] for results regarding the multifractal analysis of self-similar measures.

When S is conformal and satisfies the bounded distortion property, it is classical (see [1] for instance) that there exists a unique ergodic measure ν_0 on $\{1,\ldots,m\}^{\mathbb{N}}$ such that for every $k\in\mathbb{N}$, for every $\underline{i}\in\{1,\ldots,m\}^k$, one has

$$C^{-1} \leq \frac{\nu_0([\underline{i}])}{|f_{\underline{i}}(K)|^{\dim(S)}} \leq C, \tag{18}$$

where $C > 0$ is independent of $\underline{i}$. In addition, when the IFS is self-similar, ν_0 is simply the self-similar measure corresponding to the similarity dimension (see Remark 2.9). We now introduce the following condition.

Definition 3.10. Let S be a conformal IFS satisfying equation (15), let K be its attractor, let ν_0 be as in equation (18), and set $\mu_0 = \nu_0 \circ \pi^{-1}$, where π is the canonical projection from $\{1,\ldots,m\}^{\mathbb{N}}$ to K. We say that S satisfies condition (A) if for every $q > 1$,

$$\tau_{\mu_0}(q) = (q-1)\dim(S).$$

We mention that condition (A) has been proved by Barral and Feng in [4] to be equivalent to the so-called asymptotically weak separation condition without exact-overlaps mentioned at the end of §3.1 (see Proposition 4.11 thereafter).

Thus (due to Remark 4.22 below), the following result holds true.

LEMMA 3.11. *Assume that S is a conformal IFS satisfying the bounded distortion property and condition (A). Then, for every $s > 0$, $\dim_H \mu_s = h_s$. In particular, Theorem 3.8 applies to every $\phi : \mathbb{N} \to \mathbb{R}^+$.*

Remark 3.12.

- As mentioned above, the advantage of the formulation of condition (A) is that the multifractal spectrum of self-similar measures, Gibbs measures, etc. is a topic well studied. For instance, as a consequence of the estimates regarding the L^q spectrum of self-similar measures on $\mathbb{R}$, established by Shmerkin in [36], Theorem 3.8 applies to any self-similar IFS satisfying the exponential separation condition on $\mathbb{R}$ with similarity dimension smaller than 1 (which, for instance, is ensured as soon as the parameters defining the IFS are algebraic and the IFS has no exact-overlaps). In addition, should these estimates have an analog in higher dimension under suitable assumptions, then Theorem 3.8 would apply to IFSs satisfying the same conditions.
- We also emphasize that if condition (A) is satisfied, every projection of ergodic measure has the dimension one would expect (see Remark 4.22), so, although the class of IFSs satisfying condition (A) is large, condition (A) is somewhat an overkill for our problem.

In §4, we recall and establish some geometric properties of a weakly conformal IFS. We also establish Proposition 4.10, which justifies that one can apply Theorem 3.1 to weakly conformal IFS satisfying the AWSC.

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The mass transference principle for weakly conformal measures, which is key to prove Theorem 3.1, is established in §5.

In §6, Theorem 3.1 is established.

Section 7 is dedicated to the proof of Theorem 3.3.

4. *Geometric properties of weakly conformal IFS*

In this section, all the basic geometric properties of a weakly conformal IFS needed in the rest of this article are recalled or established. More precisely, in §4.1, we recall the key Lemma 4.1, which will be used instead of an usual bounded distortion property one often requires when dealing with conformal IFSs. Section 4.2 is dedicated to the proof of Proposition 2.6. In §4.3, we recall the definition of the Lyapunov exponent in the case of a weakly conformal IFS and some basic properties of this exponent. Finally, in §4.4, we recall some facts about the asymptotically weak separation condition.

4.1. *Some general estimates.*

Let $m \geq 2$ be an integer. One collects some useful geometric results on a C^1 weakly conformal IFS.

Consider a C^1 weakly conformal IFS $S = \{f_1, \ldots, f_m\}$ with attractor K and for every $x \in K$, $k \in \mathbb{N}$, and $\underline{i} = (i_1, \ldots, i_k) \in \Lambda^k$, write

$$c_{\underline{i}}(x) = \|f'_{\underline{i}}(x)\|.$$

Let us recall the following result established as [**19**, Lemma 5.4].

LEMMA 4.1. [**19**] *For any $c > 1$, there exists a constant $D(c) > 0$ such that, for every $k \in \mathbb{N}$, for every $\underline{i} \in \Lambda^k$ and every $x, y \in K$,*

$$D(c)^{-1}c^{-k}\|f'_{\underline{i}}(x)\| \cdot \|x - y\| \leq \|f_{\underline{i}}(x) - f_{\underline{i}}(y)\| \leq D(c)c^k\|f'_{\underline{i}}(x)\| \cdot \|x - y\| \tag{19}$$

$$D(c)^{-1}c^{-k}\|f'_{\underline{i}}(x)\| \leq |f_{\underline{i}}(K)| \leq D(c)c^k\|f'_{\underline{i}}(x)\|. \tag{20}$$

Remark 4.2. Let $X \subset U$ be a compact set and U as in §2.2.2 (that is, as in the definition of a weakly conformal IFS). It is proved in [**19**] that equation (19) actually holds for any $(x, y) \in X^2$.

Note that for every $k \in \mathbb{N}$ and every $x \in K$, one has

$$c^{\pm k}\|f'_{\underline{i}}(x)\| = \|f'_{\underline{i}}(x)\|^{1+(\pm k \log c/\log \|f'_{\underline{i}}(x)\|)}.$$

Moreover, since there exists two constants $C_1, C_2 > 0$ such that for every $1 \leq i \leq m$ and every $x \in K$,

$$C_1 \leq \|f'_i(x)\| \leq \|f'_i(x)\| \leq C_2,$$

there also exists two constants $0 < t_1 \leq t_2$ such that

$$t_1 \leq \frac{k}{\log \|f'_{\underline{i}}(x)\|} \leq t_2.$$

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Combining this fact with Lemma 4.1, for any $\theta > 0$, there exists $\widetilde{C}_\theta > 0$ such that for every $k \in \mathbb{N}$, every $\underline{i} \in \Lambda^k$, and every $x, y \in K$,

$$\widetilde{C}_\theta^{-1} c_{\underline{i}}(x)^{1+\theta} \|x-y\| \leq \|f_{\underline{i}}(x) - f_{\underline{i}}(y)\| \leq \widetilde{C}_\theta c_{\underline{i}}(x)^{1-\theta} \|x-y\|. \tag{21}$$

In particular, there also exists $\widehat{C}_\theta$ such that for every $\underline{i} \in \Lambda^*$ and every $x \in K$, one has

$$\widehat{C}_\theta^{-1} c_{\underline{i}}^{1+\theta}(x)|K| \leq |f_{\underline{i}}(K)| \leq \widehat{C}_\theta c_{\underline{i}}^{1-\theta}(x)|K|. \tag{22}$$

Let us remark that equation (22) also implies that there exist $0 < \alpha \leq \beta < 1$ as well as $C_\alpha, C_\beta > 0$ such that, for any $k \in \mathbb{N}$,

$$C_\alpha \alpha^k \leq |f_{\underline{i}}(K)| \leq C_\beta \beta^k. \tag{23}$$

4.2. *Proof of Proposition 2.6.* As mentioned in §2, the proof of Proposition 2.6 is very standard and does not diverge much from the proof in the conformal case. Unfortunately, one cannot derive directly the result from classical cases (see [**8**, **34**], for instance) and those computations in the weakly conformal case do not seem to be written explicitly in the literature so for the seek of completeness, it is done below.

Proof. Assume first that the limit exists in $\mathbb{R} \cup \{-\infty\}$, and let us show that it is independent of the choice of z and that the limit is $> -\infty$. Let $c > 1$ be a real number. Recalling Lemma 4.1, for any $k \in \mathbb{N}$, one has

$$\begin{aligned}\log\bigg(\sum_{\underline{i}\in\Lambda^k} D(c)^{-s} c^{-sk} |f_{\underline{i}}(K)|^s\bigg) &\leq \log\bigg(\sum_{\underline{i}\in\Lambda^k} \|f_{\underline{i}}'(z)\|^s\bigg)\\ &\leq \log\bigg(\sum_{\underline{i}\in\Lambda^k} D(c)^s c^{sk} |f_{\underline{i}}(K)|^s\bigg).\end{aligned} \tag{24}$$

Since equation (24) holds for any $c > 1$, one gets that

$$\lim_{k\to+\infty} \frac{1}{k}\bigg(\log\bigg(\sum_{\underline{i}\in\Lambda^k} \|f_{\underline{i}}'(z)\|^s\bigg) - \log\bigg(\sum_{\underline{i}\in\Lambda^k} |f_{\underline{i}}(K)|^s\bigg)\bigg) = 0, \tag{25}$$

which proves that this quantity does not depend on z. Moreover, there exists $b > 0$ so that for any $k \in \mathbb{N}$, any $\underline{i} \in \Lambda^k$, and any $x \in K$,

$$\|f_{\underline{i}}'(x)\| \geq b^k.$$

This implies that if $P_z(s)$ is well defined, then $P_z(s) > -\infty$.

Let us now prove that the limit exists. For $k \in \mathbb{N}$, write

$$g_k = \log\bigg(\sum_{\underline{i}\in\Lambda^k} |f_{\underline{i}}(K)|^s\bigg). \tag{26}$$

As in the conformal case, the existence of the pressure relies on a sub-additivity argument.

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LEMMA 4.3. *For any $\varepsilon > 0$, there exists a constant $M_\varepsilon > 0$ such that for any $n, m \in \mathbb{N}$, one has*

$$g_{n+m} \leq M_\varepsilon + m\varepsilon + g_n + g_m. \tag{27}$$

Furthermore, any sequence $(g_n)_{n\in\mathbb{N}}$ verifying equation (27) is such that $(g_n/n)_{n\in\mathbb{N}}$ converges in $\mathbb{R} \cup \{-\infty\}$.

Proof. Let us start by proving the second statement. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence satisfying equation (27). Fix $\varepsilon > 0$ and M_ε satisfying equation (27). For any $q \in \mathbb{N}^*$, $b \in \mathbb{N}^*$, $0 \leq r < q$, one has

$$\begin{aligned} g_{bq+r} &\leq bg_q + g_r + (bq+r)\varepsilon + (b+1)M_\varepsilon \\ \Rightarrow \frac{g_{bq+r}}{bq+r} &\leq \frac{bq}{bq+r} \cdot \frac{g_q}{q} + \frac{(b+1)M_\varepsilon + g_r}{bq+r} + \varepsilon. \end{aligned}$$

Fixing q large enough independently of b so that $((b+1)M_\varepsilon/bq) \leq \varepsilon$, for any large $b \in \mathbb{N}^*$, one has

$$\frac{g_{bq+r}}{bq+r} \leq (1+\varepsilon)\frac{g_q}{q} + 2\varepsilon.$$

This implies that

$$\limsup_{n\to+\infty} \frac{g_n}{n} \leq (1+\varepsilon) \liminf_{n\to+\infty} \frac{g_n}{n} + 2\varepsilon.$$

Letting $\varepsilon \to 0$ proves the statement.

One now shows that g_n satisfies equation (27).

Let $k \in \mathbb{N}$ and $\underline{i} \in \Lambda^k$. Let us begin by the following lemma.

LEMMA 4.4. *Let $D(c)$ be again defined as in Lemma 4.1, then one has, for any $\underline{j} \in \Lambda^*$,*

$$\tfrac{1}{2}D(c)^{-2}c^{-2k}|f_{\underline{i}}(K)| \cdot |f_{\underline{j}}(K)| \leq |f_{\underline{i}\underline{j}}(K))| \leq 2D(c)^2c^{2k}|f_{\underline{i}}(K)| \cdot |f_{\underline{j}}(K)|. \tag{28}$$

Proof. Let us start by establishing the lower bound. Let $x, y \in K$ be such that

$$\|f_{\underline{j}}(x) - f_{\underline{j}}(y)\| \leq |f_{\underline{j}}(K)| \leq 2\|f_{\underline{j}}(x) - f_{\underline{j}}(y)\|. \tag{29}$$

By Lemma 4.1, one has

$$D(c)^{-1}c^{-k}\|f'_{\underline{i}}(f_{\underline{j}}(x))\| \cdot \|f_{\underline{j}}(x) - f_{\underline{j}}(y)\| \leq \|f_{\underline{i}\underline{j}}(x) - f_{\underline{i}\underline{j}}(y)\| \leq |f_{\underline{i}\underline{j}}(K)| \tag{30}$$

and

$$\|f'_{\underline{i}}(f_{\underline{j}}(x))\| \geq D(c)^{-1}c^{-k}|f_{\underline{i}}(K)|. \tag{31}$$

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Combining equations (29), (30), and (31), one obtains

$$\tfrac{1}{2}D(c)^{-2}c^{-2k}|f_{\underline{i}}(K)|\cdot|f_{\underline{j}}(K)|\leq|f_{\underline{i}\underline{j}}(K)|.$$

Let us focus now on the upper bound. Let $x, y\in K$ be such that

$$\|f_{\underline{i}\underline{j}}(x)-f_{\underline{i}\underline{j}}(y)\|\geq\tfrac{1}{2}|f_{\underline{i}\underline{j}}(K)|. \tag{32}$$

Using again Lemma 4.1, one has

$$\begin{aligned}\|f_{\underline{i}\underline{j}}(x)-f_{\underline{i}\underline{j}}(y)\| &\leq D(c)c^{k}\|f'_{\underline{i}}(f_{\underline{j}}(x))\|\cdot\|f_{\underline{j}}(x)-f_{\underline{j}}(y)\|\\ &\leq D(c)^{2}c^{2k}|f_{\underline{i}}(K)|\cdot|f_{\underline{j}}(K)|.\end{aligned} \tag{33}$$

The upper bound is obtained by combining equations (32) and (33). □

By Lemma 4.4, for any $c>1$ and any $n, n'\in\mathbb{N}$, one has

$$\begin{aligned} g_{n+n'} = & \log\Big(\sum_{\underline{i}\in\Lambda^{n+n'}}|f_{\underline{i}}(K)|^{s}\Big)=\log\Big(\sum_{\underline{i}\in\Lambda^{n},\underline{j}\in\Lambda^{n'}}|f_{\underline{i}\underline{j}}(K)|^{s}\Big)\\ \leq & \log\Big(\sum_{\underline{i}\in\Lambda^{n},\underline{j}\in\Lambda^{n'}}2^{s}D(c)^{2s}c^{2sn}|f_{\underline{i}}(K)|^{s}|f_{\underline{j}}(K)|^{s}\Big)\\ = & \; n\cdot 2s\log(c)+\log(2^{s}D(c)^{2s})+\log\bigg(\Big(\sum_{\underline{i}\in\Lambda^{n}}|f_{\underline{i}}(K)|^{s}\Big)\times\big(\sum_{\underline{j}\in\Lambda^{n'}}|f_{\underline{j}}(K)|^{s}\big)\bigg)\\ \leq & \; 2sn\log(c)+\log(2^{s}D(c)^{2s})+g_{n}+g_{n'}.\end{aligned}$$

Fixing $c=e^{\varepsilon/2s}$, one has $2s\log(c)=\varepsilon$ and setting $M_{\varepsilon}=\log(2^{s}D(c)^{2s})$ shows that $(g_{n})_{n\in\mathbb{N}}$ satisfies the condition of Lemma 4.3. □

Lemma 4.3 together with equation (25) concludes the proof of Proposition 2.6.

4.3. *Lyapunov exponent and dimension of weakly conformal measures.* Let $m\geq 2$ and let us fix a C^{1} weakly conformal IFS $S=\{f_{1},\ldots,f_{m}\}$ with attractor K.

Given $x=(x_{n})_{n\in\mathbb{N}}\in\Lambda^{\mathbb{N}}$, the following quantity, called Lyapunov exponent of S at x, defines a logarithmic shrinking rate associated with S at x. See [**19**, Proposition 5.6].

PROPOSITION 4.5. [**19**] *For $x=(x_{n})_{n\in\mathbb{N}}\in\Lambda^{\mathbb{N}}$, the Lyapunov exponent of S at x is defined, when the following limit exists, as*

$$\lambda(x)=-\lim_{n\to+\infty}\frac{\log|f_{x_{1}}\circ\cdots\circ f_{x_{n}}(K)|}{n}. \tag{34}$$

Moreover, for every ergodic measure $\nu\in\mathcal{M}(\Lambda^{\mathbb{N}})$ (with respect to the shift σ), there exists $\lambda_{\nu}\geq 0$ such that for ν-almost any $x=(x_{n})_{n\in\mathbb{N}}$,

$$\lambda(x)=\int\lambda(y)\,d\nu(y):=\lambda_{\nu}. \tag{35}$$

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Remark 4.6.

- By equation (23), the Lyapunov exponents are uniformly bounded above and below by some positive constant.
- When S is self-similar and $0 < c_1, \ldots, c_m < 1$ are the contracting ratios associated with the similarities $f_1, \ldots, f_m$, the Lyapunov exponent of ν as in Proposition 4.5 is simply

$$\lambda_\nu = - \sum_{1 \leq i \leq m} p_i \log c_i .$$

The following consequence of Proposition 4.5 will be useful later (see Proposition 6.7).

COROLLARY 4.7. *Let $((p_1^{(k)}, \ldots, p_m^{(k)}))_{k \in \mathbb{N}} \in ([0,1]^m)^{\mathbb{N}}$ be a sequence of probability vectors such that $(p_1^{(k)}, \ldots, p_m^{(k)}) \to (p_1, \ldots, p_m)$. Denote for $k \in \mathbb{N}\nu$, $\nu_k \in \mathcal{M}(\Lambda^{\mathbb{N}})$ the measures defined for any cylinder $[(i_1, \ldots i_n)]$ by*

$$\nu_k([(i_1, \ldots, i_k)]) = p_{i_1}^{(k)} \cdot \cdots \cdot p_{i_n}^{(k)} \text{ and } \nu([(i_1, \ldots, i_n)]) = p_{i_1} \cdot \cdots \cdot p_{i_n}.$$

Then, $\nu_k \underset{k \to +\infty}{\to} \nu$ weakly, so that

$$\lim_{k \to +\infty} \lambda_{\nu_k} = \lambda_\nu .$$

Let us also recall the following fundamental result established by Feng and Hu [**19**, Theorem 2.2].

THEOREM 4.8. [**19**] *Let $(p_1, \ldots, p_m) \in [0,1]^m$ be a probability vector, $\nu \in \mathcal{M}(\Lambda^{\mathbb{N}})$ defined for any $\underline{i} \in \Lambda^*$ by $\nu([\underline{i}]) = p_{\underline{i}}$ and $\mu = \nu \circ \pi^{-1}$.*

There is an $h \geq 0$ such that for μ-almost every $x \in K$, there exists $\mu^{\pi^{-1}(\{x\})} \in \mathcal{M}(\Lambda^{\mathbb{N}})$ such that:

(1) $\mu^{\pi^{-1}(\{x\})}(\pi^{-1}(\{x\})) = 1$;

(2) *for $\mu^{\pi^{-1}(\{x\})}$-almost $y = (y_1, \ldots, y_n, \ldots)$,*

$$\frac{-\log \mu^{\pi^{-1}(\{x\})}([y_1, \ldots, y_n])}{n} \to h; \tag{36}$$

(3) *for every Borel set $A \subset \Lambda^{\mathbb{N}}$,*

$$\nu(A) = \int_K \mu^{\pi^{-1}(\{x\})}(A)\, d\mu(x); \tag{37}$$

(4) *denoting λ as the Lyapunov exponent associated with ν as in equation (35), μ is exact-dimensional (Definition 5) and*

$$\dim(\mu) = \frac{-h - \sum_{1 \leq i \leq m} p_i \log p_i}{\lambda}.$$

4.4. *The AWSC condition.*

4.4.1. *Definition and known results.* To provide a larger class of weakly conformal IFSs to which Theorem 3.1 applies, we recall the definition of the AWSC [**18**]. First, given an

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IFS $S = \{f_i\}_{i\in\Lambda}$, let us define

$$\Lambda^{(k)} = \{\underline{i} = (i_1, \ldots, i_n) \in \Lambda^* : 2^{-k-1} < |f_{\underline{i}}(K)| \leq 2^{-k}\}. \quad (38)$$

Definition 4.9. One says that $S = \{f_i\}_{i\in\Lambda}$ satisfies the AWSC [18] when, writing for $k \in \mathbb{N}$,

$$t_k(S) = \max_{x\in\mathbb{R}^d} \#\{f_{\underline{i}} : \underline{i} \in \Lambda^{(k)} \text{ and } f_{\underline{i}}(K) \cap B(x, 2^{-k}) \neq \emptyset\}, \quad (39)$$

one has

$$\frac{\log t_k(S)}{k} \to 0.$$

Let us also note here that when the IFS S has no exact overlaps (i.e., for any $\underline{i} \neq \underline{j} \in \Lambda^*$, $f_{\underline{i}} \neq f_{\underline{j}}$), one also has

$$t_k(S) = \max_{x\in\mathbb{R}^d} \#\{\underline{i} : \underline{i} \in \Lambda^{(k)} \text{ and } f_{\underline{i}}(K) \cap B(x, 2^{-k}) \neq \emptyset\}. \quad (40)$$

The introduction of the AWSC condition is motivated by the following result.

PROPOSITION 4.10. *Let S be a weakly conformal IFS satisfying the AWSC with no exact overlaps. Then, its attractor K satisfies*

$$\dim_H K = \dim(S).$$

As mentioned in §3.3, the advantage of the ASWC is that it can be reformulated in terms of a condition regarding the multifractal spectrum of some natural measures (given by Lemma 4.21) associated with a weakly conformal IFS.

The following proposition was established by Barral and Feng in [4] in the case of self-similar measures but readily adapts in the case of weakly conformal measures.

PROPOSITION 4.11. [4] *Let $S = \{f_1, \ldots, f_m\}$ be a conformal IFS and let K be its attractor. Assume that there exists a measure $\nu_0 \in \mathcal{M}(\{1, \ldots, m\}^{\mathbb{N}})$ such that for every $\underline{i} \in \{1, \ldots, m\}^k$, one has*

$$C^{-1} \leq \frac{\nu_0([\underline{i}])}{|f_{\underline{i}}(K)|^{\dim(S)}} \leq C, \quad (41)$$

where $C > 0$ is independent of $\underline{i}$. Set $\mu_0 = \nu_0 \circ \pi^{-1}$. If for every $q \geq 1$, one has

$$\tau_{\mu_0}(q) = (q-1)\dim(S),$$

where τ_{μ_0} is the scaling function defined by equation (17), then $\dim_H y\mu_0 = \dim(S)$, S satisfies the AWSC, and S has no exact-overlaps.

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Proposition 4.11 combined with a well-known result established by Shmerkin in [**36**] regarding the multifractal spectrum of self-similar measures on $\mathbb{R}$ satisfying the exponential separation yields the following result established as [**4**, Theorem 1.3].

THEOREM 4.12. [**4**] *Let S be a self-similar IFS on $\mathbb{R}$ satisfying the exponential separation condition [**24**, Theorem 1.4]. Then, S satisfies the AWSC if and only if* $\dim(S) \leq 1$.

4.4.2. *Some technical results regarding the AWSC.* For $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$, recall equation (38) and define

$$\widetilde{\Lambda}^{(k)} = \{\underline{i} = (i_1, \ldots, i_n) \in \Lambda^* : |f_{\underline{i}}(K)| \leq 2^{-k} < |f_{(i_1,\ldots i_{k-1})}(K)|\}. \tag{42}$$

Moreover, for $x \in K$, we set

$$T_k(x) = \{f_{\underline{i}} : \ f_{\underline{i}}(K) \cap B(x, 2^{-k}) \neq \emptyset, \ \underline{i} \in \widetilde{\Lambda}^{(k)}\},$$
$$T'_k(x) = \{f_{\underline{i}} : \ f_{\underline{i}}(K) \cap B(x, 2^{-k}) \neq \emptyset, \ \underline{i} \in \Lambda^{(k)}\}.$$

Note that S satisfies AWSC if and only if $\lim_{k\to+\infty} \max_{x\in\mathbb{R}^d} (\log \#T_k(x)/k) = 0$.

PROPOSITION 4.13. *One has*

$$\lim_{k\to+\infty} \max_{x\in\mathbb{R}^d} \frac{\log \#T_k(x)}{k} = 0 \iff \lim_{k\to+\infty} \max_{x\in\mathbb{R}^d} \frac{\log \#T'_k(x)}{k} = 0.$$

Proof. By equation (23), there exists $0 < \alpha < \frac{1}{2} < \beta \leq 1$ such that for every $k \in \mathbb{N}$,

$$\alpha^k \leq |f_{\underline{i}}(K)| \leq \beta^k.$$

Remark 4.14.

(1) For every $k \in \mathbb{N}$ and every $\underline{i} = (i_1, \ldots, i_n) \in \Lambda^{(k)}$, one has

$$C(\alpha, \beta)^{-1} k \leq k\frac{-\log 2}{\log \beta} + 1 \leq n \leq 2k\frac{-\log 2}{\log \alpha} \leq C(\alpha, \beta)k.$$

(2) For every $c > 1$, by Lemma 4.4, for every $\underline{i} = (i_1, \ldots, i_n) \in \widetilde{\Lambda}^{(k)}$,

$$D(c)^{-2} \min_{1\leq j\leq m} |f_j(K)| c^{-2C(\alpha,\beta)k} 2^{-k} \leq |f_{\underline{i}}(K)| \leq 2^{-k}.$$

In particular, for any $k \in \mathbb{N}$ large enough, one has

$$c^{-1/2C(\alpha,\beta)k} 2^{-k} \leq |f_{\underline{i}}(K)| \leq 2^{-k}. \tag{43}$$

LEMMA 4.15. *For every $\varepsilon_0 > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that for every $k \geq k_{\varepsilon_0}$, for every $\underline{i} = (i_1, \ldots, i_n) \in \Lambda^{(k)}$, there exists $0 \leq p \leq \varepsilon_0 k$ such that $(i_1, \ldots, i_{n-p}) \in \widetilde{\Lambda}^{(k)}$.*

Proof. Fix $\varepsilon = \varepsilon_0/2C(\alpha, \beta)$ and $c > 1$ such that $c^{1-\varepsilon}\beta^\varepsilon < 1$. By Lemma 4.4, for any $(i_1, \ldots, i_n) \in \Lambda^*$ and $0 \leq p \leq n$,

$$|f_{(i_1,\ldots,i_n)}(K)| \leq D(c)^2 c^{n-p} |f_{(i_1,\ldots,i_{n-p})}(K)| \times |f_{(i_{n-p+1},\ldots,i_n)}(K)|.$$

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In particular, for $p \geq n\varepsilon$,

$$|f_{(i_1,\ldots,i_{n-p})}(K)| \geq \frac{D(c)^{-2}c^{-(n-p)}}{|f_{(i_{n-p+1},\ldots,i_n)}(K)|}2^{-k} \geq 2^{-k}\frac{D(c)^{-2}}{(c^{1-\varepsilon}\beta^{\varepsilon})^n}.$$

This yields, for k large enough and $p = \lfloor \varepsilon C(\alpha, \beta)k \rfloor + 1 \leq 2C(\alpha, \beta)\varepsilon k$, that

$$|f_{(i_1,\ldots,i_{n-p})}(K)| > 2^{-k}.$$

As a consequence, there must exist $p \leq 2C(\alpha, \beta)\varepsilon k$ such that $(i_1, \ldots, i_{n-p}) \in \widetilde{\Lambda}^{(k)}$. □

LEMMA 4.16. *For every $c > 1$, for every $\varepsilon > 0$, for every k large enough (depending on c) and every $x \in \mathbb{R}^d$, one has*

$$\#T_k(x) \leq k\left\lfloor \frac{(C(\alpha, \beta)/2)\log c}{\log 2}\right\rfloor C_d c^{kd(C(\alpha,\beta)/2)} \max_{k\leq k"\leq k(1+\lfloor((C(\alpha,\beta)/2)\log c)/\log 2\rfloor)} \max_{y\in\mathbb{R}^d} \#T'_{k'}(y) \tag{44}$$

$$\textit{and} \quad \#T'_k(x) \leq m^{k\varepsilon}\#T_k(x).$$

Proof. Remark that, for each k' such that

$$c^{-(1/2)C(\alpha,\beta)k}2^{-k} \leq 2^{-k'} \leq 2^{-k},$$

there exists a constant C_d (which depends on d, α, and β) so that each ball $B(x, 2^{-k})$ can be covered by less than

$$C_d c^{kd(C(\alpha,\beta)/2)}$$

balls of radius $2^{-k'}$. This implies that

$$\#\{f_{\underline{i}} : \underline{i} \in T_k(x) \cap \Lambda^{(k')}\} \leq C_d c^{kd(C(\alpha,\beta)/2)} \max_{y\in\mathbb{R}^d} \#T'_{k'}(y).$$

Since one has

$$\widetilde{\Lambda}^{(k)} \subset \bigcup_{k'=k}^{k(1+\lfloor((C(\alpha,\beta)/2)\log c)/\log Z\rfloor)} \Lambda^{(k')},$$

it holds that

$$\#T_k(x) \leq k\left\lfloor \frac{(C(\alpha, \beta)/2)\log c}{\log 2}\right\rfloor C_d c^{kd(C(\alpha,\beta)/2)} \max_{k\leq k"\leq k(1+\lfloor((C(\alpha,\beta)/2)\log c)/\log 2\rfloor)} \max_{y\in\mathbb{R}^d} \#T'_{k'}(y). \tag{45}$$

Moreover, by Lemma 4.15, there exists $\phi_k : T'_k(x) \to T_k(x)$ defined by

$$\phi_k((i_1, \ldots, i_n)) = (i_1, \ldots i_{n-p})$$

with $0 \leq p \leq k\varepsilon$. The mapping ϕ_k verifies that, for every $(i_1, \ldots i_{n'}) \in T_k(x)$,

$$\#\phi_k^{-1}(\{(i_1, \ldots i_{n'})\}) \leq m^{k\varepsilon}.$$

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This implies that

$$\#T'_k(x) \leq m^{k\varepsilon} \#T_k(x). \qquad \square$$

Taking the log of the estimates of Lemma 4.16 and letting k tend to infinity concludes the proof. $\square$

4.4.3. *AWSC and dimension regularity.* The following notion was introduced by Barral and Feng in [4].

Definition 4.17. [4] Let $S = \{f_1, \ldots, f_m\}$ be a weakly conformal IFS. For $P = (p_1, \ldots, p_m) \in [0, 1]^m$ a probability vector, denote again by $\nu_P \in \mathcal{M}(\Lambda^*)$ the measure satisfying for every $(i_1, \ldots, i_n) \in \Lambda^*$, $\nu_P([i_1, \ldots, i_n]) = p_{i_1} \times \cdots \times p_{i_n}$ and $\mu_P = \nu_P \circ \pi^{-1}$. The IFS S is said to be dimension regular if, for every probability vector P,

$$\dim(\mu_P) = \min\left\{\frac{-\sum_{1\leq i\leq m} p_i \log(p_i)}{\lambda_{\nu_P}}, d\right\}, \tag{46}$$

where λ_{ν_P} is defined by equation (35).

Remark 4.18.

- When S is self-similar, calling $0 < c_1, \ldots, c_m < 1$ the contraction ratios of the similarities $f_1, \ldots, f_m$, for any probability vector $(p_1, \ldots, p_m)$, μ, and ν as in Definition 4.17, one has

$$\dim(\mu) = \min\left\{\frac{-\sum_{1\leq i\leq m} p_i \log(p_i)}{\lambda_{\nu}}, d\right\} = \min\left\{\frac{\sum_{1\leq i\leq m} p_i \log(p_i)}{\sum_{i=1}^{m} p_i \log(c_i)}, d\right\}. \tag{47}$$

- As proved in [24], any self-similar IFS on $\mathbb{R}$ satisfying the exponential separation condition [24, Theorem 1.4] is dimension regular.

We will prove the following result which implies Proposition 4.10.

PROPOSITION 4.19. *Assume that $S = \{f_1, \ldots, f_m\}$ satisfies the AWSC without exact overlaps. Then, S is dimension regular and* $\dim(S) = \dim_H(K)$.

Had the IFS been conformal and satisfying some bounded distortion properties, the proof of Proposition 4.19 would follow directly from the existence of appropriated Gibbs measures. Unfortunately, such measures do not always exist in the weakly conformal case, but some measures that are close enough from satisfying the desired properties still exist as established by the following lemma.

LEMMA 4.20. *Let $\varepsilon > 0$ and $s \geq 0$ be real numbers. There exists $k \in \mathbb{N}$ and a probability vector $(p_{\underline{i}})_{\underline{i}\in\Lambda^k}$ such that the weakly conformal measure ν associated with $S' = \{f_{\underline{i}}\}_{\underline{i}\in\Lambda^k}$ and $(p_{\underline{i}})_{\underline{i}\in\Lambda^k}$ verifies, for any $p \in \mathbb{N}$ and $\underline{i}_1, \ldots, \underline{i}_p \in \Lambda^k$,*

$$e^{-kp\varepsilon} \frac{|f_{\underline{i}_1\cdots\underline{i}_p}(K)|^s}{e^{pkP(s)}} \leq \nu([\underline{i}_1 \cdots \underline{i}_p]) \leq e^{kp\varepsilon} \frac{|f_{\underline{i}_1\cdots\underline{i}_p}(K)|^s}{e^{pkP(s)}}. \tag{48}$$

Proof. Fix $\varepsilon > 0$ and $c > 1$ small enough so that $8s \log c \leq \varepsilon$.

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By Lemma 4.3, there exists $k \in \mathbb{N}$ so large that the constant named $D(c)$ in Lemma 4.1 verifies $(\log D(c)/k) \leq \log c$ and

$$\left| \frac{1}{k} \log \sum_{\underline{i} \in \Lambda^k} |f_{\underline{i}}(K)|^s - P(s) \right| \leq \frac{\varepsilon}{2}. \tag{49}$$

Writing again $g_k = \log \sum_{\underline{i} \in \Lambda^k} |f_{\underline{i}}(K)|^s$, let us define the probability vector $(p_{\underline{i}})_{\underline{i} \in \Lambda^k}$ by setting

$$p_{\underline{i}} = \frac{|f_{\underline{i}}(K)|^s}{e^{g_k}}.$$

Let ν be the weakly conformal measure associated with $S' = \{f_{\underline{i}}\}_{\underline{i} \in \Lambda^k}$ and $(p_{\underline{i}})_{\underline{i} \in \Lambda^k}$. Applying Lemma 4.4, for any $p \in \mathbb{N}$, $\underline{i}_1, \ldots, \underline{i}_p \in \Lambda^k$,,

$$D(c)^{-2p} c^{-2kp} \leq \frac{|f_{\underline{i}_1} \circ \cdots \circ f_{\underline{i}_p}(K)|}{\prod_{j=1}^{p} |f_{\underline{i}_j}(K)|} \leq D(c)^{2p} c^{2kp}. \tag{50}$$

Also,

$$D(c)^{2sp} c^{2skp} = e^{pk2s\cdot((\log D(c)/k)+\log c)} \leq e^{(\varepsilon/2)pk}. \tag{51}$$

As a consequence, for any $p \in \mathbb{N}$ and any $\underline{i}_1, \ldots, \underline{i}_p \in \Lambda^k$, one has

$$\nu([\underline{i}_1 \cdots \underline{i}_p]) = p_{\underline{i}_1} \cdot \cdots \cdot p_{\underline{i}_p} = \frac{\prod_{j=1}^{p} |f_{\underline{i}_j}(K)|^s}{e^{pg_k}} = \frac{\prod_{j=1}^{p} |f_{\underline{i}_j}(K)|^s}{e^{kp((g_k/k)-P(s))} e^{pkP(s)}}.$$

Using equations (49), (50), and (51) concludes the proof. □

Remark 4.21. The measure ν can be extended over $\Lambda^{\mathbb{N}}$ by the usual arguments. Moreover, for any $\underline{i} = (i_1, \ldots, i_n) \in \Lambda^*$, write $n_1 = k\lfloor n/k \rfloor$ and $n_2 = k(\lfloor n/k \rfloor + 1)$. Consider $\underline{j} \in \Lambda^{n_1}$ such that $[\underline{i}] \subset [\underline{j}]$ and $\underline{\ell} = (\ell_1, \ldots, \ell_{n_2-n}) \in \Lambda^{n_2-n}$, one has

$$e^{-n_2\varepsilon} \frac{|f_{(i_1,\ldots,i_n,\ell_1,\ldots,\ell_{n_2-n})}(K)|^s}{e^{n_2 P(s)}} \leq \nu([\underline{i}\underline{\ell}]) \leq \nu([\underline{i}]) \leq \nu([\underline{j}]) \leq e^{n_1\varepsilon} \frac{|f_{(i_1,\ldots,i_{n_1})}(K)|^s}{e^{n_1 P(s)}}. \tag{52}$$

By Lemma 4.4, there exists a constant $C > 0$ such that, uniformly on $\underline{i}$, $\underline{j}$, $\underline{i}\underline{\ell}$, one has

$$C^{-1} \leq \min\left\{ \frac{|f_{\underline{j}}(K)|}{|f_{\underline{i}}(K)|}, \frac{|f_{\underline{i}}(K)|}{|f_{\underline{i}\underline{\ell}}(K)|} \right\} \leq \max\left\{ \frac{|f_{\underline{j}}(K)|}{|f_{\underline{i}}(K)|}, \frac{|f_{\underline{i}}(K)|}{|f_{\underline{i}\underline{\ell}}(K)|} \right\} \leq C.$$

Hence, there exists a constant $\gamma_{s,\varepsilon}$ such that for any $\underline{i} = (i_1, \ldots, i_n) \in \Lambda^*$, one has

$$\gamma_{s,\varepsilon}^{-1} e^{-n\varepsilon} \frac{|f_{\underline{i}}(K)|^s}{e^{nP(s)}} \leq \nu([\underline{i}]) \leq \gamma_{s,\varepsilon} e^{n\varepsilon} \frac{|f_{\underline{i}}(K)|^s}{e^{nP(s)}}. \tag{53}$$

Let us now prove Proposition 4.19.

Proof. Call K the attractor of S. Let us show first that if any system S satisfying the AWSC also verifies that, for any weakly conformal measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ associated with a probability vector $(p_1, \ldots, p_m)$ and S,

$$\dim(\mu) = \frac{-\sum_{1\leq i\leq m} p_i \log p_i}{\lambda_\nu}, \tag{54}$$

where ν is the measure associated on $\Lambda^{\mathbb{N}}$, then $\dim(S) = \dim_H(K)$.

Fix $\varepsilon > 0$ and consider $k \in \mathbb{N}$, $S' = \{f_{\underline{i}}\}_{\underline{i}\in\Lambda^k}$, and ν as in Lemma 4.20 applied with $s = \dim(S)$. Note that, since S satisfies the AWSC, so does S'. Then, considering the measure $\mu = \nu \circ \pi^{-1}$, where π is the canonical projection, one has

$$\dim(S) - \varepsilon \leq \dim(\mu) = \frac{-\sum_{\underline{i}\in\Lambda^k} p_{\underline{i}} \log p_{\underline{i}}}{\lambda_\nu} \leq \dim(S) + \varepsilon.$$

This proves that $\dim_H(K) \geq \dim(S) - \varepsilon$. Since it always holds that $\dim_H(K) \leq \dim(S)$ (see [**15**]) and ε is arbitrary,

$$\dim_H(K) = \dim(S).$$

Let us show that, for any system satisfying the AWSC, equation (54) holds for every weakly conformal measure μ.

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure associated with S and a probability vector $(p_1, \ldots, p_m)$ and $\nu \in \mathcal{M}(\Lambda^{\mathbb{N}})$ such that $\mu = \nu \circ \pi^{-1}$.

It comes from the proof of Theorem 4.8 [**19**] (applied to μ) that for any $\varepsilon > 0$, for μ-almost any $x \in K$ such that $\mu^{\pi^{-1}(\{x\})}$ exists and satisfies the two first items of Theorem 4.8, there exists n_0 large enough so that, for any $n \geq n_0$, there exists $\underline{i}_1, \ldots, \underline{i}_{N_n}$ such that:

- for any $1 \leq j \leq N_n$,

$$e^{-n(\lambda+\varepsilon)} \leq |f_{\underline{i}_j}(K)| \leq e^{-n(\lambda-\varepsilon)}; \tag{55}$$

- one has

$$\mu^{\pi^{-1}(\{x\})}\bigg(\bigcup_{1\leq j\leq N_n} [\underline{i}_j]\bigg) \geq \frac{1}{2}; \tag{56}$$

- for any $1 \leq j \leq N_n$,

$$e^{-n(h+\varepsilon)} \leq \mu^{\pi^{-1}(\{x\})}([\underline{i}_j]) \leq e^{-n(h-\varepsilon)}. \tag{57}$$

Assume that $h > 0$ and take $0 < \varepsilon < \min\{h/2, \lambda/2\}$.

Combining equations (56) and (57), one gets

$$N_n \geq \tfrac{1}{2} e^{n(h-\varepsilon)}. \tag{58}$$

Note that $\#\{k : e^{-n(\lambda+\varepsilon)} \leq 2^{-k} \leq e^{-n(\lambda-\varepsilon)}\} \leq (2n\varepsilon/\log 2)$. As a consequence, there exists $k \in [n(\lambda-\varepsilon)/\log 2, n(\lambda+\varepsilon)/\log 2]$ such that

$$\#\Lambda^{(k)} \cap \{[\underline{i}_j]\}_{1\leq j\leq N_n} \geq \frac{N_n}{2n\varepsilon/\log 2} \geq \frac{(1/2)e^{nh/2}}{2n\varepsilon/\log 2}. \tag{59}$$

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Since for any $\leq j \leq N_n$, $[\underline{i}_j] \cap \pi^{-1}(\{x\}) \neq \emptyset$, one also has $f_{\underline{i}_j}(K) \subset B(x, e^{-n(\lambda-\varepsilon)})$, so that, writing $n' = \lfloor n(\lambda - \varepsilon)/\log 2 \rfloor$, one has

$$\#\{\underline{i} \in \Lambda^{(n')} : f_{\underline{i}}(K) \cap B(x, 2^{-n'})\} \geq \frac{(1/2)e^{nh/2}}{2n\varepsilon/\log 2}. \tag{60}$$

In particular, recalling equation (40) and Proposition 4.13,

$$\frac{\log t_k}{k} \nrightarrow 0$$

and S does not satisfy the AWSC. As a consequence, S satisfies the AWSC implying $h = 0$, which, recalling the last item of Theorem 4.8, concludes the proof. □

Remark 4.22. For simplicity, we established the result for weakly conformal measures, but Theorem 4.8, that is, [**19**, Theorem 2.1], is actually stated for ergodic measures and a careful reader will notice that the same proof combined with [**19**, Theorem 2.1] actually shows that if the IFS S satisfies AWSC, for every ergodic measure $\nu \in \mathcal{M}(\Lambda^{\mathbb{N}})$ (with respect to the shift σ), one has

$$\dim_H \nu \circ \pi^{-1} = \frac{h(\nu)}{\lambda_\nu},$$

where λ_ν is the Lyapunov exponent associated with ν and $h(\nu)$ is defined as the positive number for which, for ν-almost $(x_n)_{n\in\mathbb{N}}$, one has

$$h(\nu) = \lim_{n\to+\infty} \frac{-\log \nu([(x_1, \ldots, x_n])}{n}.$$

We also state the following corollary, which will be useful later in the manuscript.

COROLLARY 4.23. *If S is weakly conformal, then any weakly conformal measure μ satisfies equation (54). So by Corollary 4.7,* $\dim_H \mu$ *depends continuously on the choice of the probability vector.*

5. *Mass transference principle and quasi-Bernoulli measures*

The classical mass transference principle of Beresnevitch and Velani [**7**] (and many others [**14**, **22**, **26**]) relies on the fact that the ambient measure is Ahlfors-regular. However, to establish Theorem 3.1, one needs a comparable theorem when the measure is inhomogeneous (that is, not Ahlfors-regular).

Such theorems were first established by Barral and Seuret [**5**] and a general version (in terms of the measure involved) was given in [**10**, Theorem 2.2]. The key geometric notion developed in [**10**] to handle inhomogeneous mass transference principles is the following.

Definition 5.1. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $s \geq 0$. The s-dimensional μ-essential Hausdorff content of a set $A \subset \mathcal{B}(\mathbb{R}^d)$ is defined as

$$\mathcal{H}^{\mu,s}_\infty(A) = \inf\{\mathcal{H}^s_\infty(E) : \ E \subset A, \ \mu(E) = \mu(A)\}. \tag{61}$$

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As in the self-similar case treated in [10, Theorem 2.6], precise estimates of $\mathcal{H}^{\mu,s}_{\infty}(A)$ are established when μ is a C^1 weakly conformal measure in Theorem 5.5 below.

We will need the following notion of asymptotically covering sequences of balls, developed in [13] (and also used in [10]), to establish the desired mass transference principle.

Definition 5.2. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. The sequence $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ of closed balls of $\mathbb{R}^d$ satisfying $|B_n| \to 0$ is said to be μ-asymptotically covering (μ-a.c.) when there exists a constant $C > 0$ such that for every open set $\Omega \subset \mathbb{R}^d$ and $g \in \mathbb{N}$, there is an integer $N_\Omega \in \mathbb{N}$ as well as $g \leq n_1 \leq \cdots \leq n_{N_\Omega}$ such that:

(i) for all $1 \leq i \leq N_\Omega$, $B_{n_i} \subset \Omega$;
(ii) for all $1 \leq i \neq j \leq N_\Omega$, $B_{n_i} \cap B_{n_j} = \emptyset$;
(iii) also,

$$\mu\bigg(\bigcup_{i=1}^{N_\Omega} B_{n_i}\bigg) \geq C\mu(\Omega). \tag{62}$$

The following proposition is proved in [13], the second item will be used to apply our main theorem to self-conformal measures. For more details about this notion derived from a covering property proved in the KGB lemma in [7], one refers to [13].

PROPOSITION 5.3. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n\in\mathbb{N}}$ be a sequence of balls of $\mathbb{R}^d$ with $\lim_{n\to+\infty} r_n = 0$.*

(1) *If $\mathcal{B}$ is μ-a.c., then $\mu(\limsup_{n\to+\infty} B_n) = 1$.*
(2) *If there exists $v < 1$ such that $\mu(\limsup_{n\to+\infty}(vB_n)) = 1$, then $\mathcal{B}$ is μ-a.c.*
(3) *If μ is doubling, then $\mathcal{B}$ is μ-a.c. if and only if there exists $0 < v \leq 1$ such that $\mu(\limsup_{n\to+\infty}(vB_n)) = 1$.*

The mass transference principle associated with these notions is the following [10, Theorem 2.2].

THEOREM 5.4. [10] *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ be a μ-a.c. sequence of closed balls of $\mathbb{R}^d$ such that $|B_n| \to 0$, and $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ a sequence of open sets such that $U_n \subset B_n$ for all $n \in \mathbb{N}$. Let $0 \leq s < \underline{\dim}_H(\mu)$ such that for every n large enough, $\mathcal{H}^{\mu,s}_{\infty}(U_n) \geq \mu(B_n)$.*

Then,

$$\dim_H\Big(\limsup_{n\to+\infty} U_n\Big) \geq s. \tag{63}$$

To apply Theorem 5.4, precise estimates of essential contents of open sets must be achieved. The next subsection is dedicated to this problem when the measure is self-conformal and in the last subsection of §5, the mass transference principle for weakly conformal measures is established.

5.1. *Essential content for weakly conformal measures.* Estimates on essential contents for weakly conformal measures are now established. These estimates are similar to that established in [10, Theorem 2.6] for self-similar measures, but the introduction of the

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weakly conformal settings brings a number of geometrical difficulties, so for the sake of completeness, we provide all the details regarding these estimates.

THEOREM 5.5. *Let S be a C^1 weakly conformal IFS of $\mathbb{R}^d$.*

Let K be the attractor of S and μ be a measure on K which satisfies that, for every $\underline{i} \in \bigcup_{k\geq 1}\{1, \ldots, m\}^k$, $\mu \circ f_{\underline{i}}^{-1}$ is absolutely continuous with respect to μ. Then, we have the following.

For any $0 \leq s < \underline{\dim}_H(\mu)$, for any $0 < \varepsilon \leq \frac{1}{2}$, there exists a constant $c = c(d, \mu, s, \varepsilon) > 0$ depending on the dimensions d, μ, s, and ε only, such that for any ball $B = B(x, r)$ centered on K and $r \leq 1$, for any open set Ω, one has

$$\begin{aligned} c(d, \mu, s, \varepsilon)|B|^{s+\varepsilon} &\leq \mathcal{H}^{\mu,s}_{\infty}(B) \leq \mathcal{H}^{\mu,s}_{\infty}(B) \leq |B|^s, \\ c(d, \mu, s, \varepsilon)\mathcal{H}^{s+\varepsilon}_{\infty}(\Omega \cap K) &\leq \mathcal{H}^{\mu,s}_{\infty}(\Omega) \leq \mathcal{H}^{s}_{\infty}(\Omega \cap K). \end{aligned} \tag{64}$$

For any $s > \overline{\dim}_H(\mu)$, $\mathcal{H}^{\mu,s}_{\infty}(\Omega) = 0$.

- The system S is not assumed to verify any separation condition.
- When the maps are similarities, one still has, for any $s > \overline{\dim}_H(\mu)$, $\mathcal{H}^{\mu,s}_{\infty}(\Omega) = 0$, but for $s < \underline{\dim}(\mu)$, there exists a constant $c(d, \mu, s)$ such that the following more precise estimates hold true [10, Theorem 2.6]:

$$\begin{aligned} c(d, \mu, s)|B|^{s} &\leq \mathcal{H}^{\mu,s}_{\infty}(B) \leq \mathcal{H}^{\mu,s}_{\infty}(B) \leq |B|^s, \\ c(d, \mu, s)\mathcal{H}^{s}_{\infty}(\Omega \cap K) &\leq \mathcal{H}^{\mu,s}_{\infty}(\Omega) \leq \mathcal{H}^{s}_{\infty}(\Omega \cap K). \end{aligned} \tag{65}$$

- When the measure μ is weakly conformal (Definition 2.3) or the projection of a Gibbs measure, μ is exact dimensional (see [19, Theorem 2.1] for instance), which implies that $\underline{\dim}_H\mu = \overline{\dim}_H\mu$. In this case, Theorem 5.5 provides a complete description of the essential Hausdorff content, except at $s = \dim_H \mu$ (this case must depend in general on the separation property of the IFS).

Before proving Theorem 5.5, we prove that projections of quasi-Bernoulli measures satisfy the assumptions of Theorem 5.5.

Let $\nu \in \mathcal{M}(\{1, \ldots, m\}^{\mathbb{N}})$ be a measure. We call ν a quasi-Bernoulli measure if there exists $C > 0$ such that for every $\underline{i}, \underline{j} \in \bigcup_{k\geq 1}\{1, \ldots, m\}^k$, one has

$$C^{-1} \leq \frac{\nu([\underline{i}\underline{j}])}{\nu([\underline{i}]) \times \nu([\underline{j}])} \leq C.$$

PROPOSITION 5.6. *Let $S = \{f_1, \ldots, f_m\}$ be a weakly conformal IFS, let K be its attractor and π the canonical projection from $\{1, \ldots, m\}^{\mathbb{N}}$ to K. Let ν be a quasi-Bernoulli measure on $\{1, \ldots, m\}^{\mathbb{N}}$ such that for every $1 \leq i \leq m$, $\nu([i]) \neq 0$ and write $\mu = \nu \circ \pi^{-1}$. Then, for every $n \in \mathbb{N}$ and every $\underline{i} \in \{1, \ldots, m\}^n$, $\mu \circ f_{\underline{i}}^{-1}$ is absolutely continuous with respect to μ.*

Proof. Note that since for every $1 \leq i \leq m$, $\nu([i]) \neq 0$, one has for every $n \in \mathbb{N}$ and every $\underline{i} = (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$,

$$\nu([\underline{i}]) \geq \frac{1}{C^n}\prod_{k=1}^{n} \nu([i_k]) > 0.$$

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In addition, for every Borel set $E \subset \mathbb{R}^d$, one has

$$\begin{aligned}\nu(\pi^{-1}(E) \cap [\underline{i}]) &= \nu\Big\{\underline{i}(x_n)_{n\in\mathbb{N}} : f_{\underline{i}}\Big(\lim_{n\to+\infty} f_{x_1} \circ \cdots \circ f_{x_n}(0)\Big) \in E\Big\} \\ &= \nu\Big\{\underline{i}(x_n)_{n\in\mathbb{N}} : \lim_{n\to+\infty} f_{x_1} \circ \cdots \circ f_{x_n}(0) \in f_{\underline{i}}^{-1}(E)\Big\} \\ &= \nu\{\underline{i}(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in \pi^{-1} \circ f_{\underline{i}}^{-1}(E)\}.\end{aligned}$$

This implies that

$$C^{-1}\nu(\underline{i})\nu(\pi^{-1} \circ f_{\underline{i}}^{-1}(E)) \leq \nu(\pi^{-1}(E) \cap [\underline{i}]) \leq C\nu(\underline{i})\nu(\pi^{-1} \circ f_{\underline{i}}^{-1}(E)),$$

that is,

$$C^{-1}\nu(\underline{i})\mu(f_{\underline{i}}^{-1}(E)) \leq \nu(\pi^{-1}(E) \cap [\underline{i}]) \leq C\nu(\underline{i})\mu(f_{\underline{i}}^{-1}(E)).$$

Since, for every $n \in \mathbb{N}$,

$$\mu(E) = \nu(\pi^{-1}(E)) = \sum_{\underline{j}\in\{1,\ldots,m\}^n} \nu(\pi^{-1}(E) \cap [\underline{j}]),$$

we have

$$C^{-1} \sum_{\underline{j}\in\{1,\ldots,m\}^n} \nu(\underline{j})\mu(f_{\underline{j}}^{-1}(E)) \leq \mu(E) \leq C \sum_{\underline{j}\in\{1,\ldots,m\}^n} \nu(\underline{j})\mu(f_{\underline{j}}^{-1}(E)).$$

So,

$$\mu(E) = 0 \Longrightarrow C^{-1}\nu(\underline{i})\mu(f_{\underline{i}}^{-1}(E)) = 0 \Longrightarrow \mu(f_{\underline{i}}^{-1}(E)) = 0,$$

which proves that $\mu \circ f_{\underline{i}}^{-1}$ is absolutely continuous with respect to μ. □

Let us now prove Theorem 5.5.

Proof. As mentioned above, the proof of Theorem 5.5 is similar to the proof of (65) from [**10**], only it is made significantly more technical by the assumption that the IFS is weakly conformal.

Let us recall the well-known Besicovitch covering theorem.

THEOREM 5.7. [**28**] *There exists $Q_d \in \mathbb{N}^\star$, a constant depending only on the dimension d, such that for every $E \subset [0, 1]^d$, for every set $\mathcal{F} = \{B(x, r_{(x)}) : x \in E, r_{(x)} > 0\}$, there exists $\mathcal{F}_1, \ldots, \mathcal{F}_{Q_d}$ finite or countable sub-families of $\mathcal{F}$ such that:*

- *for all $1 \leq i \leq Q_d$, for all $L \neq L' \in \mathcal{F}_i$, one has $L \cap L' = \emptyset$.*
- *E is covered by the families $\mathcal{F}_i$, that is,*

$$E \subset \bigcup_{1\leq i\leq Q_d} \bigcup_{L\in\mathcal{F}_i} L. \tag{66}$$

The proof of Theorem 5.7 relies on the following geometric lemma, which will also be used.

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LEMMA 5.8. [28] *Let $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ be a family of balls and B a ball such that:*

(i) *for all $n \geq 1$, $|B_n| \geq \frac{1}{2}|B|$;*

(ii) *for all $n_1 \neq n_2 \geq 1$, the center of B_{n_1} does not belong to B_{n_2}.*

Then, B intersects less than Q_d elements of $\mathcal{B}$, where Q_d is the same constant as in Theorem 5.7.

Let us first prove the above estimates for balls. We fix $S = \{f_1, \ldots, f_m\}$ a weakly conformal IFS and we denote by K its attractor.

PROPOSITION 5.9. *Let μ be a measure supported on K satisfying that for every $\underline{i} \in \bigcup_{n\geq 1}\{1, \ldots, m\}^n$, $\mu(f_{\underline{i}}^{-1})$ is absolutely continuous with respect to μ. Then, for any $0 < \varepsilon \leq \underline{\dim}_H(\mu)$, any $0 \leq \varepsilon' \leq \frac{1}{2}$ such that $\underline{\dim}_H(\mu) - \varepsilon + \varepsilon' > 0$, there exists a constant $\chi(d, \mu, \varepsilon, \varepsilon') > 0$ such that for any ball $B = B(x, r)$ with $x \in K$ and $r \leq 1$, one has*

$$\chi(d, \mu, \varepsilon, \varepsilon')|B|^{\underline{\dim}_H(\mu)-\varepsilon+\varepsilon'} \leq \mathcal{H}^{\mu,\underline{\dim}_H(\mu)-\varepsilon}_\infty(B) \leq \mathcal{H}^{\mu,\underline{\dim}_H(\mu)-\varepsilon}_\infty(B) \leq |B|^{\underline{\dim}_H(\mu)-\varepsilon}.$$

In addition, for any $s > \overline{\dim}_H(\mu)$, $\mathcal{H}^{\mu,s}_\infty(B) = 0$.

Proof. Note first that, by definition of $\mathcal{H}^{\mu,s}_\infty$ (Definition 5.1), for any $s > \overline{\dim}_H(\mu)$, $\mathcal{H}^{\mu,s}_\infty(B) = 0$.

Let us consider $0 \leq s < \underline{\dim}_H(\mu)$ and start by a few remarks.

Set $\alpha = \underline{\dim}_H(\mu)$, and let $\varepsilon > 0$ and $\rho > 0$ be two real numbers. One defines

$$E^{\alpha,\rho,\varepsilon}_\mu = \{x \in \mathbb{R}^d : \text{ for all } r \leq \rho,\ \mu(B(x, r)) \leq r^{\alpha-\varepsilon}\}.$$

By definition of $\underline{\dim}_H \mu$ (Definition 2.2), for every $\varepsilon > 0$,

$$\mu\bigg(\bigcup_{\rho>0} E^{\alpha,\rho,\varepsilon}_\mu\bigg) = 1.$$

Let $\varepsilon > 0$ and $0 < \rho_\varepsilon \leq 1$ be two real numbers such that $\mu(E^{\alpha,\rho_\varepsilon,\varepsilon}_\mu) \geq \frac{1}{2}$ and write $E = E^{\alpha,\rho_\varepsilon,\varepsilon}_\mu$.

Write $c_{\underline{i}} = |f_{\underline{i}}(K)|$. Let us fix $\underline{i} = (i_1, \ldots, i_k) \in \Lambda^*$. For any $x \in K$ and $r > 0$, by equations (21) and (22) applied with $\theta = \varepsilon'$, one has

$$f_{\underline{i}}(B(x, r)) \supset B(f_{\underline{i}}(x_0), \widehat{C}_{\varepsilon'} c_{\underline{i}}(x_0)^{1-\varepsilon'} r) \supset B\bigg(f_{\underline{i}}(x_0), \frac{\widehat{C}_{\varepsilon'}^{-2/(1-\varepsilon')}}{|K|^{-(1+\varepsilon')/(1-\varepsilon')}} c_{\underline{i}}^{(1+\varepsilon')/(1-\varepsilon')} r\bigg).$$

Remember that $\varepsilon' \leq \frac{1}{2}$. Since $(1 + \varepsilon')/(1 - \varepsilon') \leq 1 + 4\varepsilon'$,

$$f_{\underline{i}}(B(x, r)) \supset B(f_{\underline{i}}(x_0), \widehat{C}_{\varepsilon'}^{-2/(1-\varepsilon')} \cdot |K|^{(1+\varepsilon')/(1-\varepsilon')} c_{\underline{i}}^{1+4\varepsilon'} r). \tag{67}$$

Writing $\mu_{\underline{i}} = \mu(f_{\underline{i}}^{-1})$, equation (67) yields

$$\begin{aligned} E_{\underline{i}} &:= f_{\underline{i}}(E) \\ &= \{f_{\underline{i}}(x) \in K : \text{ for all } r \leq \rho_\varepsilon,\ \mu(B(x, r)) \leq r^{\alpha-\varepsilon}\} \end{aligned}$$

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$$\subset \Big\{ f_{\underline{i}}(x), x \in K : \text{ for all } r \le \rho_\varepsilon,$$
$$\mu(f_{\underline{i}}^{-1}(B(f_{\underline{i}}(x_0), \widehat{C}_{\varepsilon'}^{-2/(1-\varepsilon')} \cdot |K|^{(1+\varepsilon')/(1-\varepsilon')} c_{\underline{i}}^{1+4\varepsilon'} r)))$$
$$\le \bigg(\frac{\widehat{C}_{\varepsilon'}^{-2/(1-\varepsilon')} \cdot |K|^{(1+\varepsilon')/(1-\varepsilon')} c_{\underline{i}}^{1+4\varepsilon'} r}{\widehat{C}_{\varepsilon'}^{-2/(1-\varepsilon')} \cdot |K|^{(1+\varepsilon')/(1-\varepsilon')} c_{\underline{i}}^{1+4\varepsilon'}} \bigg)^{\alpha-\varepsilon} \Big\}$$
$$= \Big\{ y \in f_{\underline{i}}(K) : \text{ for all } r' \le \widehat{C}_{\varepsilon'}^{-2/(1-\varepsilon')} \cdot |K|^{(1+\varepsilon')/(1-\varepsilon')} c_{\underline{i}}^{1+4\varepsilon'} \rho_\varepsilon,$$
$$\mu_{\underline{i}}(B(y, r')) \le \bigg(\frac{r'}{\widehat{C}_{\varepsilon'}^{-2/(1-\varepsilon')} \cdot |K|^{(1+\varepsilon')/(1-\varepsilon')} c_{\underline{i}}^{1+4\varepsilon'}} \bigg)^{\alpha-\varepsilon} \Big\}. \tag{68}$$

Notice also that

$$\mu_{\underline{i}}(E_{\underline{i}}) = \mu(E) \ge \tfrac{1}{2}.$$

We are now ready to estimate the μ-essential content of a ball B centered in K. Let us write

$$\gamma(S, \varepsilon') = \widehat{C}_{\varepsilon'}^{-2/(1-\varepsilon')} \cdot |K|^{(1+\varepsilon')/(1-\varepsilon')}. \tag{69}$$

Let $B = B(x, r)$ with $x \in K$ and $r \le c_0 := \min_{z \in K} \min_{1 \le i \le m} \|f_i'(z)\|$. Since $x \in K$, there exists $\underline{i} = (i_1, \ldots, i_k) \in \Lambda^*$ such that:

- $x \in f_{\underline{i}}(K)$;
- $|f_{\underline{i}}(K)| \le \frac{1}{3}|B|$;
- $|f_{(i_1,\ldots,i_{k-1})}(K)| \ge \frac{1}{3}|B|$.

By equation (22), for any $y \in K$, one has

$$|f_{\underline{i}}(K)| \ge \widehat{C}_{\varepsilon'}^{-1} \|f_{\underline{i}}'(y)\|^{1+\varepsilon'} |K| \tag{70}$$

and

$$\|f_{\underline{i}}(y)\| = \|f'_{(i_1,\ldots,i_{n-1})}(f_n(x)) \circ f'_{i_n}(x)\| \ge \|f'_{(i_1,\ldots,i_{n-1})}(f_n(x))\| c_0$$
$$\ge |f_{(i_1,\ldots,i_{n-1}}(K)|^{1/(1-\varepsilon')} \widehat{C}_{\varepsilon'}^{-1/(1-\varepsilon')} \cdot |K|^{-1/(1-\varepsilon')} c_0. \tag{71}$$

Combining equations (70) and (71), one obtains

$$c_{\underline{i}} = |f_{\underline{i}}(K)| \ge \widehat{C}_{\varepsilon'}^{-1-(1+\varepsilon')/(1-\varepsilon')} |K|^{-2\varepsilon'/(1-\varepsilon')} c_0^{1+\varepsilon'} |f_{(i_1,\ldots,i_{n-1})}(K)|^{(1+\varepsilon')/(1-\varepsilon')}$$
$$\ge \widehat{C}_{\varepsilon'}^{-1-(1+\varepsilon')/(1-\varepsilon')} |K|^{-2\varepsilon'/(1-\varepsilon')} c_0^{1+\varepsilon'} r^{1+4\varepsilon'}. \tag{72}$$

Note that $E_{\underline{i}} \subset B$.

Consider a set $A \subset B$ verifying $\mu(A) = \mu(B)$. One aims to give a lower bound for the Hausdorff content of A which depends only on B, d, ε, ε', and the measure μ.

Consider a sequence of balls $(L_n = B(x_n, \ell_n))_{n \ge 1}$ covering $A \cap E_{\underline{i}}$ such that $\ell_n < \gamma(S, \varepsilon') \rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$ and $x_n \in A \cap E_{\underline{i}}$.

Since $\mu_{\underline{i}}$ is absolutely continuous with respect to μ, it holds that $\mu_{\underline{i}}(A) = 1$.

By equation (68) applied to each ball L_n, $n \in \mathbb{N}$, one has $(|L_n|/\gamma(S,\varepsilon')c_{\underline{i}}^{1+4\varepsilon'})^{\alpha-\varepsilon} \geq \mu_{\underline{i}}(L_n)$, so that, recalling equation (72),

$$\begin{aligned}\sum_{n\in\mathbb{N}} |L_n|^{\alpha-\varepsilon} &\geq \sum_{n\in\mathbb{N}} (\gamma(S,\varepsilon')c_{\underline{i}}^{1+4\varepsilon'})^{\alpha-\varepsilon}\mu_{\underline{i}}(L_n) \geq (\gamma(S,\varepsilon')c_{\underline{i}}^{1+4\varepsilon'})^{\alpha-\varepsilon}\mu_{\underline{i}}\bigg(\bigcup_{n\in\mathbb{N}} L_n\bigg)\\ &\geq (\gamma(S,\varepsilon')c_{\underline{i}}^{1+4\varepsilon'})^{\alpha-\varepsilon}\mu_{\underline{i}}(E_{\underline{i}}) \geq \tfrac{1}{2}(\gamma(S,\varepsilon')c_{\underline{i}}^{1+4\varepsilon'})^{\alpha-\varepsilon}\\ &\geq \kappa(\mu,\varepsilon',\varepsilon)r^{(1+4\varepsilon')^2(\alpha-\varepsilon)} \geq \kappa(\mu,\varepsilon',\varepsilon)r^{(1+16\varepsilon')(\alpha-\varepsilon)}, \end{aligned} \tag{73}$$

where $\kappa(\mu,\varepsilon',\varepsilon) = \frac{1}{2}\gamma(S,\varepsilon')^{\alpha-\varepsilon} \cdot (\widehat{C}_{\varepsilon'}^{-1-(1+\varepsilon')/(1-\varepsilon')}|K|^{-2\varepsilon'/(1-\varepsilon')}c_0^{1+\varepsilon'})^{(1+4\varepsilon')(\alpha-\varepsilon)}$.

This series of inequalities holds for any sequence of balls $(L_n)_{n\in\mathbb{N}}$ with radius less than $\gamma(S,\varepsilon')\rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$ centered in $A \cap E_{\underline{i}}$. One now proves that one can freely remove those constraints on the center and the radius of the balls used to cover $A \cap E_{\underline{i}}$, up to a multiplicative constant.

Consider balls $(L_n = B(x_n,\ell_n))_{n\geq 1}$ covering $A \cap E_{\underline{i}}$ such that $\ell_n < \gamma(S,\varepsilon')\rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$ but x_n does not necessarily belongs to $A \cap E_{\underline{i}}$.

Let $n \in \mathbb{N}$. One constructs recursively a sequence of balls $(L_{n,j})_{1\leq j\leq J_n}$ such that the following properties hold for any $1 \leq j \leq J_n$:

- $L_{n,j}$ is centered on $A \cap E_{\underline{i}} \cap L_n$;
- $A \cap E_{\underline{i}} \cap L_n \subset \bigcup_{1\leq j\leq J_n} L_{n,j}$;
- for all $1 \leq j \leq J_n$, $|L_{n,j}| = |L_n|$;
- the center of $L_{n,j}$ does not belong to any $L_{n,j'}$ for $1 \leq j' \neq j \leq J_n$.

To achieve this, simply consider $y_1 \in A \cap E_{\underline{i}} \cap L_n$ and set $L_{1,n} = B(y_1,\ell_n)$. If $A \cap E_{\underline{i}} \cap L_n \not\subseteq L_{1,n}$, consider $y_2 \in A \cap E_{\underline{i}} \cap L_n \setminus L_{1,n}$ and set $L_{2,n} = B(y_2,\ell_n)$. If $A \cap E_{\underline{i}} \cap L_n \not\subseteq L_{1,n} \cup L_{2,n}$, consider $y_3 \in A \cap E_{\underline{i}} \cap L_n \setminus L_{1,n} \cup L_{2,n}$ and set $L_{3,n} = B(y_3,\ell_n)$, and so on.

Note that for any $1 \leq j \leq J_n$, any ball $L_{j,n}$ has radius ℓ_n, intersects L_n (which also has radius ℓ_n), and, because $y_j \notin \bigcup_{1\leq j'\neq j\leq J_n} L_{j',n}$, it holds that for any $j \neq j'$, $\frac{1}{3}L_{n,j} \cap \frac{1}{3}L_{n,j'} = \emptyset$. A volume argument yields that $J_n \leq Q_{d,1/3}$, where $Q_{d,1/3}$ is constant which only depends on the dimension d and the contraction factor $\frac{1}{3}$.

Hence, denoting by $(\widetilde{L}_n)_{n\in\mathbb{N}}$ the collection of the corresponding balls centered on $A \cap E_{\underline{i}}$ associated with all the balls L_n, one has by equation (73) applied to $(\widetilde{L}_n)_{n\in\mathbb{N}}$,

$$\sum_{n\in\mathbb{N}} |L_n|^{\alpha-\varepsilon} \geq \frac{1}{Q_{d,1/3}} \sum_{n\in\mathbb{N}} |\widetilde{L}_n|^{\alpha-\varepsilon} \geq \frac{\kappa(\mu,\varepsilon',\varepsilon)}{Q_{d,1/3}} r^{(1+4\varepsilon')(\alpha-\varepsilon)}.$$

Remark also that any ball of radius smaller than $c_{\underline{i}}$ can be covered by at most $(2c_{\underline{i}}^{-4\varepsilon'}/\gamma(S,\varepsilon')\rho_\varepsilon)^d$ balls of radius $\gamma(S,\varepsilon')\rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$. Moreover, by equation (72),

$$c_{\underline{i}}^{-4\varepsilon'} \leq (\widehat{C}_{\varepsilon'}^{-1-(1+\varepsilon')/(1-\varepsilon')}|K|^{-2\varepsilon'/(1-\varepsilon')}c_0^{1+\varepsilon'})^{-4\varepsilon'} r^{-4\varepsilon'\cdot(1+4\varepsilon')}.$$

Setting

$$\widehat{\kappa}(\mu,\varepsilon,\varepsilon',d) = \left(\frac{2(\widehat{C}_{\varepsilon'}^{-1-(1+\varepsilon')/(1-\varepsilon')}|K|^{-2\varepsilon'/(1-\varepsilon')}c_0^{1+\varepsilon'})^{-4\varepsilon'}}{\gamma(S,\varepsilon')\rho_\varepsilon}\right)^d,$$

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any ball of radius less than $c_{\underline{i}}$ can be covered by less than $\widehat{\kappa}(\mu, \varepsilon, \varepsilon', d) r^{-4d\varepsilon' \cdot (1+4\varepsilon')}$ balls of radius less than $\gamma(S, \varepsilon')\rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$.

This proves that for any sequence of balls $\widehat{L}_n$ with $|\widehat{L}_n| \leq c_{\underline{i}}$ covering $A \cap E_{\underline{i}}$, recalling equation (73), it holds that

$$\sum_{n\in\mathbb{N}} |\widehat{L}_n|^{\alpha-\varepsilon} \geq Q_{d,1/3}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)^{-1} r^{4d\varepsilon' \cdot (1+4\varepsilon')} \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon)} \tag{74}$$

$$\geq Q_{d,1/3}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)^{-1} \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon)+4d\varepsilon' \cdot (1+4\varepsilon')}. \tag{75}$$

Recalling that $|E_{\underline{i}}| \leq c_{\underline{i}}$ and Definition 4, since equation (74) is valid for any covering $(\widehat{L}_n)_{n\in\mathbb{N}}$ of $A \cap E_{\underline{i}}$ with $|L_n| \leq c_{\underline{i}}$, one has

$$\begin{aligned} |B|^{\alpha-\varepsilon} &\geq \mathcal{H}_\infty^{\alpha-\varepsilon}(A) \geq \mathcal{H}_\infty^{\alpha-\varepsilon}(A \cap E_{\underline{i}}) \\ &\geq Q_{d,1/3}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)^{-1} \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon)+4d\varepsilon' \cdot (1+4\varepsilon')}. \end{aligned} \tag{76}$$

Taking the inf over all the set $A \subset B$ satisfying $\mu(A) = \mu(B)$, one obtains

$$|B|^{\alpha-\varepsilon} \geq \mathcal{H}_\infty^{\mu,s}(B) \geq Q_{d,1/3}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)^{-1} \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon)+4d\varepsilon' \cdot (1+4\varepsilon')}.$$

The results stands for balls of diameter less than c_0.

Set

$$\varepsilon_0' = 16\varepsilon'(\alpha - \varepsilon) + 4d\varepsilon' \cdot (1 + 4\varepsilon')$$

and write

$$\gamma(d, \mu, \varepsilon, \varepsilon_0') = c_0^{\alpha-\varepsilon+\varepsilon_0'} Q_{d,1/3}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon_0', d)^{-1} \kappa(\mu, \varepsilon_0', \varepsilon).$$

For any ball of radius less than 1 centered on K, one has

$$|B|^{\alpha-\varepsilon} \geq \mathcal{H}_\infty^{\mu,\alpha-\varepsilon}(B) \geq \gamma(d, \mu, \varepsilon, \varepsilon_0') r^{\alpha-\varepsilon+\varepsilon_0'}.$$ □

The estimates of Theorem 5.5 are now established in the case of general open sets.

Recall that by item (5) of Proposition 5.14, for any $s > \dim(\mu)$ and any set E, $\mathcal{H}_\infty^{\mu,s}(E) = 0$.

Let us fix $s < \dim(\mu)$, $\varepsilon' > 0$ and set $\varepsilon' = \min\{(\dim(\mu) - s)/2, \frac{1}{2}\} > 0$.

Since $K \cap \Omega \subset \Omega$ and $\mu(K \cap \Omega) = \mu(\Omega)$, it holds that

$$\mathcal{H}_\infty^{\mu,s}(\Omega) \leq \mathcal{H}_\infty^s(\Omega \cap K).$$

It remains to show that there exists a constant $c(d, \mu, s, \varepsilon')$ such that for any open set Ω, the converse inequality

$$c(d, \mu, s, \varepsilon') \mathcal{H}_\infty^{s+\varepsilon'}(\Omega \cap K) \leq \mathcal{H}_\infty^{\mu,s}(\Omega)$$

holds.

Let $E \subset \Omega$ be a Borel set such that $\mu(E) = \mu(\Omega)$ and

$$\mathcal{H}_\infty^s(E) \leq 2\mathcal{H}_\infty^{\mu,s}(\Omega). \tag{77}$$

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Let $\{L_n\}_{n\in\mathbb{N}}$ be a covering of E by balls verifying

$$\mathcal{H}^s_\infty(L) \leq \sum_{n\geq 0} |L_n|^s \leq 2\mathcal{H}^s_\infty(E). \tag{78}$$

The covering $(L_n)_{n\in\mathbb{N}}$ will be modified into a covering $(\widetilde{L}_n)_{n\in\mathbb{N}}$ verifying the following properties:

- $K \cap \Omega \subset \bigcup_{n\in\mathbb{N}} \widetilde{L}_n$;
- $\bigcup_{n\in\mathbb{N}} L_n \subset \bigcup_{n\in\mathbb{N}} \widetilde{L}_n$;
- $$\sum_{n\geq 0} |\widetilde{L}_n|^{s+\varepsilon'} \leq 8.2^{s+\varepsilon'} \frac{Q_d^2}{\gamma(d,\mu,\varepsilon,\varepsilon')} \sum_{n\geq 0} |L_n|^s,$$

where Q_d and $\gamma(d,\mu,\varepsilon,\varepsilon')$ are the constants arising from Theorem 5.7 and Proposition 5.9.

The last item together with equations (77) and (78) then immediately imply that

$$\frac{\gamma(d,\mu,\varepsilon,\varepsilon')}{8.2^{s+\varepsilon'} Q_d^2} \mathcal{H}^{s+\varepsilon'}_\infty(K \cap \Omega) \leq \mathcal{H}^{\mu,s}_\infty(\Omega).$$

Setting $c(d,\mu,\varepsilon,\varepsilon') = \gamma(d,\mu,\varepsilon,\varepsilon')/8.2^{s+\varepsilon'} Q_d^2$ will then conclude the proof.

Let us start the construction of the sequence $(\widetilde{L}_n)_{n\in\mathbb{N}}$.

Let $\Delta = (K \setminus \bigcup_{n\in\mathbb{N}} B_n) \cap \Omega$. For every $x \in \Delta$, fix $0 < r_x \leq 1$ such that $B(x,r_x) \subset \Omega$. One of the following alternatives must occur:

(1) for any ball L_n such that $L_n \cap B(x,r_x) \neq \emptyset$, $|L_n| \leq r_x$; or

(2) there exists $n_x \in \mathbb{N}$ such that $L_{n_x} \cap B(x,r_x) \neq \emptyset$ and $|L_{n_x}| \geq r_x$.

Consider the set S_1 of points of X for which the first alternative holds.

By Theorem 5.7, it is possible to extract from the covering of S_1, $\{B(x,r_x), x \in S_1\}$, Q_d families of pairwise disjoint balls, $\mathcal{F}_1, \ldots, \mathcal{F}_{Q_d}$ such that

$$S_1 \subset \bigcup_{1\leq i\leq Q_d} \bigcup_{L\in\mathcal{F}_i} L.$$

Now, any ball L_n intersecting a ball $L \in \bigcup_{1\leq i\leq Q_d} \mathcal{F}_i$ must satisfy $|L_n| \leq L$. In particular, since for any $1 \leq i \leq Q_d$, the balls of $\mathcal{F}_i$ are pairwise disjoint, applying Lemma 5.8 to the ball of $\mathcal{F}_i$ intersecting L, we get that the ball L_n intersects at most Q_d balls of $\mathcal{F}_i$ and hence at most Q_d^2 balls of $\bigcup_{1\leq i\leq Q_d} \mathcal{F}_i$.

Let $L \in \bigcup_{1\leq i\leq Q_d} \mathcal{F}_i$. One aims at replacing all the balls L_n intersecting L by the ball $2L$.

For any $1 \leq i \leq Q_d$ and any ball $L \in \mathcal{F}_i$, denote by $\mathcal{G}_L$ the set of balls L_n intersecting L. Since $E \subset \bigcup_{n\in\mathbb{N}} L_n$ and $\mu(E) = \mu(\Omega)$, one has $E \cap L \subset \bigcup_{B\in\mathcal{G}_L} B$ and $\mu(E \cap L) = \mu(L)$. By Definition 5.1 and Proposition 5.9, this implies that

$$\gamma(d,\mu,\varepsilon,\varepsilon')|L|^{s+\varepsilon'} \leq \mathcal{H}^{\mu,s}_\infty(L) \leq \sum_{B\in\mathcal{G}_L} \mathcal{H}^{\mu,s}_\infty(B) \leq \sum_{B\in\mathcal{G}_L} |B|^s. \tag{79}$$

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Replace the balls of $\mathcal{G}_L$ by the ball $\widehat{L} = 2L$ (recall that $\bigcup_{B\in\mathcal{G}_L} B \subset 2L$). The new sequence of balls so obtained by the previous construction applied to all the balls $L \in \bigcup_{\leq i\leq Q_d} \mathcal{F}_i$ is denoted by $(\widehat{L}_k)_{1\leq k\leq K}$, where $0 \leq K \leq +\infty$.

It follows from the construction and equation (79) that $S_1 \subset \bigcup_{1\leq k\leq K} \widehat{L}_k$ and

$$\sum_{1\leq k\leq K} \bigg(\frac{|\widehat{L}_k|}{2}\bigg)^{s+\varepsilon'} \leq \frac{Q_d^2}{\gamma(d,\mu,\varepsilon,\varepsilon')} \sum_{n\geq 0} |L_n|^s. \tag{80}$$

However, since for any $x \in S_2 = \Delta \setminus S_1$, there exists $n_x \in \mathbb{N}$ such that $L_{n_x} \cap B(x, r_x) \neq \emptyset$ and $r_x \leq |L_{n_x}|$, one has $S_2 \subset \bigcup_{n\in\mathbb{N}} 2L_n$, so that

$$\bigg(\bigcup_{n\in\mathbb{N}} L_n\bigg) \cup \bigg(K \cap \Omega \setminus \bigcup_{n\in\mathbb{N}} L_n\bigg) \subset \bigg(\bigcup_{1\leq k\leq K} \widehat{L}_k\bigg) \cup \bigg(\bigcup_{n\in\mathbb{N}} 2L_n\bigg).$$

Putting the elements of $(\widehat{L}_k)_{1\leq k\leq K}$ and $(2L_n)_{n\geq 0}$ in a single sequence $(\widehat{L}_n)_{n\geq 0}$, writing $(\widetilde{L}_n := 2\widehat{L}_n)_{n\in\mathbb{N}}$, by construction, $K \cap \Omega \subset \bigcup_{n\in\mathbb{N}} \widetilde{L}_n$ and due to equation (80),

$$\begin{aligned}\mathcal{H}^{s+\varepsilon'}_\infty(K\cap\Omega) &\leq \sum_{n\geq 0} |\widetilde{L}_n|^{s+\varepsilon'} \leq 2^{s+\varepsilon'}\bigg(\frac{Q_d^2}{\gamma(d,\mu,\varepsilon,\varepsilon')} + 1\bigg)\sum_{n\geq 0}|L_n|^s \\ &\leq 8.2^{s+\varepsilon'} \frac{Q_d^2}{\gamma(d,\mu,\varepsilon,\varepsilon')} \mathcal{H}^{\mu,s}_\infty(\Omega).\end{aligned}$$

The proof is concluded now by setting

$$c(d,\mu,s,\varepsilon') = \frac{\gamma(d,\mu,\dim(\mu)-s,\varepsilon')}{Q_d^2 8.2^{s+\varepsilon'}}. \qquad \square$$

Remark 5.10.

(1) The part of the proof of Theorem 5.5 which handles the case of open sets only relies on the fact that there exists $\gamma(d,\mu,\varepsilon,\varepsilon')$ such that for any $x \in K$, for any $\rho > 0$, there exists $0 < r_x \leq \rho$ so that, writing $B = B(x, r_x)$,

$$\begin{aligned}\gamma(d,\mu,\varepsilon,\varepsilon')|B|^{\underline{\dim}_H(\mu)-\varepsilon+\varepsilon'} &\leq \mathcal{H}^{\mu,\underline{\dim}_H(\mu)-\varepsilon}_\infty(B) \\ &\leq \mathcal{H}^{\mu,\underline{\dim}_H(\mu)-\varepsilon}_\infty(B) \leq |B|^{\underline{\dim}_H(\mu)-\varepsilon}.\end{aligned} \tag{81}$$

(2) It is easily verified that the estimates of Proposition 5.9 hold in particular if, for $s \geq 0$, there exists a constant $C > 0$ such that for any $x \in \mathrm{supp}(\mu)$, any $0 < r < R$, $(\mu(B(x,r))/\mu(B(x,R))) \leq C.(r/R)^s$. This condition is naturally linked to the lower Assouad dimension $\dim_L(\mu)$ of μ defined as [20]

$$\begin{aligned}\dim_L(\mu) = \inf\bigg\{s \geq 0 :\ &\text{for all } x \in \mathrm{supp}(\mu), \\ &\text{for all } 0 < r < R,\ \frac{\mu(B(x,r))}{\mu(B(x,R))} \leq C\bigg(\frac{r}{R}\bigg)^s\bigg\}.\end{aligned} \tag{82}$$

More precisely, the estimates of Proposition 5.9 and Theorem 5.5 hold for any $s < \dim_L(\mu)$.

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5.2. *Mass transference principle in the weakly conformal case.* Combining Theorem 5.4 with Theorem 5.5 and Proposition 5.3 yields the following result.

THEOREM 5.11. *Let $S = \{f_1, \ldots, f_m\}$ be a C^1 weakly conformal IFS of a compact X with attractor K and μ be a measure on K such that for every $\underline{i} \in \bigcup_{k\geq 1}\{1, \ldots, m\}^k$, $\mu \circ f_{\underline{i}}^{-1}$ is absolutely continuous with respect to μ.*

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of closed balls centered on K with $\lim_{n\to+\infty} |B_n| = 0$.

(1) *Suppose that $(B_n)_{n\in\mathbb{N}}$ is μ-a.c. Then, there exists a gauge function ζ such that $\lim_{r\to 0^+}(\log(\zeta(r))/\log(r)) \geq (\underline{\dim}_H(\mu)/\delta)$ and $\mathcal{H}^{\zeta}(\limsup_{n\to\infty} B_n^{\delta}) > 0$. In particular,*

$$\dim_H\left(\limsup_{n\to+\infty} B_n^{\delta}\right) \geq \frac{\underline{\dim}_H(\mu)}{\delta}. \tag{83}$$

(2) *Suppose that $\mu(\limsup_{n\to+\infty} B_n) = 1$. Then, equation (83) still holds but the existence of the gauge function is not ensured. Furthermore, if μ is doubling, then $(B_n)_{n\in\mathbb{N}}$ is μ-a.c., so that the conclusion of item (1) holds.*

Remark 5.12. One emphasizes that, for the purpose of this article, the results are stated for balls but Theorems 5.5 and 5.4 allow to deal with more general open sets. For instance, given $1 \leq \tau_1 \leq \cdots \leq \tau_d$, if U_n is an open rectangle of sidelength $\prod_{i=1}^{n} |B_n|^{\tau_i}$, one needs to estimate the (classical) Hausdorff content of the union of the cubes $C \subset U_n$ of length-side $|B_n|^{\tau_d}$ (the smallest side of U_n) for which $C \cap K \neq \emptyset$. This is achievable as soon as the rectangle has sides in 'natural directions' for the IFS we consider (for instance, horizontal rectangles on a self-similar carpet).

Proof. One proves the first item of Theorem 5.11.

Fix $\mu \in \mathcal{M}(\mathbb{R}^d)$ supported on K satisfying that for every $\underline{i} \in \bigcup_{k\geq 1}\{1, \ldots, m\}^k$, $\mu \circ f_{\underline{i}}^{-1}$ is absolutely continuous with respect to μ. Let $(B_n)_{n\in\mathbb{N}}$ be a μ-a.c. sequence of balls centered on K satisfying $|B_n| \to 0$. Let us fix $\varepsilon > 0$.

Let us start with a lemma whose proof can be found in [**12**, Lemma 4.9].

LEMMA 5.13. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Let $\mathcal{B} = (B_n := B(x_n, r_n))_{n\in\mathbb{N}}$ be a μ-a.c. sequence of balls of $\mathbb{R}^d$. Then, for every $\varepsilon > 0$, there exists a μ-a.c. sub-sequence $(B_{\phi(n)})_{n\in\mathbb{N}}$ of $\mathcal{B}$ such that for every $n \in \mathbb{N}$, $\mu(B_{\phi(n)}) \leq (r_{\phi(n)})^{\underline{\dim}_H(\mu)-\varepsilon}$.*

By Lemma 5.13, up to an extraction, one can assume that $\mu(B_n) \leq |B_n|^{\dim(\mu)-\varepsilon/4}$.

The following proposition is proved in [**10**, Proposition 3.12].

PROPOSITION 5.14. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $s \geq 0$, and $A \subset \mathbb{R}^d$ be a Borel set. The s-dimensional $\mathcal{H}^{\mu,s}_{\infty}(\cdot)$ outer measure satisfies the following properties.*

(1) *If $|A| \leq 1$, the mapping $s \geq 0 \mapsto \mathcal{H}^{\mu,s}_{\infty}(A)$ is decreasing from $\mathcal{H}^{\mu,0}_{\infty}(A) = 1$ to $\lim_{t\to+\infty} \mathcal{H}^{\mu,t}_{\infty}(A) = 0$.*
(2) $0 \leq \mathcal{H}^{\mu,s}_{\infty}(A) \leq \min\{|A|^s, \mathcal{H}^{s}_{\infty}(A)\}$.
(3) *For every subset $B \subset A$ with $\mu(A) = \mu(B)$, $\mathcal{H}^{\mu,s}_{\infty}(A) = \mathcal{H}^{\mu,s}_{\infty}(B)$.*
(4) *For every $\delta \geq 1$, $\mathcal{H}^{\mu,s/\delta}_{\infty}(A) \geq (\mathcal{H}^{\mu,s}_{\infty}(A))^{1/\delta}$.*
(5) *For every $s > \overline{\dim}_H(\mu)$, $\mathcal{H}^{\mu,s}_{\infty}(A) = 0$.*

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Also, by Theorem 5.5 and item (5) of Proposition 5.14, there exists a constant $c(d, \mu, \dim(\mu) - \varepsilon/2, \varepsilon/4)$ such that, for any $n \in \mathbb{N}$, for any $\delta > 1$,

$$\mathcal{H}^{\mu,(\underline{\dim}_H(\mu)-\varepsilon)/\delta}({B_n}^{\delta}) \geq c\bigg(d, \mu, \underline{\dim}_H(\mu) - \varepsilon, \frac{\varepsilon}{2}\bigg)|B_n|^{\underline{\dim}_H(\mu)-\varepsilon/2}.$$

Taking n large enough so that $|B_n|^{-\varepsilon/4} \geq c(d, \mu, \underline{\dim}_H(\mu)) - \varepsilon/2, \varepsilon/4)$, one gets

$$\mathcal{H}^{\mu,(\underline{\dim}_H(\mu)-\varepsilon/\delta)}({B_n}^{\delta}) \geq |B_n|^{\underline{\dim}_H(\mu)-\varepsilon/4} \geq \mu(B_n). \tag{84}$$

Defining $\mathcal{U}_\delta = ({B_n}^{\delta})_{n\in\mathbb{N}}$, using equation (84) and Theorem 5.4 with $s_\varepsilon = (\underline{\dim}_H(\mu) - \varepsilon/\delta)$ and letting $\varepsilon \to 0$ finishes the proof of the first item.

Assume now that the sequence $(B_n)_{n\in\mathbb{N}}$ satisfies only $\mu(\limsup_{n\to+\infty} B_n)$. Then, by Proposition 5.3, $(2B_n)_{n\in\mathbb{N}}$ is μ-a.c.

Since, for any $\varepsilon > 0$,

$$\limsup_{n\to+\infty}(2B_n)^{\delta+\varepsilon} \subset \limsup_{n\to+\infty} B_n^{\delta},$$

applying the first item of Theorem 5.11 to $(2B_n)_{n\in\mathbb{N}}$, one gets

$$\dim_H\Big(\limsup_{n\to+\infty} B_n^{\delta}\Big) \geq \frac{\underline{\dim}_H(\mu)}{\delta + \varepsilon}.$$

Since ε was arbitrary, the second item is proved. □

Remark 5.15. The proof of Theorem 5.11 actually shows more. With the notation of [**10**, Definition 2.5], it is proved that $s(\mu, \mathcal{B}, \mathcal{U}_\delta) \geq (\dim_H(\mu)/\delta)$ so that [**10**, Theorem 2.11] holds for self-conformal measures instead of self-similar measures.

6. *Proof of Theorem 3.1*

Before starting the proof of Theorem 3.1, we explain why the very natural strategy which consists in approximating the attractor K by the sub-attractor of IFSs satisfying the strong separation condition does not yield any interesting conclusion in most (if not every) cases. Let $S = \{f_1, \ldots, f_m\}$ be a self-similar IFS and let K be its attractor. Recall that S is said to satisfy the strong separation condition (SSC) if, for every $1 \leq i \neq j \leq m$, we have

$$f_i(K) \cap f_j(K) = \emptyset.$$

Let us start by proving the following result.

PROPOSITION 6.1. *Let $S = \{f_1, \ldots, f_m\}$ be a self-similar IFS satisfying the strong separation condition and denote by K its attractor. Then, for every*

$$x_0 \notin \bigcup_{\underline{i}\in\Lambda^*} f_{\underline{i}}^{-1}(K),$$

for every $\delta > 1$, one has

$$W(x_0, \delta) = \emptyset,$$

where $W(x_0, \delta)$ is defined as in Theorem 3.1.

Proof. Let us write c_i as the contraction ratio of the map f_i.

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Since S satisfies the strong separation condition, there exists an open set U such that (see [3] for instance):

(a) $K \subset U$;

(b) for every $\underline{i} \in \Lambda^*$, $f_{\underline{i}}(U) \subset U$;

(c) for every $n \in \mathbb{N}$, for every $\underline{i} \neq \underline{j} \in \Lambda^n$, $f_{\underline{i}}(U) \cap f_{\underline{j}}(U) = \emptyset$.

Moreover, there must exist $N \in \mathbb{N}$ so large that for every $n \geq N$, for every $\underline{i} \in \Lambda^n$, one has $f_{\underline{i}}(x_0) \in U$. Now, fix $r > 0$ small enough so that for every $\underline{i} \in \Lambda^N$, one has

$$B(f_{\underline{i}}(x_0), r) \subset U \setminus K$$

and set

$$C = \min\left\{\frac{r}{c_{\underline{i}}}, \underline{i} \in \Lambda^N\right\}.$$

Fix $k \in \mathbb{N}$, $\underline{i}, \underline{i}' \in \Lambda^N$, and $\underline{j}, \underline{j}' \in \Lambda^k$. We want to show that there exists a constant $C' > 0$ such that, writing $\underline{h} = \underline{j}\underline{i}$, one has

$$d(f_{\underline{h}}(x_0), K) \geq C' c_{\underline{h}}.$$

Assume that $\underline{j} \neq \underline{j}'$, then by items (b) and (c), one has

$$f_{\underline{j}}(B(f_{\underline{i}}(x_0), r)) \cap f_{\underline{j}'}(K) = \emptyset,$$

which implies that

$$d(f_{\underline{j}\underline{i}}(x_0), f_{\underline{j}'\underline{i}'}(K)) \geq r c_{\underline{j}} \geq C c_{\underline{j}} c_{\underline{i}}.$$

To deal with the case where $\underline{j} = \underline{j}'$, we recall first that, since $x_0 \notin \bigcup_{\underline{i} \in \Lambda^*} f_{\underline{i}}^{-1}(K)$, the following quantity is strictly positive:

$$\widetilde{C} = \min\left\{\frac{d(f_{\underline{i}}(x), f_{\underline{i}'}(K))}{c_{\underline{i}}}, \underline{i}, \underline{i}' \in \Lambda^N\right\}.$$

Hence, in this case, we get

$$d(f_{\underline{j}\underline{i}}(x_0), f_{\underline{j}'\underline{i}'}(K)) \geq c_{\underline{j}} d(f_{\underline{i}}(x), f_{\underline{i}'}(K)) \geq \widetilde{C} c_{\underline{j}} c_{\underline{i}}.$$

In any case, we established that there exists a constant C' such that for every $n \geq N$ and $\underline{h}, \underline{h}' \in \Lambda^n$, we have

$$d(f_{\underline{h}}(x_0), f_{\underline{h}'}(K)) \geq C' c_{\underline{h}}.$$

This yields

$$d(f_{\underline{h}}(x_0), K) \geq C' c_{\underline{h}}.$$

In particular, recalling that $\delta > 1$, one has

$$B(f_{\underline{h}}(x_0), c_{\underline{h}}^{\delta}) \cap K = \emptyset,$$

which implies that

$$W(x_0, \delta) = \emptyset. \qquad \square$$

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We now establish that, in most cases, one cannot handle the case of overlapping self-similar IFSs by approximating the IFS by the sub-attractor of IFSs satisfying the strong separation condition.

PROPOSITION 6.2. *Let $S = \{f_1, \ldots, f_m\}$ be a self-similar IFS and K its attractor. Assume that S satisfies the following properties:*

- $\dim_H K = \dim(S)$*;*
- *S does not satisfy the open set condition;*
- *there exists an exact-dimensional measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ such that* $\mathrm{supp}(\mu) = K$ *and* $\dim_H y\mu = \dim_H K$.

Then, there exists a set $E \subset K$ with $\dim_H E = \dim_H K$ such that for every $x \in E$, every sub-IFS $\widetilde{S} = \{f_{\underline{i}_1}, \ldots, f_{\underline{i}_k}\}$ satisfying the strong separation condition (where $\underline{i}_1, \ldots, \underline{i}_k$ are words on $\{1, \ldots, m\}$), one has for every $\delta > 1$,

$$\widetilde{W}(x, \delta) = \emptyset,$$

where $\widetilde{W}(x, \delta)$ denotes the set $W(x, \delta)$ defined using the IFS $\widetilde{S}$.

Note that the hypotheses of Proposition 6.2 are quite weak. It applies, for instance, to every self-similar IFS on the real line satisfying the exponential separation condition with similarity dimension smaller than 1 and not the open set condition. More generally, it also applies to many examples of IFS which do not satisfy the open set condition to which Theorem 3.1 is applied in the present paper.

Proof. Let us call

$$\mathcal{S} = \left\{ \widetilde{S} = \{f_{\underline{i}_1}, \ldots, f_{\underline{i}_k}\} : \underline{i}_1, \ldots, \underline{i}_k \in \bigcup_{n \geq 0} \{1, \ldots, m\}^n, \widetilde{S} \text{ satisfies SSC} \right\}.$$

Note that $\mathcal{S}$ is countable and denote by $(K_n)_{n\in\mathbb{N}}$ the sequence of attractors associated with the elements of $\mathcal{S}$. Let us recall that, due to [35], since $\dim_H K = \dim(S)$ and S does not satisfy the open set condition,

$$\mathcal{H}^{\dim_H K}(K) = 0.$$

Also, recalling that for every $n \in \mathbb{N}$, $K_n \subset K$ satisfies the strong separation condition, one has $\mathcal{H}^{\dim_H K_n}(K_n) > 0$. In particular, this implies that $\dim_H K_n < \dim_H K$ and also that

$$\dim_H \bigcup_{\underline{i} \in \bigcup_{n \geq 0} \{1,\ldots,m\}^n} f_{\underline{i}}^{-1}(K_n) := \widetilde{K}_n < \dim_H K.$$

Let μ be a $\dim_H K$-exact dimensional measure with $\mathrm{supp}(\mu) = K$. The above argument proves that for every $n \in \mathbb{N}$, $\mu(\widetilde{K}_n) = 0$, so that

$$\mu\bigg(K \setminus \bigcup_{n \geq 0} \widetilde{K}_n\bigg) = 1.$$

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In particular, setting $E = K \setminus \bigcup_{n\geq 0} \widetilde{K}_n$, one has $\dim_H E = \dim_H K$. Moreover, for every $x \in E$ and every $\widetilde{S} \in \mathcal{S}$, by Proposition 6.1, for every $\delta > 1$,

$$\widetilde{W}(x, \delta) = \emptyset,$$

where $\widetilde{W}(x, \delta)$ is defined as in Theorem 3.1 but using the IFS $\widetilde{S}$. □

Remark 6.3. It would be interesting to extend the above result to a sub-attractor satisfying the open set condition but this question seems related to some open question related to the size of the set of so-called forbidden points in the case of a self-similar IFS. One refers to [3] for more details.

6.1. *Proof of item (1) of Theorem 3.1.* Write $s = \dim_H(K)$. The notation of the proof of Theorem 5.5 is adopted in this section.

LEMMA 6.4. *For any $x_0 \in U$ and any $\delta < 1$,*

$$\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta}) = K.$$

Proof. Note first that, since K is the (compact) attractor of S,

$$\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta}) \subset K.$$

We now prove the converse inclusion.

Let $c > 1$. By Lemma 4.1 and Remark 4.2 applied with

$$X = \bigcup_{\underline{i}\in\Lambda^*} f_{\underline{i}}(x_0) \cup K,$$

there exists $D(c) > 0$ such that for any $y \in K$ and any $\underline{i} = (i_1, \ldots, i_n) \in \Lambda^*$,

$$\|f_{\underline{i}}(x_0) - f_{\underline{i}}(y)\| \leq D(c)c^n \|f'_{\underline{i}}(y)\| \cdot \|x_0 - y\|. \tag{85}$$

By Lemma 4.1 and equation (20), one has

$$\|f'_{\underline{i}}(y)\| \leq D(c)c^n |f_{\underline{i}}(K)|. \tag{86}$$

Combining equations (85) and (86), one gets

$$\|f_{\underline{i}}(x_0) - f_{\underline{i}}(y)\| \leq \max_{z\in K} d(x_0, z) D(c)^2 c^{2n} |f_{\underline{i}}(K)|. \tag{87}$$

Recall that there exist $0 < t_1 < t_2$ so that, uniformly on n and $\underline{i} \in \Lambda^n$,

$$t_1 \leq \frac{\log \|f_{\underline{i}}\|}{n} \leq t_2.$$

Set $\varepsilon = \delta - 1 > 0$, taking $c = e^{t_1\varepsilon/4}$ and writing $\kappa(S, \varepsilon, x_0) = \max_{z\in K} d(x, z)D(c)^2$, one gets

$$\|f_{\underline{i}}(x_0) - f_{\underline{i}}(y)\| \leq \kappa(S, \varepsilon, x_0)|f_{\underline{i}}(K)|^{1-\varepsilon/2}. \tag{88}$$

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In particular, since $|f_{\underline{i}}(K)| \to 0$, for n large enough, for any $\underline{i} \in \Lambda^n$,

$$f_{\underline{i}}(K) \subset B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{1-\varepsilon}).$$

Recalling that

$$K = \bigcup_{\underline{i} \in \Lambda^n} f_{\underline{i}}(K),$$

one concludes that $K \subset \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{1-\varepsilon})$. □

Remark 6.5. In the case where $S = \{f_1, \ldots, f_m\}$ is a self-similar system, a more precise statement can be given. Denote by $0 < c_1, \ldots, c_m < 1$ the contracting ratio of respectively $f_1, \ldots, f_m$. In the self-similar case, for any $z \in K$ and any $\underline{i} \in \Lambda^*$,

$$d(f_{\underline{i}}(x_0), f_{\underline{i}}(z)) = c_{\underline{i}} d(x, z) \leq c_{\underline{i}} \max_{y \in K} d(y, x_0).$$

Writing $C(x_0, S) = \max_{y \in K} d(y, x)$, this implies that $f_{\underline{i}}(K) \subset B(f_{\underline{i}}(x_0), C(x_0, S)c_{\underline{i}})$ and

$$K = \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), C(x_0, S)c_{\underline{i}}).$$

6.2. *Proof of item (2) of Theorem 3.1.*

6.2.1. *Variational principle and C^1 weakly conformal IFS.*

A modified version of a proposition of Feng and Hu used in the proof of their variational principle [19, Theorem 2.13] is needed to prove item (3) of Theorem 3.1. The following subsection is dedicated to this modification.

The result from Feng and Hu we wish to modify as follows.

PROPOSITION 6.6. (Feng and Hu [19]) *Let $m \geq 2$ be an integer and $S = \{f_1, \ldots, f_m\}$ a weakly conformal IFS. For any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ as well as words $\underline{i}_1, \ldots, \underline{i}_{n_\varepsilon} \in \Lambda^*$ such that:*

- *for any $1 \leq j < j' \leq n_\varepsilon$, $f_{\underline{i}_j}(K) \cap f_{\underline{i}_{j'}}(K) = \emptyset$;*
- *writing $S_\varepsilon = \{f_{\underline{i}_1}, \ldots, f_{\underline{i}_{n_\varepsilon}}\}$, there exists a probability vector $P_\varepsilon = (p_1, \ldots, p_{n_\varepsilon})$ such that the weakly conformal measure μ_ε associated with P_ε and S_ε satisfies $\dim_H(\mu_\varepsilon) \geq \dim_H(K) - \varepsilon$.*

Let us remark that, due to the the first item, the IFS $S_\varepsilon = \{T_1, \ldots, T_{n_\varepsilon}\}$ satisfies the SSC and might not have K as attractor. We wish to modify this proposition so that the attractor of the IFS S_ε can be taken equal to K.

Note also that in Proposition 6.6, because S_ε satisfies the SSC, by Corollary 4.23, the dimension of a weakly conformal measure associated with S_ε depends continuously on the choice of the probability vector. Moreover, writing ν_ε the canonical measure on the coding associated with μ_ε, then the Lyapunov exponent $\lambda_{\nu_\varepsilon} > 0$ (see Definition 4.5) satisfies for ν_ε-almost every $(x_n)_{n \in \mathbb{N}}$ that

$$\lim_{n \to +\infty} \frac{\log |T_{x_1} \circ \cdots \circ T_{x_n}(K)|}{n} = -\lambda_{\nu_\varepsilon}.$$

As announced above, one proves the following modified version of Proposition 6.6.

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PROPOSITION 6.7. *Let $\varepsilon_0 > 0$. There exists an IFS S_{ε_0} and a weakly conformal measure μ_{ε_0} (associated with S_{ε_0}) such that* $\text{supp}(\mu_{\varepsilon_0}) = K$ *and* $\dim_H(\mu_{\varepsilon_0}) \geq \dim_H K - \varepsilon_0$.

Remark 6.8. Similar to the proof of [19, Theorem 2.13], Proposition 6.7 yields a measure on $\Lambda^{\mathbb{N}}$ and taking weak limits of ergodic averages of this measure gives an ergodic measure fully supported on K with dimension larger $s - \varepsilon_0$.

Proof. Fix $\varepsilon = (\varepsilon_0/2) > 0$. Consider $S_\varepsilon = \{f_{\underline{i}_1}, \ldots, f_{\underline{i}_{n_\varepsilon}}\}$, P_ε, μ_ε as in Theorem 6.6 and $0 < \varepsilon' < (1/5n_\varepsilon m) \cdot \min_{1\leq i\leq m} p_i$.

Set

$$\begin{cases} g_j = f_j & \text{for } 1 \leq j \leq m, \\ g_j = f_{\underline{i}_{j-m}} & \text{for } m+1 \leq j \leq n_\varepsilon + m. \end{cases}$$

Also set $\widetilde{S}_\varepsilon = \{g_1, \ldots, g_{m+n_\varepsilon}\}$ and note that $\widetilde{S}_\varepsilon$ has attractor K. Denote by $\widetilde{P}_{\varepsilon,\varepsilon'} = (\widetilde{p}_1, \ldots, \widetilde{p}_{m+n_\varepsilon})$ the probability vector defined as

$$\begin{cases} \widetilde{p}_j = \varepsilon' & \text{for } 1 \leq j \leq m, \\ \widetilde{p}_j = p_{j-m} - \dfrac{m}{n_\varepsilon}\varepsilon'. & \end{cases}$$

Let $\mu_{\varepsilon,\varepsilon'}$ be the weakly conformal measure associated with $\widetilde{S}_\varepsilon$ and $\widetilde{P}_{\varepsilon,\varepsilon'}$. Applying Theorem 4.8 to $\mu_{\varepsilon,\varepsilon'}$, let us prove that the corresponding h (see second item of Theorem 4.8) tends to 0 as ε' tends to 0.

Set $\Theta = \{1, \ldots, n_\varepsilon + m\}$ and $\Theta^* = \bigcup_{k>0} \Theta^k$. Let us denote by π_Θ the canonical projection of $\Theta^{\mathbb{N}}$ on K. One endows $\Sigma_\Theta = \Theta^{\mathbb{N}}$ with the metric d_Θ defined for any $x = (x_n)$, $y = (y_n) \in \Sigma_\Theta$ by

$$d_\Theta(x, y) = e^{-\min\{i\in\mathbb{N}: x_i \neq y_i\}} \quad \text{and} \quad d_\Theta(x, x) = 0. \tag{89}$$

Let us remark that the metric d allows one to define on $\Theta^{\mathbb{N}}$ the Hausdorff dimension and the Packing dimension in a similar way than on $\mathbb{R}^d$.

Let $\nu_{\varepsilon,\varepsilon'} \in \mathcal{M}(\Theta^{\mathbb{N}})$ be the Bernoulli product verifying $\nu_{\varepsilon,\varepsilon'} \circ \pi_\Theta^{-1} = \mu_{\varepsilon,\varepsilon'}$.

By the strong law of large numbers, for every $x = (x_n)_{n\in\mathbb{N}}$ in a set $\widetilde{\Sigma}_\Theta$ of $\nu^{\varepsilon,\varepsilon'}$-full measure, there exists $N_x \in \mathbb{N}$ such that for any $n \geq N_x$, any $1 \leq i \leq n_\varepsilon + m$,

$$\left| \frac{\#\{1 \leq j \leq n \ : x_j = i\}}{n} - \widetilde{p}_i \right| \leq \varepsilon'. \tag{90}$$

For $n \in \mathbb{N}$, write

$$A_n = \{x \in \widetilde{\Sigma}_\Theta : N_x \leq n\}.$$

By Theorem 4.8, there exists N such that, using the notation involved,

$$\mu_{\varepsilon,\varepsilon'}\left(B_N = \left\{ y : \dim_H(\mu_{\varepsilon,\varepsilon'}^{\pi_\Theta^{-1}(\{y\})}) = h \text{ and } \mu_{\varepsilon,\varepsilon'}^{\pi_\Theta^{-1}(\{y\})}(A_N) \geq \frac{1}{2} \right\}\right) \geq \frac{1}{2}.$$

We fix such an N.

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The following lemma is useful to estimate the number of cylinders of generation n which intersects A_N.

LEMMA 6.9. *Consider* $N \in \mathbb{N}$, $y \in K$, *and* $x = (x_n)_{n\in\mathbb{N}}, \widetilde{x} = (\widetilde{x}_n)_{n\in\mathbb{N}} \in \pi_\Theta^{-1}(\{y\})$. *Assume that for every* $1 \leq k \leq N$,

$$(x_k \textit{ or } \widetilde{x}_k \in \{1, \ldots, m\}) \Rightarrow (x_k = \widetilde{x}_k).$$

Then, for every $0 \leq j \leq N$ *such that* $x_j \geq m+1$, *one also has*

$$x_j = \widetilde{x}_j.$$

Proof. We proceed by contradiction. Suppose that the claim is not true and let $x_{j_0} \geq m+1$ be such for any $1 \leq i < j_0, \widetilde{x}_i = x_i$ and $\widetilde{x}_{j_0} \neq x_{j_0}$. Write

$$z = \lim_{k\to+\infty} g_{x_{j_0+1}} \circ g_{x_{j_0+2}} \circ \cdots \circ g_{x_{j_0+k}}(0)$$

and

$$\widetilde{z} = \lim_{k\to+\infty} g_{\widetilde{x}_{j_0+1}} \circ g_{\widetilde{x}_{j_0+2}} \circ \cdots g_{\widetilde{x}_{j_0+k}}(0).$$

Then, recalling that $x, \widetilde{x} \in \pi_\Theta^{-1}(\{y\})$,

$$g_{x_1} \circ \cdots \circ g_{x_{j_0-1}} \circ g_{x_{j_0}}(z) = g_{x_0} \circ \cdots \circ g_{x_{j_0-1}} \circ g_{\widetilde{x}_{j_0}}(\widetilde{z}) = y,$$

which implies that

$$g_{x_{j_0}}(z) = g_{\widetilde{x}_{j_0}}(\widetilde{z}).$$

Recalling that the system $\{g_{m+1}, \ldots, g_{m+N_\varepsilon}\}$ satisfies the SSC, one also has

$$g_{x_{j_0}}(K) \cap g_{\widetilde{x}_{j_0}}(K) = \emptyset,$$

which yields a contradiction. □

Continuing the proof of the proposition, we note that, by equation (90), for every $y \in B_N$, $x = (x_n)_{n\in\mathbb{N}} \in \pi_\theta^{-1}(\{y\}) \cap A_N$, and $N' \geq N$,

$$\#\{1 \leq k \leq N' : x_k \in \{1, \ldots, m\}\} \leq 2m\varepsilon' N'. \tag{91}$$

Lemma 6.9 together with equation (91) yields for N large enough,

$$\#\{\underline{i} \in \Theta^{N'} : [\underline{i}] \cap A_N \cap \pi_\Theta^{-1}(\{y\}) \neq \emptyset\} \leq \sum_{k=0}^{\lfloor 2m\varepsilon' N' \rfloor + 1} \binom{N'}{k} m^k.$$

Since $\varepsilon' < (1/5m)$ so that $2m\varepsilon' N' < (N'/2)$,

$$\#\{\underline{i} \in \Theta^{N'} : [\underline{i}] \cap A_N \cap \pi_\Theta^{-1}(\{y\}) \neq \emptyset\} \leq (\lfloor 2m\varepsilon' N' \rfloor + 2) \binom{N'}{\lfloor 2m\varepsilon' N' \rfloor + 1} m^{\lfloor 2m\varepsilon' N' \rfloor + 1}.$$

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Using the Stierling formula, provided that ε' was chosen small enough at the start and N (so N' too) large enough, there exists a constant $C > 0$ such that

$$\begin{aligned}
&\#\{\underline{i} \in \Theta^{N'} : [\underline{i}] \cap A_N \cap \pi_\Theta^{-1}(\{y\}) \neq \emptyset\} \\
&\quad \leq C(\lfloor 2m\varepsilon' N' \rfloor + 2) \frac{(N')^{\lfloor 2m\varepsilon' N' \rfloor + 1} \cdot m^{\lfloor 2m\varepsilon' N' \rfloor + 1}}{((\lfloor 2m\varepsilon' N' \rfloor + 1)/e)^{\lfloor 2m\varepsilon' N' \rfloor + 1} \sqrt{2\pi(\lfloor 2m\varepsilon' N' \rfloor + 1)}} \\
&\quad \leq C(\lfloor 2m\varepsilon' N' \rfloor + 2) \left(\frac{mN'}{2m\varepsilon' N'/e}\right)^{\lfloor 2m\varepsilon' N' \rfloor + 1} \frac{1}{\sqrt{2\pi(\lfloor 2m\varepsilon' N' \rfloor + 1)}} \\
&\quad \leq C(\lfloor 2m\varepsilon' N' \rfloor + 2) \left(\frac{e}{2\varepsilon'}\right)^{3mN'\varepsilon'} \\
&\quad = C(\lfloor 2m\varepsilon' N' \rfloor + 2) e^{3mN'\varepsilon' \log(e/2\varepsilon')} \leq e^{\sqrt{\varepsilon'} N'}.
\end{aligned} \tag{92}$$

Since equation (92) holds for any $N' \geq N$, one obtains that

$$\dim_P(A_N \cap \pi_\Theta^{-1}(\{y\})) \leq \sqrt{\varepsilon'}.$$

Recalling that, by definition of B_N, the measure $\mu^{\pi_\theta^{-1}(\{y\})}$ is h-exact dimensional, that $\mu^{\pi_\theta^{-1}(\{y\})}(\pi_\theta^{-1}(\{y\})) = 1$, and that $\mu^{\pi_\theta^{-1}(\{y\})}(A_N) \geq \frac{1}{2}$, one has

$$\begin{aligned}
h &= \inf\{\dim_H(A),\ \text{A Borel set satisfying } \mu^{\pi_\theta^{-1}(\{y\})}(A) > 0\} \\
&\leq \dim_P(A_N \cap \pi_\Theta^{-1}(\{y\})) \leq \sqrt{\varepsilon'}.
\end{aligned}$$

By Remark 4.6 and the fourth item of Theorem 4.8, there exists a constant $\widetilde{C}$ depending on the system S such that

$$\dim_H(\mu_{\varepsilon,\varepsilon'}) \geq \frac{\dim_H(\nu_{\varepsilon,\varepsilon'})}{\lambda_{\nu_{\varepsilon,\varepsilon'}}} - \widetilde{C}\sqrt{\varepsilon'}, \tag{93}$$

where $\lambda_{\nu_{\varepsilon,\varepsilon'}}$ is given by Definition 4.5. Also, by Corollary 4.7, for any Bernoulli product $\nu \in \mathcal{M}(\Theta)$ associated with a probability vector $\widehat{P} \in (0,1)^{n_\varepsilon + m}$, the Lyapunov exponent depends continuously on the vector $\widehat{P}$. Recalling that $\Theta^{\mathbb{N}}$ is endowed with the metric given by equation (89), it is also classical that $\dim_H(\nu)$ depends continuously on the choice of $\widehat{P}$. Since $\lim_{\varepsilon' \to 0} P_{\varepsilon,\varepsilon'} = \{0\}^m \times P_\varepsilon$,

$$\lim_{\varepsilon' \to 0} \frac{\dim_H(\nu_{\varepsilon,\varepsilon'})}{\lambda_{\nu_{\varepsilon,\varepsilon'}}} = \frac{\dim_H(\nu_\varepsilon)}{\lambda_{\nu_\varepsilon}}. \tag{94}$$

Equation (94) combined with equation (93) proves that for ε' small enough, one has

$$\dim_H(\mu_{\varepsilon,\varepsilon'}) \geq \frac{\dim_H(\nu_\varepsilon)}{\lambda_{\nu_\varepsilon}} - 2\varepsilon \geq s - 2\varepsilon,$$

which concludes the proof of Proposition 6.7. □

We can now finish the proof of item (3) of Theorem 3.1.

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6.2.2. *Proof of item (2) of Theorem 3.1.* Let us recall that, by Proposition 2.6 and Definition 2.8, $\dim(S)$ verifies, for any $z \in K$,

$$P(\dim(S)) = \lim_{k\to+\infty} \frac{1}{k} \log \sum_{\underline{i}\in\Lambda^k} |f_{\underline{i}}(K)|^{\dim(S)} = 0.$$

Fix $x_0 \in K$, $\delta \geq 1$ and write

$$\mathcal{L}(\delta) = \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta}).$$

Let us first show that $\dim_H(\mathcal{L}(\delta)) \leq (\dim(S)/\delta)$.

Let α and β be as in equation (23). If one must change the constants α and β, one can assume that there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and every $\underline{i} \in \Lambda^k$,

$$\alpha^k \leq |f_{\underline{i}}(K)| \leq \beta^k.$$

For every $k \in \mathbb{N}$, recalling equation (38), for every $\underline{i} = (i_1, \ldots, i_n) \in \Lambda^{(k)}$, one has

$$\alpha^n \leq 2^{-k} \leq \beta^{n-1} \Rightarrow n \leq k\frac{-\log(2)}{\log(\beta)} + 1 \leq 2k\frac{-\log(2)}{\log(\beta)}.$$

In particular, every integer $p \in \mathbb{N}$ and every $(i_{n+1}, \ldots, i_{n+p}) \in \Lambda^p$ such that $(i_1, \ldots, i_{n+p}) \in \Lambda^{(k)}$ must satisfy

$$\beta^{n+p-1} \geq \alpha^n \Rightarrow p \leq n \times \left(\frac{\log\alpha}{\log\beta} - 1\right) + 1 \leq 2n\left(\frac{\log\alpha}{\log\beta} - 1\right) \leq k \times C(\alpha, \beta).$$

This implies that, for any $\nu \in \mathcal{M}(\Lambda^{\mathbb{N}})$,

$$\sum_{\underline{i}\in\Lambda^{(k)}} \nu([\underline{i}]) \leq kC(\alpha, \beta). \tag{95}$$

Consider $\varepsilon > 0$. Let us recall that Lemma 4.20 applied with

$$\varepsilon' = \frac{(\varepsilon/2)\log 2}{2(-\log(2))/\log(\beta)}$$

and $s = \dim(S)$ combined with Remark 4.21 yields a constant $\gamma_{\varepsilon'} > 0$ and a measure $\nu_{\varepsilon'} \in \mathcal{M}(\Lambda^{\mathbb{N}})$ such that for any $k \in \mathbb{N}$ and every $\underline{i} = (i_1, \ldots, i_n) \in \Lambda^{(k)}$,

$$\gamma_{\varepsilon'}^{-1} e^{-k(\varepsilon/2)\log 2} |f_{\underline{i}}(K)|^{\dim(S)} \leq \nu_{\varepsilon'}([\underline{i}]) \leq \gamma_{\epsilon'} e^{k(\varepsilon/2)\log 2} |f_{\underline{i}}(K)|^{\dim(S)}. \tag{96}$$

For any $\delta \geq 1$,

$$\begin{aligned}\sum_{\underline{i}\in\in\bigcup_{k\geq k_0}\Lambda^k} (|f_{\underline{i}}(K)|^{\delta})^{(\dim(S)+\varepsilon)/\delta} &= \sum_{\underline{i}=(i_1,\ldots,i_n)\in\bigcup_{k\geq k_0}\Lambda^{(k)}} |f_{\underline{i}}(K)|^{\dim(S)+\varepsilon} \\ &\leq \sum_{k\geq k_0} \sum_{\underline{i}=(i_1,\ldots,i_n)\in\Lambda^{(k)}} 2^{-k\varepsilon}\gamma_{\varepsilon'} e^{k(\varepsilon/2)\log 2} \nu_{\varepsilon'}([\underline{i}]) \\ &\leq \gamma_{\varepsilon'} C(\alpha, \beta) \sum_{k\geq k_0} k 2^{-k(\varepsilon/2)} < +\infty. \end{aligned} \tag{97}$$

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As a consequence,

$$\dim_H(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta})) \leq \frac{\dim(S)+\varepsilon}{\delta},$$

and letting ε tend to 0 establishes the upper bound.

Now we prove that

$$\dim_H(\mathcal{L}(\delta)) \geq \frac{\dim(S)}{\delta}.$$

Let $\varepsilon > 0$ and μ_ε be a weakly conformal measure as in Proposition 6.7. For any $k \in \mathbb{N}$, the balls $\{B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|)\}_{\underline{i}\in\Lambda^k}$ are centered on $K = \mathrm{supp}(\mu)$ and their limsup covers K. This implies that $\mu_\varepsilon(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|)) = 1$.

Applying Theorem 5.11, one gets

$$\frac{s-\varepsilon}{\delta} \leq \dim_H \Big(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta})\Big).$$

Letting $\varepsilon \to 0$ finishes the proof.

7. *Application to homogeneous self-similar IFS*

7.1. *Proof of Theorem 3.3.*

A self-similar IFS $S = \{f_1, \ldots, f_m\}$ is said to be c-homogeneous, for $0 < c < 1$, if for every $\underline{i} \in \Lambda^*$,

$$|f_{\underline{i}}(K)| = c^n |K|.$$

Such overlapping IFSs have been studied recently by various authors. Among the recent results on the topic, a Khintchine-type result is established in [2]. Although it is very tempting to try to combine such a result with Theorem 5.11, it turns out that the study on the Hausdorff dimension of limsup sets obtained via a more general approximation function in the case of homogeneous self-similar IFSs does not require this result and can be achieved only using Theorem 5.11.

Let us fix $m \geq 2$ an integer, $0 < c < 1$, and a c-homogeneous IFS $S = \{f_1, \ldots, f_m\}$.

LEMMA 7.1. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a mapping such that*

$$\sum_{n\geq 1}\sum_{\underline{i}\in\Lambda^n} \psi(n)^{\dim(S)} = +\infty,$$

then, for every $\varepsilon > 0$, there exists an infinity of integers $(n_k)_{k\in\mathbb{N}}$ such that $\psi(n_k) \geq c^{(1+\varepsilon)n_k}$.

Proof. Assume that it is not the case: there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for every $n > N$, $\psi(n) < c^{(1+\varepsilon)n}$. Let us denote $\upsilon \in \mathcal{M}(\{1, \ldots, m\}^{\mathbb{N}})$, the measure defined for every $\underline{i} = (1, \ldots, i_n) \in \Lambda^*$ by

$$\upsilon([\underline{i}]) = c^{n\dim(S)}.$$

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In this case, one has

$$\begin{aligned}\sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n}\psi(n)^{\dim(S)} &\leq \sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n} c^{n(1+\varepsilon)\dim(S)}\\ &= \sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n} c^{n\varepsilon\dim(S)}\nu([\underline{i}])\\ &= \sum_{n\geq N} c^{n\varepsilon\dim(S)}\nu\bigg(\bigcup_{\underline{i}\in\Lambda^n}[\underline{i}]\bigg) = \sum_{n\geq N} c^{n\varepsilon\dim(S)} < +\infty.\end{aligned}$$

This is a contradiction. □

This lemma implies in particular that for every $\varepsilon > 0$, there exists an infinity of integers $(n_k)_{k\in\mathbb{N}}$ for every $\underline{i} \in \Lambda^{n_k}$, writing $1-\varepsilon = 1/(1+\varepsilon')$,

$$\psi(n_k)^{1-\varepsilon} \geq |f_{\underline{i}}(K)|.$$

Since for every $k \in \mathbb{N}$ and every $x_0 \in K$,

$$K \subset \bigcup_{\underline{i}\in\Lambda^{n_k}} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|),$$

one obtains the following corollary.

COROLLARY 7.2. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a mapping such that*

$$\sum_{n\geq 1}\sum_{\underline{i}\in\Lambda^n}\psi(n)^{\dim(S)} = +\infty,$$

then, for every $x_0 \in K$ and every $\varepsilon > 0$,

$$K = \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), \psi(n)^{1-\varepsilon}).$$

We now prove Theorem 3.3.

Recall equation (14) and assume first that $s_\phi \geq 1$. Then by Lemma 7.1, for every $\varepsilon > 0$,

$$K = \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), \phi(n)^{1-\varepsilon}).$$

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be given by Proposition 6.7, one has

$$\mu\Big(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), \phi(n)^{1-\varepsilon})\Big) = 1,$$

which, by Theorem 5.11, implies that, writing $\delta_\varepsilon = 1/(1-\varepsilon)$,

$$\dim_H W(x_0,\phi) \geq \frac{\dim_H K - \varepsilon}{1/(1-\varepsilon)}.$$

Since this holds for every $\varepsilon > 0$, one has

$$\dim_H W(x_0,\phi) \geq \dim_H K,$$

and hence $\dim_H W(x_0,\phi) = \dim_H K = \min\{1, s_\phi\}\dim(S)$.

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We now assume that $s_\phi < 1$. We first prove that

$$\dim_H W(x_0, \phi) \le s_\phi \dim(S).$$

Let $\varepsilon > 0$. By definition of s_ϕ (see equation (14)),

$$\sum_{n\ge 1}\sum_{\underline{i}\in\Lambda^n} \phi(n)^{(s_\phi+\varepsilon)\dim(S)} < +\infty,$$

which yields $\mathcal{H}^{(s_\phi+\varepsilon)\dim(S)}(W(x_0,\phi)) = 0$, since this holds for every $\varepsilon > 0$,

$$\dim_H W(x_0, \phi) \le s_\phi \dim(S).$$

Let us show that

$$\dim_H W(x_0, \phi) \ge s_\phi \dim(S).$$

Fix again $\varepsilon > 0$. Since

$$\sum_{n\ge 1}\sum_{\underline{i}\in\Lambda^n} \phi(n)^{(s_\phi-\varepsilon)\dim(S)} = +\infty,$$

by Lemma 7.1, one has

$$K = \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), \psi(n)^{(s_\phi-\varepsilon)(1-\varepsilon)}).$$

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be given by Proposition 6.7. One has

$$\mu(W(x_0, \phi^{(s_\phi-\varepsilon)(1-\varepsilon)})) = 1.$$

Writing $\delta_\varepsilon = 1/(s_\phi-\varepsilon)(1-\varepsilon)$ and applying Theorem 5.11, one gets

$$\dim_H W(x_0,\phi) \ge \frac{\dim(S)}{\delta_\varepsilon} = (s_\phi-\varepsilon)(1-\varepsilon)\dim(S).$$

Letting $\varepsilon \to 0$ yields

$$\dim_H W(x_0, \phi) \ge s_\phi \dim(S),$$

which concludes the proof.

8. *The classical shrinking target problem: proof of Theorem 3.8*

Let $S = \{f_1, \ldots, f_m\}$ be a conformal IFS satisfying the bounded distortion property (Definition 15) and $s > 0$, ν_s, μ_s, and ϕ as in Theorem 3.8. Write again $\Lambda = \{1, \ldots, m\}$. First, let us recall that $\mu_s = \nu_s \circ \pi^{-1}$ is exact dimensional with

$$\dim_H \mu_s = h_s = \frac{h(\nu_s)}{\lambda_{\nu_s}},$$

where λ_{ν_s} is the Lyapunov exponent associated with ν_s (defined in equation (35)) and $h(\nu_s) \ge 0$ is such that for ν_s-almost every $(x_n)_{n\in\mathbb{N}} \in \Lambda^{\mathbb{N}}$,

$$\lim_{n\to+\infty} \frac{-\log \nu_s([x_1,\ldots,x_n])}{n} = h(\nu_s).$$

Let us collect some remarks.

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Remark 8.1. By equations (35) and (16), we have

$$\lambda_{\nu_s} = \frac{h(\nu_s) - P(s)}{s} \quad \text{and} \quad h(\nu_s) = s\lambda_{\nu_s} + P(s).$$

In particular, we have

$$\dim_H \mu_s = \frac{sh(\nu_s)}{h(\nu_s) - P(s)}$$

and $s \leq \dim_H \mu_s$.

In the rest of the section, given $\delta \geq 1$, we write

$$W_{x_0,\phi,\delta} = \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta} \phi(|\underline{i}|)).$$

8.1. *Lower bound for* $\dim_H W_{x_0,\phi,\delta}$.

PROPOSITION 8.2. *Let s be the root of $P(s) = s\alpha/\delta$. Then, for every $\varepsilon > 0$,*

$$\mu_s\Big(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{s/\dim_H \mu} \phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu})\Big) = 1.$$

Proof. Fix $\varepsilon_0 > 0$. By definition of the Lyapunov exponent λ_{ν_s}, there exists $N \in \mathbb{N}$ and a set $E \subset \Lambda^{\mathbb{N}}$ such that $\nu_s(E) \geq 1 - \varepsilon_0$ and for every $(x_n)_{n\in\mathbb{N}} \in E$, for every $n \geq N$, we have

$$e^{-n(1+\varepsilon_0)\lambda_{\nu_s}} \leq |f_{x_1,\ldots,x_n}(K)| \leq e^{-n(1-\varepsilon_0)\lambda_{\nu_s}}.$$

By definition of α, there exists an infinite set of integers $\mathcal{N} \subset \mathbb{N}$ such that for every $n \in \mathcal{N}$, we have that

$$\phi(n) \geq e^{-n(1+\varepsilon/2)\alpha}.$$

Thus, for every $(x_n)_{n\in\mathbb{N}} \in E$, writing $\underline{i} = (x_1, \ldots, x_n)$,

$$\begin{aligned} &|f_{\underline{i}}(K)|^{s/\dim_H \mu_s} \phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu_s} \\ &\quad \geq e^{-n(1+\varepsilon_0)(s\lambda_\nu/\dim_H \mu_s)} e^{-n(1-\varepsilon)(1+\varepsilon/2)\times(s\alpha/\delta \dim_H \mu_s)} \end{aligned}$$

and recalling that $P(s) = s\alpha/\delta$, one has

$$\begin{aligned} &|f_{\underline{i}}(K)|^{s/\dim_H \mu_s} \phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu_s} \\ &\quad \geq e^{-n(1+\varepsilon_0-(1-\varepsilon)(1+\varepsilon/2))(s\lambda_{\nu_s}/\dim_H \mu_s)} \times e^{-n(1-\varepsilon)(1+\varepsilon/2)\times((P(s)+s\lambda_{\nu_s})/\dim_H \mu_s)}. \end{aligned}$$

Note also that

$$\left(1 + \varepsilon_0 - (1-\varepsilon)\left(1 + \frac{\varepsilon}{2}\right)\right) = \varepsilon_0 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2},$$

so that

$$\left(1 + \varepsilon_0 - (1-\varepsilon)\left(1 + \frac{\varepsilon}{2}\right)\right) \frac{s\lambda_{\nu_s}}{\dim_H \mu_s} \leq C\varepsilon\lambda_{\nu_s}$$

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for some $C > 0$ independent of ε provided that ε_0 was chosen small enough to begin with. In addition, using Remark 8.1,

$$\frac{P(s)+s\lambda_{\nu_s}}{\dim_H \mu_s} = \lambda_{\nu_s} \times \frac{P(s)+s\lambda_{\nu_s}}{h(\nu_s)} = \lambda_{\nu_s}.$$

This implies that, for some $C' > 0$ independent of ε, provided that ε_0 was chosen small enough,

$$\begin{aligned} |f_{\underline{i}}(K)|^{s/\dim_H \mu_s}\phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu_s} &\geq e^{-n(1+\varepsilon_0-(1-\varepsilon)(1+\varepsilon/2))(s\lambda_{\nu_s}/\dim_H \mu_s)} \\ &\quad \times e^{-n(1-\varepsilon)(1+\varepsilon/2)\times(P(s)+s\lambda_{\nu_s}/\dim_H \mu_s)} \\ &\geq e^{-n(1-C'\varepsilon)\lambda_{\nu_s}} \geq e^{-n(1-\varepsilon_0)\lambda_{\nu_s}} \geq |f_{\underline{i}}(K)|. \end{aligned}$$

The last inequality implies that

$$f_{\underline{i}}(K) \subset B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{s/\dim_H \mu_s}\phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu_s}),$$

and hence $\pi((x_n)_{n\in\mathbb{N}}) \in B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{s/\dim_H \mu_s}\phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu_s})$. Since this happens for every $n \in \mathcal{N}$, we conclude that

$$\pi((x_n)_{n\in\mathbb{N}}) \in \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{s/\dim_H \mu_s}\phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu_s}).$$

This yields

$$E \subset \pi^{-1}\Big(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{s/\dim_H \mu_s}\phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu_s})\Big)$$

so that

$$1-\varepsilon_0 \leq \nu_s(E) \leq \mu_s\Big(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{s/\dim_H \mu_s}\phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H y \mu_s})\Big).$$

Letting $\varepsilon_0 \to 0$ proves the claim. □

Applying Proposition 5.11 to $\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{s/\dim_H \mu_s}\phi(|\underline{i}|)^{s(1-\varepsilon)/\delta \dim_H \mu_s})$ and μ_s yields the following corollary.

COROLLARY 8.3. *Write $\delta' = 1/(s(1-\varepsilon)/\delta \dim_H \mu_s)$, we have*

$$\dim_H \Big(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta/(1-\varepsilon)}\phi(|\underline{i}|))\Big) \geq \frac{\dim_H \mu_s}{\delta'} = \frac{s(1-\varepsilon)}{\delta}.$$

Since this holds for every $\delta \geq 1$ and $\varepsilon > 0$, we conclude that

$$\dim_H \Big(\limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta}\phi(|\underline{i}|))\Big) \geq \frac{s}{\delta}.$$

8.2. *Upper bound for* $\dim_H W_{x_0,\phi,\delta}$. Fix $\varepsilon > 0$. By definition of α, there exists $N \in \mathbb{N}$ so large that for every $n \geq N$, one has

$$\phi(n) \leq e^{-n(\alpha/(1+\varepsilon/2))}.$$

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Hence, writing $0 < \varepsilon' = \min\{\varepsilon, (1+\varepsilon)/(1+\varepsilon/2)\}$ and recalling that $P(s) = s\alpha/\delta$, we get

$$\begin{aligned}\sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n}(|f_{\underline{i}}(K)|^\delta\phi(n))^{s(1+\varepsilon)/\delta} &\leq \sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n}|f_{\underline{i}}(K)|^{s(1+\varepsilon)}e^{-n((1+\varepsilon)/1+\varepsilon/2)s\alpha/\delta} \\ &\leq \sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n}|f_{\underline{i}}(K)|^{s(1+\varepsilon)}e^{-n((1+\varepsilon)/1+\varepsilon/2)P(s)} \\ &\leq \sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n}(|f_{\underline{i}}(K)|^s e^{-nP(s)})^{1+\varepsilon'} \\ &= \sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n}\mu_s(\underline{i})^{1+\varepsilon'}.\end{aligned}$$

Using a similar argument as in equation (97), we have

$$\sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n}\mu_s(\underline{i})^{1+\varepsilon'} < +\infty,$$

and hence

$$\sum_{n\geq N}\sum_{\underline{i}\in\Lambda^n}(|f_{\underline{i}}(K)|^\delta\phi(n))^{s(1+\varepsilon)/\delta} < +\infty,$$

so

$$\dim_H \limsup_{\underline{i}\in\Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta\phi(|\underline{i}|)) \leq (1+\varepsilon)\frac{s}{\delta}.$$

Letting $\varepsilon \to 0$ yields the desired result.

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