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ON THE INSTABILITY OF PERIODIC WAVES FOR DISPERSIVE EQUATIONS—REVISITED

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Abstract. The purpose of this paper is to present an extension of the results in [8]. We establish a more general proof for the moving kernel formula to prove the spectral stability of periodic traveling wave solutions for the regularized Benjamin–Bona–Mahony type equations. As applications of our analysis, we show the spectral instability for the quintic Benjamin–Bona–Mahony equation and the spectral (orbital) stability for the regularized Benjamin–Ono equation.

§1. Introduction

The regularized Benjamin–Bona–Mahony type equation (rBBM henceforth)

$$u_t + u_x + (f(u))_x + (\mathcal{M}u)_t = 0, \tag{1}$$

arises as a regularized version of the Korteweg–de Vries type equation as

$$u_t + (f(u))_x - (\mathcal{M}u)_x = 0, \tag{2}$$

where in both equations, $u : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a real valued function which is L-periodic at the $x-$variable. Here, $\mathcal{M}$ is expressed by the Fourier multiplier as

$$\widehat{\mathcal{M}g}(\kappa) = \theta(\kappa)\widehat{g}(\kappa), \quad \kappa\in\mathbb{Z}, \tag{3}$$

the symbol θ is assumed to be an even and continuous function on $\mathbb{R}$ satisfying

$$A_1|\kappa|^m \le \theta(\kappa) \le A_2|\kappa|^m, \quad m>0, \tag{4}$$

for all $\kappa\in\mathbb{Z}$, and $A_i \ge 0$, $i = 1,2$.

Particular cases of (1) are relevant models describing the propagation of nonlinear waves. In fact, if $\mathcal{M} = -\partial_x^2$ and $f(u) = \frac{u^2}{2}$ in (1), we obtain the Benjamin–Bona–Mahony (BBM) as a improvement model to the Korteweg–de Vries equation for modeling of small-amplitude and long-wavelength surface water waves. When $\mathcal{M} = \mathcal{H}\partial_x$ and $f(u) = \frac{u^2}{2}$ we have the regularized Bejamin–Ono equation (rBO). Here, $\mathcal{H}$ indicates the Hilbert transform in the periodic context. This equation is a model for the time evolution of long-crested waves at the interface between two immiscible fluids, which appears in various physical applications (see [18] and references therein). Important to quote that results of orbital stability for positive and periodic traveling waves associated to the BBM and rBO equations have been obtained in [4] using the arguments in [6]. In addition, several qualitative aspects have been determined in [2] for the same power nonlinearity and $\mathcal{M} = D^\alpha$ where D^α represents the fractional differential operator.

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As usual, the rBBM equation (1) admits the following conserved quantities

$$E(u) = \frac{1}{2}\int_0^L u\mathcal{M}u - W(u)dx \tag{5}$$

$$P(u) = \frac{1}{2}\int_0^L u\mathcal{M}u + u^2 dx, \tag{6}$$

and

$$M(u) = \int_0^L u dx, \tag{7}$$

where W is the primitive of f, that is, $W' = f$.

A periodic traveling wave for (1) is a solution of the form $u(x,t) = \phi(x-ct)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth L-periodic function and c is a nonzero real constant representing the wave speed. Substituting this form into (1), we obtain

$$c(\mathcal{M}+1)\phi - (\phi + f(\phi)) + A = 0. \tag{8}$$

where A is a constant of integration. In whole of this paper, we assume that ϕ enjoys the zero mean property, so that $\int_0^L \phi dx = 0$. By (8), it follows immediately that $A = \frac{1}{L}\int_0^L f(\phi)dx$.

Taking into account the conserved quantities (5), (6), and (7), we are enabled to consider the following augmented Lyapunov functional

$$G(u) = E(u) + (c-1)P(u) + AM(u). \tag{9}$$

By (8), one has $G'(\phi) = 0$, that is, ϕ is a critical point of the functional G. Moreover, by (9), the linearized operator around the wave ϕ is represented by the second Fréchet derivative of G at the point ϕ as

$$\mathcal{L} := G''(\phi) = c\mathcal{M} + c - 1 - f'(\phi). \tag{10}$$

We see that $\mathcal{L}$ is a self-adjoint operator defined in $L^2_{per}([0,L])$ with dense domain $D(\mathcal{L}) = H^m_{per}([0,L])$.

The main purpose of this paper is to revisit the approach in [8] for periodic waves which solve (8) with $A = \frac{1}{L}\int_0^L f(\phi)dx$ being this value not necessarily zero. The authors considered in [8] periodic waves with zero mean property as we are supposing and $A = 0$. This fact gives a restriction since we cannot consider the case $f(u) = \frac{u^2}{2}$ and consequently, the well-known equations BBM and rBO as above. As far as we can see, few examples can be obtained with this kind of strong restriction. In fact, for power nonlinearities as $f(u) = \frac{u^{p+1}}{p+1}$, the value of p needs to be even and satisfying $\int_0^L f(\phi)dx = 0$. As example, it has been presented in [8] only the spectral stability/instability of periodic cnoidal waves for the case $p = 2$.

Our first application is the spectral instability of cnoidal wave solutions for the quintic BBM equation, that is $\mathcal{M} = -\partial_x^2$ and $f(u) = \frac{u^5}{5}$ in (1). Here, we still consider the case $A = 0$ since the corresponding equation (8) has two family of explicit even periodic waves, namely, one of dnoidal and another one of cnoidal type. For dnoidal waves, the authors in [5] obtained that such families are orbitally stable in the energy space. According with our best knowledge, the existence and spectral stability of cnoidal waves for the quintic BBM equation never been proven in the current literature.

As a second example, we present the spectral (orbital) stability of periodic waves for the rBO equation. In our approach, we realized a connection between the positive periodic waves associated with the corresponding Benjamin–Ono equation

$$\mathcal{H}\psi + \omega\psi - \psi^2 = 0 \tag{11}$$

and the periodic wave with zero mean ϕ given by the formula

$$\psi = \frac{1}{2c}\left[\phi - (c-1) + \sqrt{(c-1)^2 + 2A}\right]. \tag{12}$$

In addition, (12) gives us a correspondence between the spectral property for the linearized operator around the periodic wave ψ and the corresponding linear operator (10).

Next, we establish the linearized spectral problem for the rBBM equation. By considering the pertubation $u(x,t) = \phi(x-ct) + w(x-ct,t)$ in (1) and using (8), we obtain that w satisfies the nonlinear equation

$$(\partial_t - c\partial_x)(w + \mathcal{M}w) + \partial_x(w + f'(\phi)w) + w\partial_x w = 0. \tag{13}$$

Substituing (13) by its linearization around ϕ, we obtain the linear equation

$$\partial_t(v + \mathcal{M}v) = \partial_x \mathcal{L}v, \tag{14}$$

where $\mathcal{L}$ is given by (10). Since ϕ depends only on x, (14) has a separation of variables of the form $w(x,t) = e^{\lambda t}\eta(x)$ with some $\lambda \in \mathbb{C}$ and $\eta : \mathbb{T} \to \mathbb{C}$ which satisfies the spectral problem

$$\partial_x \mathcal{L}\eta = \lambda(\mathcal{M}+1)\eta. \tag{15}$$

Since $\mathcal{M}+1$ is an invertible operator, we can rewrite the problem (15) as

$$J\mathcal{L}\eta = \lambda\eta, \tag{16}$$

where $J := (\mathcal{M}+1)^{-1}\partial_x$. Denoting the spectrum of $J\mathcal{L}$ by $\sigma(J\mathcal{L})$, the periodic wave ϕ is spectrally stable if $\sigma(J\mathcal{L}) \subset i\mathbb{R}$. Otherwise, that is, if $\sigma(J\mathcal{L})$ contains a point λ with $Re(\lambda) > 0$, the periodic wave ϕ is said to be spectrally unstable.

We see in the periodic framework that J is not a one-to-one operator. Hence, the classical spectral stability results in [13] cannot applied. To overcome this difficulty, the authors in [12] have considered the restricted problem

$$J\mathcal{L}\big|_{\mathbb{V}}\chi = \lambda\chi, \tag{17}$$

where $\mathcal{L}|_{\mathbb{V}}$ is a restriction of $\mathcal{L}$ on the closed subspace X_0 of periodic functions with zero mean,

$$\mathbb{V} = \left\{ f \in L^2_{per}([0,L]) : \int_0^L f(x)dx = 0 \right\}. \tag{18}$$

Thus, the new problem (17) allows to consider the definition of spectral stability as above restricted to the periodic space $\mathbb{V}$.

We denote $\frac{\partial_x}{\lambda - c\partial_x} := (\lambda - c\partial_x)^{-1}\partial_x$, with $Re(\lambda) > 0$. Then, the spectral problem (15) is equivalent to the following one

$$(\mathcal{M}+1)\eta - \frac{\partial_x}{\lambda - c\partial_x}(1 + f'(\phi))\eta = 0. \tag{19}$$

Moreover, consider the orthogonal projection $Q : L^2_{per}([0,L]) \to \mathbb{V}$ defined by $Qv = v - \frac{1}{L}\int_0^L v dx$ and the family of closed operators $\mathcal{A}^\lambda : H^m_{per}([0,L]) \cap \mathbb{V} \to \mathbb{V}$, $Re(\lambda) > 0$, given as in [8] by

$$\mathcal{A}^\lambda v = (\mathcal{M}+1)v - \frac{\partial_x}{\lambda - c\partial_x} Q(1+f'(\phi))v. \tag{20}$$

Thus, we reformulate the spectral problem (19) by considering the problem $\mathcal{A}^\lambda v = 0$ which is better related to (17). Thus, we obtain that the wave ϕ is spectrally unstable if we find $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$ such that the operator $\mathcal{A}^\lambda$ has a nontrivial kernel. To this end, we use some tools of asymptotic analytic perturbation theory based on ideas in [8] and [21]. After that, we present a more general expression for the moving kernel formula contained in [8] for periodic waves with zero mean and $A = \frac{1}{L}\int_0^L f(\phi)dx \neq 0$. In addition, for the case $f(u) = \frac{u^2}{2}$, we present a modified moving kernel formula. This new formula seems a nice tool to decide about the spectral stability of periodic waves and we apply it to determine the spectral (orbital) stability for the rBO equation in an easier manner compared with those ones in [4].

Important to mention that our results can be extended to present a revisited result concerning the spectral stability/instability for the Korteweg–de Vries type equation (2) (as in [8]). The results are similar as determined by ourselves in the present manuscript, but we decided not to show the formal results for this case since there exist several contributors in spectral stability/instability for this topic as [3], [12], [17], [22], [25], and references therein. Concerning the regularized BBM equation for both periodic/solitary waves it seems that few references containing relevant examples can be found in the current literature.

This paper is organized as follows. In Section 2, we present a verbatim of the results presented in [8]. In Section 3, we obtain a revisited moving kernel formula and the corresponding spectral stability/instability criterion for the rBBM equation. Section 4 is devoted to our applications and we show the spectral instability of cnoidal waves for the quintic BBM equation and the spectral (orbital) stability for the rBO equation.

§2. Basic framework on the spectral stability—verbatim of [8]

To simplify the notation, let us consider the modified operator $\mathcal{L}_0 := \frac{1}{c}\mathcal{L}$ and the differential operators

$$\mathcal{E}^{\lambda,\pm} := \frac{\lambda}{\lambda \pm c\partial_x}.$$

Thus, we can rewrite the operator given in (20) as

$$\mathcal{A}^\lambda = \mathcal{M} + 1 - \frac{1}{c}(1 - \mathcal{E}^{\lambda,-})Q(1+f'(\phi)). \tag{21}$$

LEMMA 2.1. (i) *For $\lambda > 0$, operators $\mathcal{E}^{\lambda,\pm} \in B(L^2_{per})$ are continuous with respect to λ and*

$$||\mathcal{E}^{\lambda,\pm}||_{B(L^2_{per})} \leq 1, \tag{22}$$

$$||\mathcal{E}^{\lambda,\pm}||_{B(L^2_{per})} \leq 1. \tag{23}$$

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(ii) $\mathcal{E}^{\lambda,\pm}$ *converges to* 0 *strongly (uniformly) in* $\mathbb{V}$ *as* $\lambda \to 0^+$.
(iii) $\mathcal{E}^{\lambda,\pm}$ *converges to* I *strongly in* L^2_{per} *as* $\lambda \to 0^+$.

Proof. (i) For each $v \in L^2_{per}([0,L])$, we obtain by Parseval Theorem that

$$||\mathcal{E}^{\lambda,\pm}v||^2_{L^2_{per}} = L^2 \sum_{n\in\mathbb{Z}} \left|\frac{\lambda}{\lambda \pm icn}\right|^2 |\widehat{v}(n)|^2 \le L^2 \sum_{n\in\mathbb{Z}} |\widehat{v}(n)|^2 = ||v||^2_{L^2_{per}}. \tag{24}$$

This last fact establishes (22). To prove (23), we see that

$$||v - \mathcal{E}^{\lambda,\pm}v||^2_{L^2_{per}} = L^2 \sum_{n\in\mathbb{Z}} \left|\frac{\pm icn}{\lambda \pm icn}\right|^2 |\widehat{v}(n)|^2 \le ||v||^2_{L^2_{per}}.$$

(ii) For $v \in \mathbb{V}$,

$$||\mathcal{E}^{\lambda,\pm}v||^2_{L^2_{per}} = L^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} \left|\frac{\lambda}{\lambda \pm icn}\right|^2 |\widehat{v}(n)|^2.$$

Since for $n \in \mathbb{Z}\setminus\{0\}$, $\left|\frac{\lambda}{\lambda\pm icn}\right|^2 \longrightarrow 0$ as $\lambda \to 0^+$, we deduce $||\mathcal{E}^{\lambda,\pm}v||^2_{L^2_{per}} \longrightarrow 0$ as $\lambda \to 0^+$.
(iii) The proof is similar to (ii). ∎

PROPOSITION 2.1. *For* $\lambda > 0$*, the operator* $\mathcal{A}^\lambda$ *converges to* $\mathcal{A}^0 := Q\mathcal{L}_0$ *strongly in* $\mathbb{V}$ *when* $\lambda \to 0^+$ *and converges to* $\mathcal{M}+1$ *strongly in* L^2_{per} *when* $\lambda \to \infty$.

Proof. For each $v \in H^m_{per}([0,L]) \cap \mathbb{V}$ we have

$$c^2||(\mathcal{A}^\lambda - Q\mathcal{L}_0)v||^2_{L^2_{per}} = ||\mathcal{E}^{\lambda,-}Q(\phi + f'(\phi)v)||^2_{L^2_{per}}$$

and

$$c^2||(\mathcal{A}^\lambda - (\mathcal{M}+1))v||^2_{L^2_{per}} = ||(1-\mathcal{E}^{\lambda,-})Q(v + f'(\phi)v)||^2_{L^2_{per}}.$$

Therefore, we conclude by Lemma 2.1 that $\mathcal{A}^\lambda$ converges to $Q\mathcal{L}_0$ in $\mathbb{V}$ and $\mathcal{A}^\lambda$ converges to $\mathcal{M}+1$ in $L^2_{per}([0,L])$ when $\lambda \to 0^+$ and $\lambda \to +\infty$, respectively. ∎

The next result establishes that all eigenvalues of $\mathcal{A}^\lambda$ (with domain $H^m_{per}([0,L])$) are isolated, namely, the spectrum of $\mathcal{A}^\lambda$ is discrete, $\sigma(\mathcal{A}^\lambda) = \sigma_p(\mathcal{A}^\lambda)$, and so the essential spectrum $\sigma_{ess}(\mathcal{A}^\lambda)$ is empty. Therefore, the spectrum of $\mathcal{A}^\lambda$ with domain $H^m_{per} \cap \mathbb{V}$ is also discrete.

PROPOSITION 2.2. *For any* $\lambda > 0$*, we have* $\sigma_{ess}(\mathcal{A}^\lambda) = \sigma_{ess}(\mathcal{M}+1) = \emptyset$.

Proof. Letting $T = \mathcal{M}+1$ and $A_\lambda = -\dfrac{1}{c}(1-\mathcal{E}^{\lambda,\pm})Q(1+f'(\phi))$, we have $\mathcal{A}^\lambda = T + A_\lambda$ with $D(T) \subset D(A_\lambda)$. Next, since T is a closed linear operator, we obtain immediately that A_λ is $T-$compact. Therefore [20, Theorem 5.35] implies $\sigma_{ess}(\mathcal{A}^\lambda) = \sigma_{ess}(T) = \emptyset$. The last equality is consequence of T has a compact resolvent. ∎

LEMMA 2.2. *Let* $c > 0$ *be fixed. There exists* $\Lambda > 0$ *such that for all* $\lambda > \Lambda$, $\mathcal{A}^\lambda$ *has no eigenvalues* $z \in \mathbb{C}$ *satisfying* $Re z \le 0$.

Proof. See [8]. ∎

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Lemma 2.2 gives us the following result:

PROPOSITION 2.3. *Let $c>0$ be fixed. There exists $\Lambda>0$ such that for all $\lambda>\Lambda$, $\mathcal{A}^\lambda$ with domain $H^m_{per}([0,L])\cap\mathbb{V}$ has no eigenvalues $z\in\mathbb{C}$ satisfying $Rez\leq 0$.*

Proof. See [8]. ∎

Now, we present some results concerning the spectra of the linear operator $\mathcal{A}^\lambda$ in $H^m_{per}([0,L])\cap\mathbb{V}$ for $\lambda>0$ small enough. In order to obtain the results contained in this subsection, we will apply the arguments of asymptotic perturbation theory in [16, Chapter 19] and [20, Chapter VIII] to the periodic context. We start with the following two definitions.

DEFINITION 2.1. An eigenvalue $\mu_0\in\sigma(Q\mathcal{L}_0)=\sigma_p(Q\mathcal{L}_0)$ is stable with respect to the family $\mathcal{A}^\lambda$ if the following two conditions hold:

(i) There is $\delta>0$ such that the region $\mathcal{Q}_\delta:=\{z\in\mathbb{C};\ 0<|z-\mu_0|<\delta\}$ satisfies $\mathcal{Q}_\delta\subset\rho(Q\mathcal{L}_0)\cap\Delta_b$, where $\rho(Q\mathcal{L}_0)$ is the resolvent set of $Q\mathcal{L}_0$ and Δ_b is the region of boundedness for the family $\mathcal{A}^\lambda$, defined by

$$\Delta_b:=\{z\in\mathbb{C};\ ||R_\lambda(z)||_{B(L^2_{per})}\leq M,\ \forall\ \ 0<\lambda\ll 1\}.$$

Here, $M=M(z)>0$ does not depend on λ and $R_\lambda(z)=(\mathcal{A}^\lambda-z)^{-1}:\mathbb{V}\to H^{\frac{m}{2}}_{per}([0,L])\cap\mathbb{V}$.

(ii) Let Γ be a simple closed curve about μ_0 such that $\Gamma\subset\mathcal{Q}_\delta\subset\rho(Q\mathcal{L}_0)\cap\rho(\mathcal{A}^\lambda)$, for all λ small and define the associated Riesz projector for $\mathcal{A}^\lambda$

$$P_\lambda=-\frac{1}{2\pi i}\oint_\Gamma R_\lambda(z)dz.$$

Then

$$\lim_{\lambda\to 0^+}||P_\lambda-P_{\mu_0}||_{B(\mathbb{V})}=0, \tag{25}$$

where P_{μ_0} is the Riesz projector for $Q\mathcal{L}_0$ and μ_0.

LEMMA 2.3. *Let $c>0$ be fixed. For all $\lambda>0$ small enough, consider $u\in H^{\frac{m}{2}}_{per}([0,L])\cap\mathbb{V}$ satisfying $(\mathcal{A}^\lambda-z)u=v$, where $z\in\mathbb{C}$ with $Rez\leq\frac{1}{2}(1-\frac{1}{c})$ and $v\in L^2_{per}$. Then, we have the estimate*

$$||u||_{H^{\frac{m}{2}}_{per}}\leq M\left(||u||_{L^2_{per,e}}+||v||_{L^2_{per}}\right) \tag{26}$$

for some constant $M>0$ which does not depend on $\lambda>0$. Here, the notation $||\cdot||_{L^2_{per,e}}$ indicates the norm $||g||_{L^2_{per,e}}:=\left(\int_0^L g(x)^2e(x)dx\right)^{1/2}$, where $e(x)=(f'(\phi(x)))^2$.

Proof. See [8] and [21]. ∎

THEOREM 2.1. *Let $c>0$ be fixed. For $z\in\mathbb{C}$ with $Rez\leq\frac{1}{2}(1-\frac{1}{c})$, we have $z\in\Delta_b$ if and only if $z\in\rho(Q\mathcal{L}_0)$.*

Proof. Consider $z\in\Delta_b$. First, it is easy to see that $C^\infty_{per}([0,L])\cap\mathbb{V}$ is a *core* for the linear operator $\mathcal{A}^\lambda$. Then, for all $u\in C^\infty_{per}([0,L])\cap\mathbb{V}$ we have

$$||(\mathcal{A}^\lambda-z)u||_{L^2_{per}}\geq\varepsilon||u||_{L^2_{per}}>0, \tag{27}$$

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for all $0 < \lambda \ll 1$. Here, $\varepsilon > 0$ is parameter which does not depend on λ. By Proposition 2.1 and (21), we obtain for $\lambda \to 0^+$ that

$$||(Q\mathcal{L}_0 - z)u||_{L^2_{per}} \geq \varepsilon ||u||_{L^2_{per}}.$$

Since $Q\mathcal{L}_0$ is self-adjoint, it follows from the last inequality that $z \in \rho(Q\mathcal{L}_0)$.

Next, we assume that $z \in \rho(Q\mathcal{L}_0)$ but $z \notin \Delta_b$. We guarantee the existence of a sequence $\{u_\lambda\} \subset C^\infty_{per}([0,L]) \cap \mathbb{V}$, with $||u_\lambda||_{L^2_{per}} = 1$ such that

$$||(\mathcal{A}^\lambda - z)u_\lambda||_{L^2_{per}} \longrightarrow 0, \qquad \text{as} \qquad \lambda \to 0^+. \tag{28}$$

Let us denote $v_\lambda = (\mathcal{A}^\lambda - z)u_\lambda$. From Lemma 2.3 we have for λ small

$$||u_\lambda||_{H^{\frac{m}{2}}_{per}} \leq M(||u_\lambda||_{L^2_{per,e}} + ||v_\lambda||_{L^2_{per}}) \leq C.$$

Hence, from the compact embedding $H^{m/2}_{per}([0,L]) \hookrightarrow L^2_{per}([0,L])$, we have (modulo a subsequence) that $u_\lambda \rightharpoonup u$ in $H^{m/2}_{per}([0,L])$ and $u_\lambda \longrightarrow u$ in $\mathbb{V}$ as $\lambda \to 0^+$. Then $\|u\|_{L^2_{per}} = 1$. Next, for each $v \in D((\mathcal{A}^\lambda)^*) = D(Q\mathcal{L}_0)$ we conclude

$$0 = \lim_{\lambda \to 0^+} \langle v, (\mathcal{A}^\lambda - z)u_\lambda \rangle_{L^2_{per}} = \lim_{\lambda \to 0^+} \langle ((\mathcal{A}^\lambda)^* - \bar{z})v, u_\lambda \rangle_{L^2_{per}} = \langle (Q\mathcal{L}_0 - \bar{z})v, u \rangle_{L^2_{per}}. \tag{29}$$

Therefore, $u \in D(Q\mathcal{L}_0)$ and $(Q\mathcal{L}_0 - z)u = 0$. Since $z \in \rho(Q\mathcal{L}_0)$, we conclude that $u = 0$ and this last fact generates a contradiction since $||u||_{L^2_{per}} = 1$. The proof of the lemma is now completed. ∎

THEOREM 2.2. *Let $\mathcal{A}^\lambda$ be the linear operator defined in (21). Suppose that $\mu_0 \in \sigma(Q\mathcal{L}_0)$ (therefore μ_0 is a discrete eigenvalue). The μ_0 is stable in the sense of the Definition 2.1.*

Proof. Let $\mu_0 \in Q\mathcal{L}_0$, then we can choose $\delta > 0$ such that the annular region

$$\mathcal{Q}_\delta = \{z \in \mathbb{C};\ 0 < |z - \mu_0| < \delta\} \subset \rho(Q\mathcal{L}_0). \tag{30}$$

From Theorem 2.1, we see that $\mathcal{Q}_\delta \subset \Delta_b$. Then for $z \in \mathcal{Q}_\delta$

$$||R_\lambda(z)||_{\mathbb{V}} \leq M(z), \quad for \quad 0 < \lambda \ll 1. \tag{31}$$

Therefore, since $\mathcal{A}^\lambda u \to Q\mathcal{L}_0 u$ for $\lambda \to 0^+$ and $\rho(Q\mathcal{L}_0) \cap \Delta_b \neq \emptyset$, we see by the arguments in [20, Chapter VIII] that for all $z \in \mathcal{Q}_\delta$ and $u \in C^\infty_{per}([0,L]) \cap \mathbb{V}$, we have

$$\lim_{\lambda \to 0^+} R_\lambda(z)u = R_0(z)u.$$

Then, the strong resolvent convergence $R_\lambda(z) \to R_0(z)$ is uniform on the circle $\Gamma = \{z : |z - \mu_0| = r < \delta\}$. Hence, the Riesz projections P_λ satisfies for $u \in C^\infty_{per}([0,L]) \cap \mathbb{V}$ that $\lim_{\lambda \to 0^+} P_\lambda u = P_{\mu_0} u$, and, since P_{μ_0} is self-adjoint we have $\lim_{\lambda \to 0^+} P^*_\lambda u = P_{\mu_0} u$. The first of these convergence implies the principle of the nonexpansion of the spectrum, that is, $\dim(P_\lambda) \geq \dim(P_{\mu_0})$ ([20, Chapter VIII, Lemma 1.23]). Next, using [20, Chapter VIII, Lemma 1.24], the two convergence above of the Riesz projectors and the condition that

$$\dim(P_\lambda) \leq \dim(P_{\mu_0}), \tag{32}$$

for all $0 < \lambda \ll 1$, are sufficient to establish the condition (ii) of the Definition 2.1, that is, the norm convergence of the projections. Thus, let us suppose that inequality (32) does not occur. Then, since P_{μ_0} is a orthogonal projection, we can find a sequence $u_\lambda \in \mathbb{V}$,

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$\|u_\lambda\|_{L^2_{per}} = 1$ such that $P_\lambda u_\lambda = u_\lambda$ and $P_{\mu_0} u_\lambda = 0$. Hence, there is a a subsequence, still denoted by $\{u_\lambda\}$, such that $u_\lambda \rightharpoonup u_0$ in L^2_{per}. Now, for $v \in L^2_{per}$ and $\lambda \to 0^+$ the relation

$$\langle v, u_\lambda\rangle_{L^2_{per}} = \langle v, (P_\lambda - P_{\mu_0}) u_\lambda\rangle_{L^2_{per}} = \langle (P^*_\lambda - P_{\mu_0}) v, u_\lambda\rangle_{L^2_{per}},$$

implies that $\langle v, u_0\rangle_{L^2_{per}} = 0$ and thus $u_0 = 0$.

On the other hand, for $z \in \mathcal{Q}_\delta - \Gamma$, we have from the first resolvent identity that

$$(\mathcal{A}^\lambda - z) P_\lambda u_\lambda = -\frac{1}{2\pi i} \oint_\Gamma [u_\lambda - (z-\eta) R_\lambda(\eta) u_\lambda]\, d\eta.$$

Therefore, from (31) and the compactness of Γ, we obtain for $0 < \lambda \ll 1$ that $\|(\mathcal{A}^\lambda - z) P_\lambda u_\lambda\|_{L^2_{per}} \leq M_0[1 + \sup_{\eta\in\Gamma} |z-\eta|]$. Hence,

$$||\mathcal{A}^\lambda u_\lambda||_{L^2_{per}} \leq ||(\mathcal{A}^\lambda - z) P_\lambda u_\lambda||_{L^2_{per}} + ||z P_\lambda u_\lambda||_{L^2_{per}} \leq C, \tag{33}$$

where $C > 0$ does not depend on $\lambda > 0$. Inequality (33) implies that u_λ is bounded in $H^m_{per}([0,L])$. So, we obtain (modulo a subsequence) that there is $u \in L^2_{per}([0,L])$ such that $u_\lambda \to u$ in $L^2_{per}([0,L])$, as $\lambda \to 0^+$, with $||u||_{L^2_{per}} = 1$. Since u_λ converges weakly to zero in L^2_{per} we obtain a contradiction by the uniqueness of the weak limit. This finishes the proof of the theorem. □

§3. The moving kernel formula—revisited

LEMMA 3.1. *Let $L > 0$ be fixed. Suppose that the smooth curve of periodic waves $c \in (0, +\infty) \mapsto \phi \in H^m_{per}([0,L]) \cap \mathbb{V}$ satisfies (8). Assume that $\ker(Q\mathcal{L}_0) = [\phi']$. For $\lambda > 0$ small enough, let $b_\lambda \in \mathbb{R}$ be the only eigenvalue of $\mathcal{A}^\lambda$ near origin. Then,*

$$\lim_{\lambda\to 0^+} \frac{b_\lambda}{\lambda^2} = -\frac{1}{||\phi'||^2_{L^2per}} \frac{1}{c} \left\langle (\mathcal{M}+1) \frac{d}{dc}\phi, \phi \right\rangle := I(c) \tag{34}$$

Proof. By Theorem 2.2, we see that for $\lambda > 0$ small enough, there exists $u_\lambda \in H^{\frac{m}{2}}_{per}([0,L]) \cap \mathbb{V}$ such that $(\mathcal{A}^\lambda - b_\lambda) u_\lambda = 0$, $b_\lambda \in \mathbb{R}$ and $\lim_{\lambda\to 0^+} b_\lambda = 0$. We assume $||u_\lambda||_{L^2_{per,e}} = 1$. Thus, from Lemma 2.3, follow that $||u_\lambda||_{H^{\frac{m}{2}}_{per}} \leq C$, for some constant $C > 0$ which does not depend on $\lambda > 0$. Since u_λ is bounded and the embedding $H^{\frac{m}{2}}_{per}([0,L]) \hookrightarrow L^2_{per}([0,L])$ is compact, modulus a subsequence we have

$$u_\lambda \rightharpoonup u_0 \text{ in } H^{\frac{m}{2}}_{per}([0,L]) \cap \mathbb{V}, \quad \text{as} \quad \lambda \to 0^+ \tag{35}$$

and,

$$u_\lambda \to u_0 \text{ in } \mathbb{V} \quad \text{as} \quad \lambda \to 0^+. \tag{36}$$

Since $\mathcal{A}^\lambda \to \mathcal{A}^0 = Q\mathcal{L}_0$ in $\mathbb{V}$ as $\lambda \to 0^+$ and $||\mathcal{A}^\lambda u_\lambda - \mathcal{A}^0 u_0||_{L^2_{per}} \leq ||\mathcal{A}^\lambda u_\lambda - \mathcal{A}^\lambda u_0||_{L^2_{per}} + ||\mathcal{A}^\lambda u_0 - \mathcal{A}^0 u_0||_{L^2_{per}}$, we have $\lim_{\lambda\to 0^+} \mathcal{A}^\lambda u_\lambda = \mathcal{A}^0 u_0$. Moreover, $\mathcal{A}^0 u_0 = Q\mathcal{L}_0 u_0 = 0$ and since $\ker(Q\mathcal{L}_0) = [\phi']$, we ensure the existence of $\alpha_0 \neq 0$ such that $u_0 = \alpha_0 \phi'$. We can assume $\alpha_0 = 1$, by normalizing the sequence. So, $u_0 = \phi'$.

In view of the equality $(\mathcal{A}^\lambda - b_\lambda)(u_\lambda - u_0) = b_\lambda u_0 + (\mathcal{A}^0 - \mathcal{A}^\lambda) u_0$, by Lemmas 2.2 and 2.3 and convergence (36), we obtain

$$u_\lambda \to u_0 \quad \text{in} \quad H^{\frac{m}{2}}_{per}([0,L]) \cap \mathbb{V}, \quad \text{as} \quad \lambda \to 0^+.$$

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Next, we are going to show that $\lim_{\lambda\to 0^+}\frac{b_\lambda}{\lambda}=0$. Indeed, given that $(\mathcal{A}^\lambda-b_\lambda)u_\lambda=0$, we obtain

$$\frac{b_\lambda}{\lambda}u_\lambda=\frac{\mathcal{A}^\lambda}{\lambda}u_\lambda+\frac{(\mathcal{A}^\lambda-\mathcal{A}^0)}{\lambda}u_\lambda. \tag{37}$$

So, since $\mathcal{A}^0\phi=0$ we have

$$\begin{aligned}\frac{b_\lambda}{\lambda}&=\left\langle\frac{(\mathcal{A}^\lambda-\mathcal{A}^0)}{\lambda}u_\lambda,\phi'\right\rangle_{L^2_{per}}\\&=-\frac{1}{c}\left\langle\frac{\partial_x}{\lambda-c\partial_x}Q(u_\lambda+f'(\phi)u_\lambda),\phi\right\rangle_{L^2_{per}}\\&=\frac{1}{c^2}\left\langle(1-\mathcal{E}^{\lambda,-})Q(u_\lambda+f'(\phi)u_\lambda),\phi\right\rangle_{L^2_{per}}.\end{aligned}$$

Therefore, by Lemma 2.1, we obtain that

$$\begin{aligned}\lim_{\lambda\to 0^+}\frac{b_\lambda}{\lambda}&=\lim_{\lambda\to 0^+}\frac{b_\lambda}{\lambda}\frac{\langle u_\lambda,\phi'\rangle_{L^2_{per}}}{\langle u_\lambda,\phi'\rangle_{L^2_{per}}}\\&=\frac{1}{c^2}\frac{1}{||\phi||^2_{L^2_{per}}}\langle Q(\phi'+f'(\phi)\phi'),\phi\rangle_{L^2_{per}}\\&=\frac{1}{c^2}\frac{1}{||\phi||^2_{L^2_{per}}}\langle\phi'+f'(\phi)\phi',\phi\rangle_{L^2_{per}}=0.\end{aligned}$$

Next, we calculate $\lim_{\lambda\to 0^+}\frac{b_\lambda}{\lambda^2}$. We write $u_\lambda=c_\lambda\phi'+\lambda v_\lambda$, with $c_\lambda=\frac{\langle u_\lambda,\phi'\rangle_{L^2_{per}}}{||\phi'||_{L^2_{per}}}$. Then $\langle v_\lambda,\phi'\rangle_{L^2_{per}}=0$ and $c_\lambda\to 1$, as $\lambda\to 0^+$. Now, we show

$$||v_\lambda||_{H^{\frac{m}{2}}_{per}}\le C, \tag{38}$$

where the constant $c>0$ does not depend on $\lambda>0$. In fact, first note that

$$\mathcal{A}^\lambda v_\lambda=\frac{b_\lambda}{\lambda}u_\lambda-c_\lambda\frac{\mathcal{A}^\lambda\phi'}{\lambda}=\frac{b_\lambda}{\lambda}u_\lambda-c_\lambda\left(\frac{\mathcal{A}^\lambda-\mathcal{A}^0}{\lambda}\right)\phi'. \tag{39}$$

So, considering $\omega_\lambda:=\left(\frac{\mathcal{A}^\lambda-\mathcal{A}^0}{\lambda}\right)\phi'$ and using (8), we have

$$\omega_\lambda=\frac{1}{c}\frac{\partial_x}{\lambda-c\partial_x}(\phi+f(\phi)-A)=-\frac{1}{c^2}(1-\mathcal{E}^{\lambda,-})(\phi+f(\phi)-A) \tag{40}$$

and $||\omega_\lambda||_{L^2_{per}}\le C$. Here, the constant $C>0$ does not depend on $\lambda>0$. So, from Lemma 2.1, we obtain for $\lambda\to 0^+$ that

$$\omega_\lambda\to-\frac{1}{c^2}(\phi+f(\phi)-A)=-\frac{1}{c}(\mathcal{M}+1)\phi. \tag{41}$$

Next, we are going to show that $||v_\lambda||_{L^2_{per,e}}\le C$ for some $C>0$. In fact, suppose otherwise. So, there is a sequence $\lambda_n\to 0^+$ such that $||v_{\lambda_n}||_{L^2_{per,e}}\ge n$. We denote $\widetilde{v}_{\lambda_n}=\frac{v_{\lambda_n}}{||v_{\lambda_n}||_{L^2_{per,e}}}$.

Then, $||\widetilde{v}_{\lambda_n}||_{L^2_{per,e}}=1$ and $\widetilde{v}_{\lambda_n}$ satisfies the equation

$$\mathcal{A}^{\lambda_n}\widetilde{v}_{\lambda_n}=\frac{1}{||v_\lambda||_{L^2_{per,e}}}\left(\frac{b_{\lambda_n}}{\lambda_n}u_{\lambda_n}-c_{\lambda_n}\frac{(\mathcal{A}^{\lambda_n}-\mathcal{A}^0)}{\lambda_n}\phi'\right).$$

We write

$$\omega_{\lambda_n}:=\left(\frac{\mathcal{A}^{\lambda_n}-\mathcal{A}^0}{\lambda_n}\right)\phi',$$

then $\omega_{\lambda_n}\to-\frac{1}{c^2}(\phi+f(\phi)-A)$, as $\lambda_n\to0^+$ and $||\omega_{\lambda_n}||_{L^2_{per}}\le C$. By Lemma 2.3, we obtain that $||\widetilde{v}_{\lambda_n}||_{H^{\frac{m}{2}}_{per}}\le C$ and modulus a subsequence we get $\widetilde{v}_{\lambda_n}\rightharpoonup\widetilde{v}_0\neq0$ in $H^{\frac{m}{2}}_{per}([0,L])\cap\mathbb{V}$ and $\widetilde{v}_{\lambda_n}\to\widetilde{v}_0$ in $\mathbb{V}$. Since $\frac{b_{\lambda_n}}{\lambda_n},\frac{1}{||v_\lambda||_{L^2_{per,e}}}\to0$, as $\lambda_n\to0^+$, we immediately conclude that $\mathcal{A}^0\widetilde{v}_0=0$. So, $\widetilde{v}_0\in\ker(Q\mathcal{L}_0)=[\phi']$, that is, we guarantee the existence of $\xi_0\in\mathbb{R}\backslash\{0\}$ such that $\widetilde{v}_0=\xi_0\phi'$. But, since $\langle\widetilde{v}_{\lambda_n},\phi'\rangle_{L^2_{per}}=0$, we obtain $\langle\widetilde{v}_0,\phi'\rangle_{L^2_{per}}=0$ which is a contradiction. Thus, we deduce that $||v_\lambda||_{L^2_{per,e}}\le C$. Finally, by using Lemma 2.3, we have (38) satisfied.

Consequently, $v_\lambda\rightharpoonup v_0$ in $H^{\frac{m}{2}}_{per}([0,L])\cap\mathbb{V}$ and $v_\lambda\to v_0\neq0$ in $\mathbb{V}$, as $\lambda\to0^+$. Moreover, from (39) and (41), we obtain $\mathcal{A}^\lambda v_\lambda\to\frac{1}{c}(\mathcal{M}+1)\phi$. So, from Lemma 2.1 and convergence $v_\lambda\to v_0$ above, we obtain $\mathcal{A}^\lambda v_\lambda\to\mathcal{A}^0v_0=Q\mathcal{L}_0v_0$. Thus, by the uniqueness of the limit, it follows that

$$Q\mathcal{L}_0v_0=\frac{1}{c}(\mathcal{M}+1)\phi. \tag{42}$$

On the other hand, since $c>0$, we can rewrite (8) as

$$(\mathcal{M}+1)\phi-\frac{1}{c}(\phi+f(\phi))+\frac{A}{c}=0. \tag{43}$$

Since $c\in(0,+\infty)\mapsto\phi$ is a smooth curve, we can differentiate (43) with respect to c to obtain

$$\begin{aligned}(\mathcal{M}+1)\frac{d}{dc}\phi-\frac{1}{c}\left(\frac{d}{dc}\phi+f'(\phi)\frac{d}{dc}\phi\right)&=-\frac{1}{c^2}(\phi+f(\phi))+\frac{A}{c^2}-\frac{1}{c}\frac{dA}{dc}\\&=-\frac{1}{c}(\mathcal{M}+1)\phi-\frac{1}{c}\frac{dA}{dc}.\end{aligned} \tag{44}$$

Using the definition of $\mathcal{L}_0$ and (44), we obtain

$$\mathcal{L}_0\left(\frac{d}{dc}\phi\right)=-\frac{1}{c}(\mathcal{M}+1)\phi-\frac{1}{c}\frac{dA}{dc}. \tag{45}$$

Thus, we have

$$Q\mathcal{L}_0\left(\frac{d}{dc}\phi\right)=-\frac{1}{c}(\mathcal{M}+1)\phi. \tag{46}$$

Now, from (42) and (46), we have the relation $Q\mathcal{L}_0\left(\frac{d}{dc}\phi\right)=-Q\mathcal{L}_0v_0$. Therefore, since $\ker(Q\mathcal{L}_0)=[\phi']$, there exists $\theta\in\mathbb{R}\backslash\{0\}$ such that $\frac{d}{dc}\phi+v_0=\theta\phi'$. Next, we define

$\overline{c_\lambda} := c_\lambda + \lambda\theta$ and $\overline{v_\lambda} := v_\lambda + \theta\phi'$ to deduce the convenient expression $u_\lambda = \overline{c_\lambda}\phi' + \lambda\overline{v_\lambda}$. Therefore, by the convergences $v_\lambda \to v_0$ in $L^2_{per,e}([0,L])$ and

$$(\mathcal{A}^\lambda - b_\lambda)(v_\lambda - v_0) = \frac{b_\lambda}{\lambda}u_\lambda - c_\lambda\omega_\lambda - \mathcal{A}^\lambda(v_0) - b_\lambda v_\lambda + b_\lambda v_0 \longrightarrow 0$$

in $L^2_{per}([0,L])$, as $\lambda \to 0^+$, we obtain by Lemma 2.3 that $||v_\lambda - v_0||_{H^{\frac{m}{2}}_{per}} \to 0$. Thus,

$$\overline{v_\lambda} = v_\lambda - \theta\phi' \to v_0 - \theta\phi' = -\frac{d}{dc}\phi, \quad \text{as } \lambda \to 0^+.$$

Now, for $\Theta_\lambda := \frac{(\mathcal{A}^\lambda - \mathcal{A}^0)}{\lambda}$, we deduce that

$$\frac{b_\lambda}{\lambda^2}u_\lambda = \frac{1}{\lambda}\Theta_\lambda u_\lambda + \frac{1}{\lambda^2}Q\mathcal{L}_0 u_\lambda = \overline{c_\lambda}\frac{1}{\lambda}\Theta_\lambda\phi' + \Theta_\lambda\overline{v_\lambda} + \frac{1}{\lambda^2}Q\mathcal{L}_0 u_\lambda.$$

Consequently, we obtain

$$\begin{aligned} \mathcal{I}(\lambda) &:= \frac{1}{\lambda^2}\langle b_\lambda u_\lambda, \phi'\rangle_{L^2_{per}} \\ &= \frac{1}{\lambda^2}\langle Q\mathcal{L}_0 u_\lambda, \phi'\rangle_{L^2_{per}} + \frac{1}{\lambda}\overline{c_\lambda}\langle\Theta_\lambda\phi', \phi'\rangle_{L^2_{per}} + \langle\Theta_\lambda\overline{v_\lambda}, \phi'\rangle_{L^2_{per}}. \end{aligned} \tag{47}$$

Thus, since the operator $Q\mathcal{L}_0 : H^m_{per}([0,L]) \cap \mathbb{V} \subset L^2_{per}([0,L]) \cap \mathbb{V} \to \mathbb{V}$ is self-adjoint, the first term on the right side of (47) is zero. Next, we will handle with the last two terms in (47) for $0 < \lambda \ll 1$ small enough. In fact, by Lemma 2.1 and the fact that $Q\left(\frac{d}{dc}\phi + f'(\phi)\frac{d}{dc}\phi\right) = \frac{d}{dc}\phi + f'(\phi)\frac{d}{dc}\phi - \frac{1}{L}\int_0^L f'(\phi)\frac{d}{dc}\phi dx$, we obtain,

$$\begin{aligned} \langle\Theta_\lambda\overline{v_\lambda}, \phi'\rangle_{L^2_{per}} &= \left\langle\frac{1}{\lambda}(\mathcal{A}^\lambda - \mathcal{A}^0)v_\lambda, \phi'\right\rangle_{L^2_{per}} \\ &= \frac{1}{c^2}\langle(1 - \mathcal{E}^{\lambda,-})Q(\overline{v_\lambda} + f'(\phi)\overline{v_\lambda}), \phi\rangle_{L^2_{per}} \\ &\to -\frac{1}{c^2}\left\langle Q\left(\frac{d}{dc}\phi + f'(\phi)\frac{d}{dc}\phi\right), \phi\right\rangle_{L^2_{per}} \\ &= -\frac{1}{c^2}\left\langle\frac{d}{dc}\phi + f'(\phi)\frac{d}{dc}\phi, \phi\right\rangle_{L^2_{per}}. \end{aligned} \tag{48}$$

Moreover,

$$\begin{aligned} \frac{1}{\lambda}\langle\Theta_\lambda\phi', \phi'\rangle_{L^2_{per}} &= \frac{1}{\lambda}\left\langle\frac{1}{c}\frac{1}{\lambda - c\partial_x}Q(\phi' + f'(\phi)\phi'), \phi'\right\rangle_{L^2_{per}} \\ &= \frac{1}{c^2}\left\langle\phi + f(\phi), \frac{1}{c}\phi\right\rangle_{L^2_{per}} - \frac{1}{c^3}\langle\phi + f(\phi), \mathcal{E}^{\lambda,+}\phi\rangle_{L^2_{per}} \\ &\longrightarrow \frac{1}{c^2}\left\langle\frac{1}{c}(\phi + f(\phi)), \phi\right\rangle_{L^2_{per}}, \end{aligned}$$

as $\lambda \to 0^+$. Again by (8), we deduce for $\lambda \to 0^+$.

$$\frac{1}{\lambda}\overline{c_\lambda}\langle\Theta_\lambda\phi', \phi'\rangle_{L^2_{per}} \longrightarrow \frac{1}{c^2}\langle(\mathcal{M}+1)\phi, \phi\rangle_{L^2_{per}}. \tag{49}$$

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Furthermore, since $\mathcal{L}_0\left(\frac{d}{dc}\phi\right) = -\frac{1}{c}(\mathcal{M}+1)\phi - \frac{1}{c}\frac{dA}{dc}$, we have that

$$\frac{1}{c}(\mathcal{M}+1)\phi - \frac{1}{c}\left(\frac{d}{dc}\phi + f'(\phi)\frac{d}{dc}\phi\right) = -(\mathcal{M}+1)\frac{d}{dc}\phi - \frac{1}{c}\frac{dA}{dc}. \tag{50}$$

Thus, combining (47) – (49), and using (50), we obtain

$$\begin{aligned}\lim_{\lambda\to 0^+}\mathcal{I}(\lambda) &= \lim_{\lambda\to 0^+}\frac{1}{\lambda}\overline{c_\lambda}\langle\Theta_\lambda\phi',\phi'\rangle_{L^2_{per}} + \lim_{\lambda\to 0^+}\langle\Theta_\lambda\overline{v_\lambda},\phi'\rangle_{L^2_{per}}\\ &= \frac{1}{c}\left\langle -(\mathcal{M}+1)\frac{d}{dc}\phi - \frac{1}{c}\frac{dA}{dc},\phi\right\rangle_{L^2_{per}}\\ &= \frac{1}{c}\left\langle -(\mathcal{M}+1)\frac{d}{dc}\phi,\phi\right\rangle_{L^2_{per}}.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\lim_{\lambda\to 0^+}\frac{b_\lambda}{\lambda^2} &= \lim_{\lambda\to 0^+}\frac{\mathcal{I}(\lambda)}{\langle u_\lambda,\phi'\rangle}\\ &= -\frac{1}{c}\frac{1}{||\phi'||^2_{L^2_{per}}}\left\langle(\mathcal{M}+1)\frac{d}{dc}\phi,\phi\right\rangle_{L^2_{per}}\\ &= I(c).\end{aligned}$$

This finishes the proof of the Lemma. ∎

Corollary 3.1. *Let $c\in(0,+\infty)\mapsto\phi$ be a smooth curve of periodic traveling solutions with the zero property satisfying (8) with $f(s)=\frac{s^2}{2}$. The moving kernel formula in Lemma 3.1 is given by*

$$I(c) = -\frac{1}{c^2}\frac{1}{||\phi'||^2_{L^2_{per}}}\left[\langle(\mathcal{M}+1)\phi,\phi\rangle_{L^2_{per}} + L\left(\frac{dA}{dc}\right)\right]. \tag{51}$$

Proof. Deriving (8) with respect to c, integrating in $[0,L]$ and using the fact that $c\in(0,+\infty)\mapsto\phi$ is a smooth curve with the zero mean property, we obtain

$$c\int_0^L\phi(\mathcal{M}+1)\frac{d}{dc}\phi dx + \int_0^L\phi(\mathcal{M}+1)\phi dx - \frac{1}{2}\frac{d}{dc}\int_0^L\phi^2dx - \frac{1}{3}\frac{d}{dc}\int_0^L\phi^3dx = 0. \tag{52}$$

The next step is to multiply (8) by ϕ, integrating in $[0,L]$ and deriving the final result with respect to c to get

$$2c\int_0^L\phi(\mathcal{M}+1)\frac{d}{dc}\phi dx + \int_0^L\phi(\mathcal{M}+1)\phi dx - \frac{1}{2}\frac{d}{dc}\int_0^L\phi^2dx - \frac{1}{2}\frac{d}{dc}\int_0^L\phi^3dx = 0. \tag{53}$$

By (52) and (53), we deduce that

$$-c\left\langle(\mathcal{M}+1)\frac{d}{dc}\phi,\phi\right\rangle + \langle(\mathcal{M}+1)\phi,\phi\rangle + \frac{1}{2}\frac{d}{dc}\int_0^L\phi^2dx = 0. \tag{54}$$

Deriving (8) with respect to c and integrating we have

$$\frac{1}{2}\frac{d}{dc}\int_0^L\phi^2dx = L\left(\frac{dA}{dc}\right). \tag{55}$$

Hence, we can combine (34) with (54) and (55) to obtain the required result in (51). ∎

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We are in position to establish the following spectral stability criterion for (1).

THEOREM 3.1. *Let $c \to \phi \in H^m_{per}([0,L]) \cap \mathbb{V}$ be a smooth curve of periodic solution to (8) with $c > 0$. We assume that*

$$\ker(Q\mathcal{L}_0) = [\phi']. \tag{56}$$

Denote by $n(Q\mathcal{L}_0)$ the number (counting multiplicity) of negative eigenvalues of the operator $Q\mathcal{L}_0$ defined on $H^m_{per}([0,L]) \cap \mathbb{V}$. The wave is spectrally unstable if one of the following two conditions is true:

(i) *$n(Q\mathcal{L}_0)$ is even and $I(c) < 0$.*
(ii) *$n(Q\mathcal{L}_0)$ is odd and $I(c) > 0$.*

Proof. See [8]. ∎

Concerning the spectral/orbital stability, we have the following result.

THEOREM 3.2. *Under the same assumptions of Theorem 3.1, if $n(Q\mathcal{L}_0) = 1$ and $I(c) < 0$ we have:*

(a) *the periodic wave ϕ is spectrally stable.*
(b) *Let $s \geq \frac{m}{2}$ be large enough. If the associated Cauchy problem for (1) is globally well-posed in $H^s_{per}([0,L]) \cap \mathbb{V}$, the periodic wave ϕ is orbitally stable in the Sobolev space $H^{\frac{m}{2}}_{per}([0,L]) \cap \mathbb{V}$.*
(c) *If we assume additionally that $n(\mathcal{L}_0) = 1$, $\ker(\mathcal{L}_0) = [\phi']$ and the associated Cauchy problem for (1) is globally well-posed in $H^s_{per}([0,L])$ for $s \geq \frac{m}{2}$ large enough, the periodic wave ϕ is orbitally stable in $H^{\frac{m}{2}}_{per}([0,L])$.*

Proof. We prove item (a). Applying [1, Proposition 3.8] for $Q\mathcal{L}_0$, we obtain by (46) and the fact $I(c) < 0$ the existence of $C_1 > 0$ such that

$$\langle \mathcal{L}_0 v, v\rangle_{L^2_{per}} = \langle Q\mathcal{L}_0 v, v\rangle \geq C_1 ||v||^2_{L^2_{per}}, \tag{57}$$

for all $v \in H^m_{per}([0,L]) \cap \{1, (\mathcal{M}+1)\phi\}^{\perp}$ such that $v \in \{\phi'\}^{\perp}$. Since (57) implies that $n\left(\mathcal{L}_0|_{\{1,(\mathcal{M}+1)\phi\}^{\perp}}\right) = 0$, the periodic wave is then spectrally stable. To prove item (b), we need to assume that the Cauchy problem is globally well-posed in $H^s_{per}([0,L]) \cap \mathbb{V}$ for $s \geq \frac{m}{2}$ large enough. We see that $\int_0^L u(x,t)dx = \int_0^L u_0(x)dx = 0$ for all $t \geq 0$, where u_0 denotes the initial data $u(x,0) = u_0(x)$ of the Cauchy problem associated to (1). Since we have the estimate (57), the orbital stability in $H^{\frac{m}{2}}_{per}([0,L]) \cap \mathbb{V}$ is determined using the approach in [1] (see also [11]). Finally, we prove (c). In fact, since $n(\mathcal{L}_0) = 1$, $\ker(\mathcal{L}_0) = [\phi']$ and $I(c) < 0$, the orbital stability in the energy space $H^{\frac{m}{2}}_{per}([0,L])$ comes immediately from the arguments in [1]. ∎

§4. Applications

In this section, we will present two applications in order to illustrate that our approach can be applied.

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4.1 Spectral instability for the quintic BBM equation

We start by proving the spectral instability of cnoidal periodic waves for the focusing quintic BBM equation,

$$u_t+u_x+u^4u_x-u_{xxt}=0, \tag{58}$$

that is, $\mathcal{M}=-\partial_x^2$ and $f(u)=\frac{u^5}{5}$ in (1).

We seek for periodic waves of the form $u(x,t)=\phi_c(x-ct)$ solving the equation

$$-c\phi''+(c-1)\phi-\frac{1}{5}\phi^5+A=0, \tag{59}$$

where A is a constant of integration. We assume that $A\equiv 0$ for all c in order to produce symmetric periodic waves with the zero mean property.

An explicit family of periodic traveling waves of (58) is given by

$$\phi(x)=\frac{a\mathrm{cn}\left(\frac{4K}{L}x,k\right)}{\sqrt{1-b\mathrm{sn}^2\left(\frac{4K}{L}x,k\right)}}, \tag{60}$$

where the period $L>0$ is fixed. Here, cn stands for the cnoidal elliptic function, sn the snoidal elliptic function and $K=K(k)$ indicate the complete elliptic integral of first kind depending on the elliptic modulus $k\in(0,1)$. Parameters a, b and c depends smoothly on the modulus $k\in(0,1)$ and they are given by

$$a=\frac{5^{\frac{1}{4}}2\sqrt{K}(2-k^2+2\sqrt{k^4-k^2+1})^{\frac{1}{4}}}{(-16K^2\sqrt{k^4-k^2+1}+L^2)^{\frac{1}{4}}}, \tag{61}$$

$$b=k^2-1-\sqrt{k^4-k^2+1}, \tag{62}$$

and

$$c=\frac{L^2}{-16K^2\sqrt{k^4-k^2+1}+L^2}, \tag{63}$$

for all $k\in(0,1)$.

Function b assumes only negative values for all $k\in(0,1)$ and thus, the denominator in (60) makes sense for all values of $k\in(0,1)$ and $x\in[0,L]$. Next, to get real solutions ϕ, we also need to assume that $q(k,L):=-16K^2(k^4-k^2+1)^{\frac{1}{2}}+L^2$, present in the denominator of a and c in (61) and (62) is a positive function in terms of the pair (k,L). For a fixed $L>0$, since $k\in(0,1)\mapsto 16K^2(k^4-k^2+1)^{\frac{1}{2}}$ is a strictly increasing function in terms of $k\in(0,1)$, we guarantee the existence of a unique $k_L\in(0,1)$ such that

$$L^2>16K^2(k^4-k^2+1)^{\frac{1}{2}}, \tag{64}$$

for all $k\in(0,k_L)$.

Now, let $\mathcal{L}:=\mathcal{L}_{(c(k,L))}$ be the linearized operator of (59) around ϕ given by

$$\mathcal{L}=-c\partial_x^2+c-1-\phi^4. \tag{65}$$

One has that $\mathcal{L}$ is an unbounded self-adjoint operator defined on $L^2_{per}([0,L])$ with domain $H^2_{per}([0,L])$.

To obtain the spectral properties for $\mathcal{L}$, it is necessary to use the classical Floquet theory. To this end, we consider the second order ordinary differential equation in a general form as

$$-\varphi'' + g(c,\varphi) = 0, \tag{66}$$

where g is a smooth function in all variables. We assume that the parameter c belongs to an open set $\mathcal{P} \subset \mathbb{R}$. The linearized operator around φ

$$\mathcal{G}h = -h'' + g'(c,\varphi)\,h, \quad c \in \mathcal{P}, \tag{67}$$

which is a Hill operator. According with [23], the spectrum of $\mathcal{L}$ is formed by an unbounded sequence of real numbers

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 \cdots < \lambda_{2n-1} \leq \lambda_{2n} \cdots,$$

where equality means that $\lambda_{2n-1} = \lambda_{2n}$ is a double eingenvalue.

By the *Oscillation Theorem* in [23], the spectrum of $\mathcal{L}$ is characterized by the number of zeros of the eigenfunctions. In fact, if χ is an eigenfunction associated to the eigenvalue λ_{2n-1} or λ_{2n}, then χ has exactly $2n$ zeros in the half-open interval $[0,\, L)$.

In our case, consider $\{\phi', y\}$ the fundamental set related to the Hill equation

$$-ch'' + (c-1)h - \phi^4 h = 0. \tag{68}$$

We see that ϕ' is an L-periodic solution of (68). The arguments in [23] establish a connection between ϕ' and y through the equality,

$$y(x+L) = y(x) + \theta\phi'(x), \tag{69}$$

where θ is a real constant given by

$$\theta = \frac{y'(L)}{\phi''(0)}. \tag{70}$$

In order to know the exact position of the zero eigenvalue associated to $\mathcal{L}$ we follow the main result put forward in [27], given below.

THEOREM 4.1. *Let ϕ' be periodic eigenfunction of $\mathcal{L}$ in (67) associated to the zero eigenvalue. If θ is the constant given by (70), then zero eigenvalue is simple if and only if $\theta \neq 0$. Moreover, since ϕ' has two zeros in the half-open interval $[0, L)$, then $\lambda_1 = 0$ if $\theta < 0$, and $\lambda_2 = 0$ if $\theta > 0$.*

To continue with our spectral analysis, it is necessary to know exactly the nonpositive spectrum of $\mathcal{L}$ by studying the inertial index $In(\mathcal{L})$ of $\mathcal{L}$, where $In(\mathcal{L})$ is a pair of integers (n,z), where n is the dimension of the negative subspace of $\mathcal{L}$ and z is the dimension of the null subspace of $\mathcal{L}$. For instance, $In(\mathcal{L}) = (1,1)$ means that the number of negative eigenvalues of $\mathcal{L}$ and denoted by $n(\mathcal{L})$ is one and the dimension of the kernel $z(\mathcal{L})$ is also one. Using Theorem 3.1 in [24], we see that it is enough to compute the index $In(\mathcal{L})$ only for an arbitrary value of $k_0 \in (0,k_L)$ and for fixed value of L such that inequality (64) holds.

Table 1. Values of θ and $\mathcal{J}$ in terms of the modulus k_0 for $L=4\pi$.

	$L=4\pi$	
k_0	θ	$\mathcal{J}$
0.0001	32.4432	6.2833
0.1	30.5161	6.3690
0.3	33.4888	6.3182
0.5	46.0200	6.0088
0.7	74.3573	5.2724
0.9	224.6495	2.8378
0.9999	$\theta \notin \mathbb{R}$	$\mathcal{J} \notin \mathbb{R}$

We put forward some values of θ for fixed values of $L>0$ and $k \in (0,k_L)$. To do so, we need to determine the solution associated with the initial value problem given by

$$\left\{\begin{array}{l} -cy''+(c-1-\phi^4)y=0 \\ y(0)=-\frac{1}{\phi''(0)} \\ y'(0)=0. \end{array}\right. \tag{71}$$

We also want to check the spectral stability of ϕ. To this end, we need to use an index formula established in [28, Theorem 4.1] and given by

$$n(Q\mathcal{L})=n(\mathcal{L})-n(\mathcal{J})-z_0, \tag{72}$$

and

$$z(Q\mathcal{L})=z(\mathcal{L})+z_0-z_\infty, \tag{73}$$

where z_0 denotes the dimension of the kernel of the quantity $\mathcal{J}=\langle \mathcal{L}^{-1}1,1\rangle$ and z_∞ the corresponding number of diverging eigenvalues. For $L>0$ such that (64) occurs, it is important to mention that $\theta \neq 0$ if, and only if, $z_\infty=0$.

We need to calculate $\mathcal{J}$. First, we use the solution y obtained in (71) and the initial value problem given by

$$\left\{\begin{array}{l} -c\overline{y}''+(c-1-\phi^4)\overline{y}=1 \\ \overline{y}(0)=\frac{1}{cy'(L)}\int_0^L y(x)dx \\ \overline{y}'(0)=0. \end{array}\right. \tag{74}$$

The initial condition $\overline{y}(0)$ is obtained by multiplying equation in (74) by y and performing two integration by parts.

According with the tables below, we can compute some values of θ and $\mathcal{J}$ in terms of the modulus $k_0 \in (0,k_L)$ for fixed periods. In Table 1, we see that for $k_0=0.9999$ the values of θ and $\mathcal{J}$ are complex numbers since inequality (64) is not satisfied for this value of the elliptic modulus. Next, we know that ϕ' is an eigenfunction associated to the zero eigenvalue and we can deduce that ϕ' has exactly two zeros in half-open interval $[0,L)$. By Theorem 4.1 and Tables 1 and 2, we have that the linearized operator $\mathcal{L}$ satisfies $In(\mathcal{L})=(2,1)$ for all $k\in(0,k_L)$.

From equality (34), we have

$$P(c)=\frac{d}{dc}\int_0^L (\phi')^2+\phi^2dx. \tag{75}$$

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Table 2. Values of θ and $\mathcal{J}$ in terms of the modulus k_0 for $L = 12\pi$.

	$L = 12\pi$	
k_0	θ	$\mathcal{J}$
0.0001	2992.02	219.919
0.1	2768.77	226.843
0.3	2490.16	236.194
0.5	2444.75	240.866
0.7	2759.26	238.559
0.9	4591.67	206.467
0.9999	502992.1	41.752

Since we know the periodic waves for (59), we can calculate $||\phi||^2_{H^1_{per}} = \int_0^L (\phi')^2 + \phi^2 dx$ to obtain $I(c)$ in (34).

Using the explicit form of ϕ in (60), we found that

$$\int_0^L \phi^2 dx = \int_0^L \frac{a^2 \mathrm{cn}^2\left(\frac{4K}{L}x,k\right)}{1-b\mathrm{sn}^2\left(\frac{4K}{L}x,k\right)} dx.$$

Let us consider the change of variables $s = \frac{4K}{L}x$. We have

$$\int_0^L \phi^2 dx = \frac{a^2 L}{4K}\int_0^{4K} \frac{\mathrm{cn}^2(s,k)}{1-b\mathrm{sn}^2(s,k)} ds = \frac{a^2 L}{K}\int_0^{K} \frac{\mathrm{cn}^2(s,k)}{1-b\mathrm{sn}^2(s,k)} ds. \tag{76}$$

Now, since $0 < -b < +\infty$ and using the formula 411.03 in [10], we obtain

$$\int_0^{K} \frac{\mathrm{cn}^2(s,k)}{1-b\mathrm{sn}^2(s,k)} ds = \frac{\pi(1-b)(1-\Lambda_0(\beta,k))}{2\sqrt{b(1-b)(b-k^2)}},$$

where

$$\Lambda_0(\beta,k) = \frac{2}{\pi}[E(k)F(\beta,k') + K(k)E(\beta,k') - K(k)F(\beta,k')]$$

and

$$\beta = \sin^{-1}\left(\frac{1}{\sqrt{1-b}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2-k^2+\sqrt{k^4-k^2+1}}}\right).$$

Here, $F(\beta,k')$ and $E(\beta,k')$ indicate the incomplete elliptic integral of first kind and second kind, respectively. Parameter $k' = \sqrt{1-k^2}$ is the elliptic complementary modulus and $E(k)$ indicate the complete elliptic integral of second kind. Function $\Lambda_0(\beta,k)$ is known as Heuman Lambda function.

Defining the function q which depends only on $k \in (0,k_L)$ as

$$q(k) := \frac{\pi(1-b)(1-\Lambda_0(\beta,k))}{2K\sqrt{b(1-b)(b-k^2)}} = \frac{\pi\sqrt{2-k^2+\sqrt{k^4-k^2-1}}(1-\Lambda_0(\beta,k))}{2K\sqrt{1-k^2+\sqrt{k^4-k^2+1}}\sqrt{1+\sqrt{k^2-k^2+1}}}$$

we can rewrite (76) as

$$\int_0^L \phi^2 dx = (a^2 L) q(k). \tag{77}$$

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On the other hand, multiplying (59) by ϕ and integrating in $[0,L]$, we obtain

$$-c\int_0^L(\phi')^2dx-(c-1)\int_0^L\phi^2dx+\frac{1}{5}\int_0^L\phi^6dx=0. \tag{78}$$

Moreover, multiplying (59) by ϕ' and integrating, we obtain the quadrature form

$$c\frac{(\phi')^2}{2}-(c-1)\frac{\phi^2}{2}+\frac{\phi^6}{30}+B=0, \tag{79}$$

where B is a nonzero constant of integration.

Integrating (79) in $[0,L]$, we found

$$c\int_0^L(\phi')^2dx-(c-1)\int_0^L\phi^2dx+\frac{1}{15}\int_0^L\phi^6dx+2LB=0. \tag{80}$$

Combining (78) and (80), we get

$$\int_0^L(\phi')^2dx=\frac{(c-1)}{2c}\int_0^L\phi^2dx-\frac{3LB}{2c}. \tag{81}$$

Since ϕ' is odd, we can express B by taking $x=0$ in (79) as

$$B=(c-1)\frac{\phi^2(0)}{2}-\frac{\phi^6(0)}{2}.$$

Using the explicit form of ϕ in (60), we obtain

$$B=\frac{32}{3}\frac{\sqrt{5}K^3\sqrt{2-k^2+2\sqrt{k^4-k^2+1}}(\sqrt{k^4-k^2+1}-2+k^2)}{(-16K^2\sqrt{k^4-k^2+1}+L^2)^{\frac{3}{2}}}. \tag{82}$$

Therefore, the relations (61), (63), (77), (81), and (82) allow us to write

$$\int_0^L(\phi')^2+\phi^2dx=\frac{a^2L(3c-1)}{2c}q(k)-\frac{3LB}{2c}:=V_L(k). \tag{83}$$

Using Maple program, we can see that for $L>4K(k^4-k^2+1)^{\frac{1}{4}}$ and function $V_L(k)$ is strictly increasing in $(0,k_L)$ (see Figures 1–4. For instance, by taking $L=4\pi$, we can conclude that $k_L=0.99$.

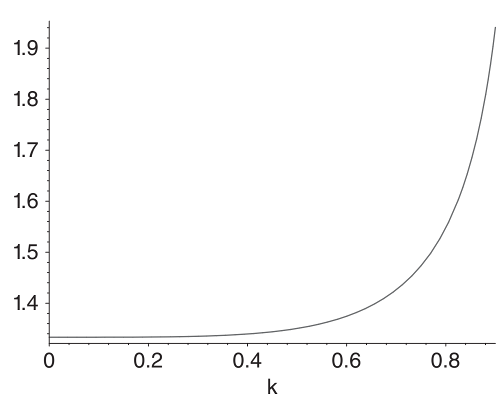

Figure 1.
Graph of $k\in(0,k_L)\mapsto c(k)$ for $L=4\pi$.

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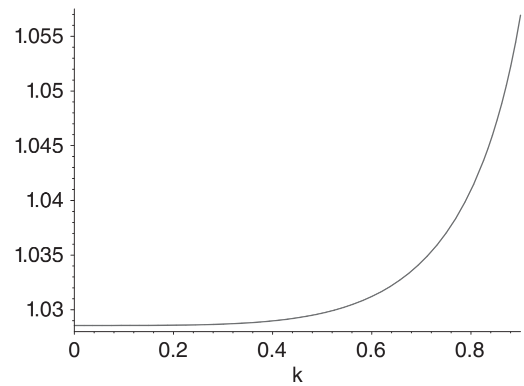

Figure 2.
Graph of $k \in (0, k_L) \mapsto c(k)$ for $L = 12\pi$.

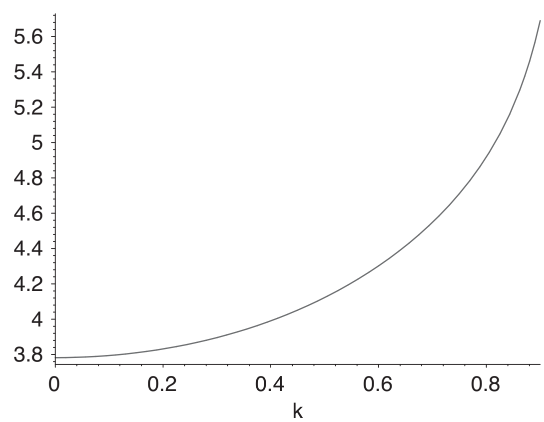

Figure 3.
Graph of $V_L(k)$ for $L = 4\pi$.

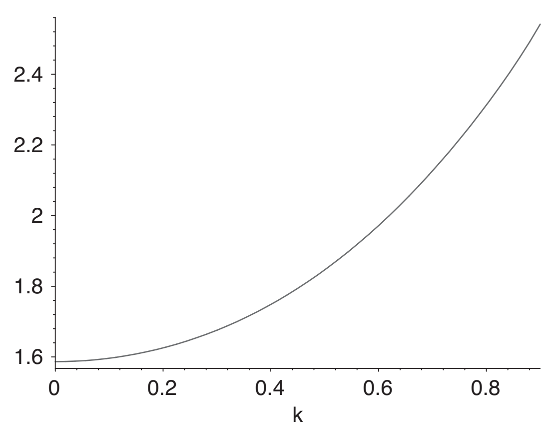

Figure 4.
Graph of $V_L(k)$ for $L = 12\pi$.

Thus,

$$P(c) = \frac{d}{dc} \int_0^L (\phi')^2 + \phi^2 dx = \frac{dV_L(k)}{dk} \frac{dk}{dc} > 0, \tag{84}$$

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for all $k \in (0, k_L)$. So, using (3.1), we conclude that $I(c) < 0$. In addition, we obtain from Table 1 that $\mathcal{J} = \langle \mathcal{L}^{-1}1, 1\rangle > 0$, that is, $n(\mathcal{J}) = 0$. Hence, we conclude from (73) that $\ker(Q\mathcal{L}) = [\phi']$. Therefore, since $n(\mathcal{L}) = 2$, $n(\mathcal{J}) = 0$ and $z_0 = 0$, we deduce from (72) that $n(Q\mathcal{L}) = 2$. Since $n(Q\mathcal{L}) = 2$ and $I(c) < 0$, we conclude that the periodic solution ϕ is spectrally unstable by Theorem 3.1.

REMARK 4.1. It is worth mentioning that the orbital instability of the periodic wave ϕ in $H^1_{per}([0, L])$ can be determined by repeating similar arguments as in [8, Section 3.1]. To this end, it makes necessary to employ the results in [15] where the authors gave sufficient conditions to prove the nonlinear (orbital) instability from the spectral instability.

4.2 Spectral stability for the rBO equation

Now, we are interested in studying the spectral (orbital) stability of zero mean periodic waves for the rBO equation

$$u_t + u_x + uu_x + \mathcal{H}u_{xt} = 0. \tag{85}$$

As we have performed in the last application, periodic traveling waves of (85) are solutions of the form $u(x,t) = \phi(x - ct)$. Substitute this kind of solution into (85), we obtain after integration

$$c\mathcal{H}\phi' + (c-1)\phi - \frac{\phi^2}{2} + A = 0, \tag{86}$$

where $A = A(c)$ is a constant of integration defined by

$$A(c) := \frac{1}{2L}\int_0^L \phi^2 dx. \tag{87}$$

Important to mention that condition (87) gives that ϕ has the zero mean property.

As before, the linearized and self-adjoint operator around the periodic wave ϕ is defined by

$$\mathcal{L}_\phi = c\mathcal{H}\partial_x + c - 1 - \phi. \tag{88}$$

Let us consider the following transformation

$$\psi = \frac{1}{2c}\left[\phi - (c-1) + \sqrt{(c-1)^2 + 2A}\right] \tag{89}$$

to convert (86) to the equation which determines positive and periodic waves for the classical Benjamin–Ono equation ($\mathcal{M} = \mathcal{H}\partial_x$ and $f(u) = \frac{u^2}{2}$ in (2)),

$$\mathcal{H}\psi' + \omega\psi - \psi^2 = 0, \tag{90}$$

where $\omega = \frac{1}{c}\sqrt{(c-1)^2 + 2A}$. Integrating (89) on $[0, L]$ and using the fact that $\phi : \mathbb{R} \to \mathbb{R}$ is a periodic function with the zero mean property, we have

$$c = \left(1 - \omega + \frac{2}{L}\int_0^L \psi dx\right)^{-1}. \tag{91}$$

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To simplify the notation, we consider $L=2\pi$. According with [4], the periodic solution for (90) is given by

$$\psi(x)=\frac{\sinh(\gamma)}{\cosh(\gamma)-\cos(x)}, \tag{92}$$

where parameter $\gamma\in(0,+\infty)$ being expressed by $\gamma=\coth^{-1}(\omega)$. For $c>1$, t he transformation (89) allows us to obtain the periodic solution for the rBO equation (86) as

$$\phi(x)=2c\left(\frac{\sinh(\gamma)}{\cosh(\gamma)-\cos(x)}-1\right). \tag{93}$$

The linearized operator around periodic wave ψ is given by

$$\mathcal{L}_\psi=\mathcal{H}\partial_x+\omega-2\psi. \tag{94}$$

The transformation (89) allows us to establish a relation between both linearized operators $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ as

$$\mathcal{L}_\phi=c\mathcal{L}_\psi, \tag{95}$$

By (92), we obtain $\int_0^{2\pi}\psi(x)dx=2\pi$ which allows to deduce, by (91) that $c=(3-\omega)^{-1}$. In addition, from the equality $\omega=\frac{1}{c}\sqrt{(c-1)^2+2A}$, it follows that $A(c)=4c^2-2c$.

In order to get the spectral conditions required to prove the spectral stability of rBO equation, we employ again the index formula in [28, Theorem 4.1]. First, we calculate $\mathcal{J}=\langle\mathcal{L}_\phi^{-1}1,1\rangle$ and $I(c)$ in (34).

Denoting $A'(c):=\frac{dA}{dc}$, we see that

$$\mathcal{L}_\phi\left(\frac{d}{dc}\phi\right)=\frac{1}{c}\left(A(c)-\phi-\frac{1}{2}\phi^2-cA'(c)\right),\quad \mathcal{L}_\phi\left(\frac{1}{c}\right)=\frac{1}{c}(c-1-\phi) \tag{96}$$

and

$$\mathcal{L}_\phi\left(\frac{1}{c}\phi\right)=-\frac{1}{c}\left(\frac{1}{2}\phi^2+A(c)\right). \tag{97}$$

Equations in (96) and (97) give us the relation

$$\mathcal{L}_\phi\left(\frac{d}{dc}\phi-\frac{1}{c}-\frac{1}{c}\phi\right)=\frac{1}{c}(1+2A(c)-c-cA'(c))=\frac{1}{c}(1-3c):=d. \tag{98}$$

Thus, for $c>1$ we obtain $d\neq 0$ and

$$\mathcal{L}_\phi\left(d^{-1}\left(\frac{d}{dc}\phi-\frac{1}{c}-\frac{1}{c}\phi\right)\right)=1. \tag{99}$$

In addition, we can compute explicitly $\mathcal{J}$ as

$$\begin{aligned}\mathcal{J}=\langle\mathcal{L}_\phi^{-1}1,1\rangle &= -\frac{2\pi}{1+2A(c)-c-cA'(c)}\\ &= \frac{2\pi}{3c-1}.\end{aligned} \tag{100}$$

Therefore, for $c>1$ we have that $\mathcal{J}>0$.

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REMARK 4.2. By differentiating the relation $\omega = \frac{1}{c}\sqrt{(c-1)^2+2A(c)}$, we obtain for $c > \frac{1}{2}$ that

$$\frac{d\omega}{dc} = \frac{d}{c\sqrt{1+2A(c)+c^2-2c}} \neq 0. \tag{101}$$

Next, since $f(u) = \frac{u^2}{2}$ in rBO equation (85), we shall use the expression given in (51) in order to obtain the exact value of $I(c)$. To so do, we need to obtain a convenient expression for $\langle(\mathcal{H}\partial_x+1)\phi,\phi\rangle_{L^2_{per}} + 2\pi A'(c)$.

In fact, by Poincaré–Wirtinger inequality, we deduce

$$\langle(\mathcal{H}\partial_x+1)\phi,\phi\rangle_{L^2_{per}} = \int_0^{2\pi}\phi\mathcal{H}\phi' dx + \int_0^{2\pi}\phi^2 dx \geq 2\int_0^{2\pi}\phi^2 dx. \tag{102}$$

Moreover, by (87), we have

$$\int_0^{2\pi}\phi^2 dx = 4\pi(4c^2-2c). \tag{103}$$

Since $A'(c) = 8c-2$, we substitute (103) into (102) to obtain that

$$\langle(\mathcal{H}\partial_x+1)\phi,\phi\rangle_{L^2_{per}} + 2\pi A'(c) \geq 4\pi(8c^2-1),$$

that is, for $c > 1$ we obtain $I(c) < 0$.

On the other hand, as we have already seen in (100) since $\mathcal{J} = \langle\mathcal{L}_\phi^{-1}1,1\rangle > 0$, we get $n(\mathcal{J}) = 0$. By the equalities in (72) and (73), we obtain $n(Q\mathcal{L}_\phi) = n(\mathcal{L}_\phi)$ and $z(Q\mathcal{L}_\phi) = z(\mathcal{L}_\phi)$, respectively. To calculate $n(\mathcal{L}_\phi)$ and $z(\mathcal{L}_\phi)$, we employ (95) joint with the arguments in [6, Section 5.1]. In fact, according with [6], the linearized operator $\mathcal{L}_\psi$ has an unique simple negative eigenvalue and zero is a simple eigenvalue whose eigenfunction is ϕ' and thus, we obtain $n(\mathcal{L}_\phi) = z(\mathcal{L}_\phi) = 1$. These facts allow us to deduce $n(Q\mathcal{L}_\phi) = z(Q\mathcal{L}_\phi) = 1$.

Finally, since $I(c) < 0$ and $n(Q\mathcal{L}_\phi) = z(Q\mathcal{L}_\phi) = 1$, we employ Theorem 3.2(a) to obtain that the periodic wave solution ϕ for the rBO equation is spectrally stable. We can also use Theorem 3.2(c) to prove that the periodic wave ϕ is orbitally stable in the energy space $H^{\frac{1}{2}}_{per}([0,L])$. To do so, it remains to check the global well-posedness in $H^s_{per}([0,L])$ for $s > \frac{1}{2}$. However, this fact has been verified in [4, Corollary 3.3].

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