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DETERMINANTS OF COVARIANCE MATRICES OF DIFFERENCED AR(1) PROCESSES

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In this note, determinants are explicitly calculated for the covariance matrices of differenced and double-differenced AR(1) variables.

1. MOTIVATION AND RESULTS

Consider the AR(1) process $y_t = \rho y_{t-1} + \varepsilon_t$, $t \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, $\varepsilon_t \sim iid(0,1)$, and $\rho \in (-1,1]$. The error variance is set to unity without loss of generality in the present context because otherwise $\text{var}(\varepsilon_t)^{-1/2} y_t$ can be considered. Let $\Delta y_t = y_t - y_{t-1}$. The covariance matrix of $\Delta y = (\Delta y_1, \ldots, \Delta y_n)'$ is a symmetric Toeplitz matrix with its (t,s) element equal to $\omega_{|t-s|} := E\Delta y_t \Delta y_s$; in the unit root case $\Delta y_t = \varepsilon_t$. Similarly, the covariance matrix of the double-differenced series $\Delta^2 y_t := \Delta y_t - \Delta y_{t-1}$ is another symmetric Toeplitz matrix whose (t,s) element depends on $|t-s|$.

Evaluating the determinant of covariance matrix for Δy_t and $\Delta^2 y_t$ is sometimes useful, especially when working with the exact Gaussian likelihood functions derived from the first- and double-differenced data. A prominent example is the simple dynamic panel data model with unobservable fixed effects $y_{it} = (1-\rho)\alpha_i + \rho y_{it-1} + \varepsilon_{it}$. One may want to first-difference the data to eliminate the nuisance fixed effects and get $\Delta y_{it} = \rho \Delta y_{it-1} + \Delta \varepsilon_{it}$ and then derive the likelihood function for $\{\Delta y_{it}\}$. (See Hsiao, Pesaran, and Tahmiscioglu, 2002.) If the model also contains incidental trends as in $y_{it} = \alpha_i + (1-\rho)\gamma_i t + \rho y_{it-1} + \varepsilon_{it}$, then a double-differencing method can eliminate the incidental trends. This possibility is investigated by Han and Phillips (2006), who find (using results in the present note) some interesting facts, e.g., that a panel unit root test based on double-differenced data can outperform point optimal tests such as Ploberger and Phillips (2002) if the time span is small relative to the cross-sectional dimension.

In the preceding dynamic panel data model, approximate or conditional maximum likelihood estimation (MLE) may yield poor estimators especially if the

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time dimension is small and the cross-sectional sample size is large because then small errors due to inaccurate approximation may accumulate as the cross-sectional dimension grows.

When asymptotics is of the only concern, one may be interested in the divergence rates of the determinants rather than their exact formulas, and so a Taylor expansion may be applied. A general theorem in Grenander and Szegö (1958) is as follows. Let A_n be a sequence of Toeplitz matrices (not necessarily symmetric) whose (t,s) element is a_{s-t}. One of Szegö's theorems states that

$$\lim_{n\to\infty} |A_n|^{1/n} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(x)\, dx \right\}, \tag{1}$$

where $f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$ is the Fourier form, under the regularity condition that $\log f(x)$ is bounded for $x \in [-\pi, \pi]$. (See Grenander and Szegö, 1958, p. 64.) This theorem is especially useful if the right-hand side of (1) is finite and bigger than zero, in which case $n^{-1}\log|A_n|$ converges to a nonzero quantity.

But in our cases $\log f(x)$ is not bounded. For a stationary autoregressive moving average (ARMA) with a unit root in the moving average (MA) part, the spectral density and thus $f(x)$ are equal to zero at $x = 0$, leading to unbounded $\log f(x)$ in the range $[-\pi, \pi]$, and the result is not applicable.

Another paper that examines asymptotics for ARMA processes with possible MA unit roots is McCabe and Leybourne (1998); it does not derive the determinant explicitly. Additionally the analysis in that paper is conditional on the first observation, whereas we are interested in unconditional MLE.

Because neither Szegö's theorem nor the McCabe and Leybourne (1998) method is applicable, the present note explicitly derives the determinants of covariances for first- and double-differenced AR(1) processes. Not only does this derivation clarify the order of the determinants as the time dimension increases, but it also reveals the functional form of the determinant in terms of the autoregressive (AR) coefficient, and so some "local-to-unity" asymptotics (i.e., asymptotics when the AR coefficient marginally departs from unity) may be analyzed. See Han and Phillips (2006) for an application.

Exact Gaussian MLE based on first-differencing is analyzed by Hsiao et al. (2002) in the dynamic panel context with short time dimensions and large cross-sectional sizes, where the determinant of covariance matrix is explicitly provided. Note that this determinant can also be derived from Galbraith and Galbraith (1974, p. 68) using L'Hôpital's rule. The double-differencing case is much more complicated, on the other hand, and has not been explicitly obtained yet as far as the author knows. Haddad (2004) obtains a closed form representation by some recursion for the inverse and the determinant of ARMA(p,q) processes, but his results require stationarity and invertibility (i.e., AR and MA roots outside the unit circle) and hence do not apply to the present case. Galbraith and Galbraith (1974) provide a generally applicable method, but the algebra is quite involved, partly because invertibility is required for the derivation. Zinde-Walsh (1988) provides a general method for computing determinants of ARMA, but applying this general methodology to the double-differenced AR(1) case may be overly complicated. Applying the usual cofactor expansion did not produce useful results, either.[1]

The method used in the present note expresses the determinants in terms of difference equations. It is tailored for the differenced AR(1) processes, and so the derivation and proof are relatively simple. The covariance matrix for the first-differenced data has the following simple form of the determinant.

THEOREM 1. *Let* $y_t = \rho y_{t-1} + \varepsilon_t$ *with* $\varepsilon_t \sim iid(0,1)$. *Let* $\Omega_n = E\Delta y \Delta y'$ *where* $\Delta y = (\Delta y_1, \ldots, \Delta y_n)'$ *with* $\Delta y_t = y_t - y_{t-1}$. *Then*

$$|\Omega_n| = 1 + n(1-\rho)/(1+\rho).$$

As noted earlier, this result is already known, but we still present it here for completeness and for illustrating our method of derivation and proof. (See the proof that follows.) We can apply the same method to the double-differenced case. The determinant of the covariance matrix in that case is given as follows.

THEOREM 2. *Let* $y_t = \rho y_{t-1} + \varepsilon_t$ *with* $\varepsilon_t \sim iid(0,1)$. *Let, this time,* $\Omega_n = E\Delta^2 y \Delta^2 y'$ *where* $\Delta^2 y = (\Delta^2 y_1, \ldots, \Delta^2 y_n)'$ *with* $\Delta^2 y_t = \Delta y_t - \Delta y_{t-1}$. *Then*

$$|\Omega_n| = (n+1)\left[1 + \frac{8n(1-\rho)(2+\rho) + n^2(1-\rho)^2(7+\rho) + n^3(1-\rho)^3}{12(1+\rho)}\right].$$

2. PROOFS AND DISCUSSION

Proof of Theorem 1. From $\Delta y_t = \varepsilon_t - (1-\rho)\sum_{j=1}^{\infty} \rho^{j-1}\varepsilon_{t-j}$ for all $\rho \in (-1,1]$, we find that $\Omega_n := E\Delta y \Delta y'$ is a symmetric Toeplitz matrix whose (t,s) element is $\omega_{|t-s|}$ such that $\omega_0 = 2/(1+\rho)$ and $\omega_j = -\rho^{j-1}(1-\rho)/(1+\rho)$ for $j \geq 1$. (Also see Karanasos, 1998.) It is easy to see that

$$\Omega_{n+1} = \begin{bmatrix} \omega_0 & \xi_n' \\ \xi_n & \Omega_n \end{bmatrix}, \quad \text{where } \xi_n = (\omega_1, \ldots, \omega_n)'; \tag{2}$$

thus $|\Omega_{n+1}| = d_n |\Omega_n|$, where $d_n = \omega_0 - \xi_n' \Omega_n^{-1} \xi_n$. By the inversion formula for partitioned matrices, we have

$$\Omega_{n+1}^{-1} = \frac{1}{d_n}\begin{bmatrix} 1 & -\xi_n' \Omega_n^{-1} \\ -\Omega_n^{-1}\xi_n & d_n \Omega_n^{-1} + \Omega_n^{-1}\xi_n \xi_n' \Omega_n^{-1} \end{bmatrix}. \tag{3}$$

Now, because $\omega_{j+1} = \rho\omega_j$ for $j \geq 1$, we have $\xi_{n+1} = (\omega_1, \rho\xi_n')'$, implying that

$$\xi_{n+1}'\Omega_{n+1}^{-1}\xi_{n+1} = d_n^{-1}(\omega_1^2 - 2\rho\omega_1\xi_n'\Omega_n^{-1}\xi_n + \rho^2\omega_0\xi_n'\Omega_n^{-1}\xi_n)$$
$$= (\omega_1 - \rho\omega_0)^2/d_n + \rho(2\omega_1 - \rho\omega_0) = A, \tag{4}$$

where $\xi_n'\Omega_n^{-1}\xi_n = \omega_0 - d_n$ is used. Thus

$$d_{n+1} = \omega_0 - \xi_{n+1}'\Omega_{n+1}^{-1}\xi_{n+1} = [(1+\rho^2)\omega_0 - 2\rho\omega_1] - (\omega_1 - \rho\omega_0)^2/d_n$$
$$= \pi_1 - \pi_2/d_n,$$

for $n \geq 1$. Because $d_n = |\Omega_{n+1}|/|\Omega_n|$ for $n \geq 1$, this difference equation leads to $|\Omega_{n+2}| = \pi_1|\Omega_{n+1}| - \pi_2|\Omega_n|$ for $n \geq 1$. Now, the exact formulas of ω_0 and ω_1 imply that $\pi_1 = 2$ and $\pi_2 = 1$, and so the preceding identity implies that $|\Omega_{n+2}| - |\Omega_{n+1}| = |\Omega_{n+1}| - |\Omega_n|$. Obviously $|\Omega_n|$ can be computed from a sequence of equal steps, and therefore it is linear in n. Its closed form is derived from the initial condition that $|\Omega_1| = \omega_0 = 2/(1+\rho)$ and $|\Omega_2| = \omega_0^2 - \omega_1^2 = (3-\rho)/(1+\rho)$, and the final result follows immediately. (For more general treatment of linear difference equations, see, e.g., Hoy, Livernois, McKenna, Rees, and Stengos, 1996.) ■

Now let us consider the double-differenced case. The notations in the preceding proof are used here too but with different meanings, which are explained in relevant places. Again $y_t = \rho y_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim iid(0,1)$.

Proof of Theorem 2. Let $\omega_j = E\Delta^2 y_t \Delta^2 y_{t-j}$. Because

$$\Delta^2 y_t = \rho\Delta^2 y_{t-1} + \Delta^2\varepsilon_t = \varepsilon_t - (2-\rho)\varepsilon_{t-1} + (1-\rho)^2\sum_{j=2}^{\infty}\rho^{j-2}\varepsilon_{t-j},$$

we have $\omega_0 = 2(3-\rho)/(1+\rho)$, $\omega_1 = -(4 - 3\rho + \rho^2)/(1+\rho)$, and $\omega_j = \rho^{j-2}(1-\rho)^3/(1+\rho)$ for $j \geq 2$. (See also Karanasos, 1998.) Let $\Delta^2 y = (\Delta^2 y_1, \ldots, \Delta^2 y_n)$ and $\Omega_n = E\Delta^2 y\Delta^2 y'$. Then Ω_n is the symmetric Toeplitz matrix whose (t,s) element is $\omega_{|t-s|}$, again. The partition (2) and the inverse (3) are still valid with the redefined $\xi_n = (\omega_1, \ldots, \omega_n)'$ and $d_n = \omega_0 - \xi_n'\Omega_n^{-1}\xi_n$. But now we have $\omega_{j+1} = \rho\omega_j$ for $j \geq 2$ (rather than $j \geq 1$), and we do not have the simplicity of the previous case. Instead, by noting that $\omega_2 = \rho\omega_1 + 1$ and $\omega_{j+1} = \rho\omega_j$ for $j \geq 2$,

$$\xi_{n+1} = \begin{bmatrix} \omega_1 \\ \rho\xi_n \end{bmatrix} + \begin{bmatrix} 0 \\ e_n \end{bmatrix}, \tag{5}$$

where e_n is the first column of I_n. The extra $(0, e_n')'$ term slightly complicates the recursion, but we can still proceed as follows.

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Similarly to the first-differencing case, using (4) for the first term in (5) and denoting $a_n = e_n' \Omega_n^{-1} \xi_n$ we get

$$\omega_0 - d_{n+1} = \xi_{n+1} \Omega_{n+1}^{-1} \xi_{n+1}$$

$$= A - \frac{2}{d_n} \omega_1 a_n + 2\rho a_n + \frac{2}{d_n} \rho a_n \xi_n' \Omega_n^{-1} \xi_n + e_n' \Omega_n^{-1} e_n + \frac{1}{d_n} a_n^2$$

$$= A - 2a_n(\omega_1 - \rho\omega_0)/d_n + a_n^2/d_n + 1/d_{n-1},$$

where the terms are simplified using $\xi_n' \Omega_n^{-1} \xi_n = \omega_0 - d_n$ and $e_n' \Omega_n^{-1} e_n = 1/d_{n-1}$. So we have

$$d_{n+1} = [(1+\rho^2)\omega_0 - 2\rho\omega_1] - (\omega_1 - \rho\omega_0 - a_n)^2/d_n - 1/d_{n-1}. \quad \textbf{(6)}$$

Also, by expanding $a_{n+1} = \xi_{n+1}' \Omega_{n+1}^{-1} e_{n+1}$ using the partitioned inverse (3), we get

$$a_{n+1} = \rho + (\omega_1 - \rho\omega_0 - a_n)/d_n. \quad \textbf{(7)}$$

By change of the variable a_n to $c_n := \omega_1 - \rho\omega_0 - a_n$, (6) and (7) are rewritten as

$$d_{n+1} = \tilde{\pi}_0 - c_n^2/d_n - 1/d_{n-1} \quad \text{and} \quad c_{n+1} = \tilde{\pi}_1 - c_n/d_n, \quad \textbf{(8)}$$

where $\tilde{\pi}_0 = (1+\rho^2)\omega_0 - 2\rho\omega_1 = 6$ and $\tilde{\pi}_1 = \omega_1 - \rho\omega_0 - \rho = -4$. Similarly to the first-differencing case, we note that $d_n = |\Omega_{n+1}|/|\Omega_n|$, and by further transforming c_n to $\mu_n := c_n|\Omega_n|$, the two difference equations in (8) are, respectively, written as

$$D_{n+1} = 6D_n - (D_n D_{n-2} + \mu_{n-1}^2)/D_{n-1}, \quad \textbf{(9)}$$

$$\mu_{n+1} = -4D_{n+1} - \mu_n, \quad \textbf{(10)}$$

where $D_n \equiv |\Omega_n|$ for notational brevity.

Solving (9) and (10) is not straightforward.[2] By lagging once the second D_n (the first one in the parentheses) on the right-hand side of (9) and replacing μ_{n-1} of (9) with $-4D_{n-1} - \mu_{n-2}$ using (10), we get the linear expression

$$D_{n+1} = 6D_n - 16D_{n-1} - 6D_{n-2} + D_{n-3} - 8\mu_{n-2} \quad \textbf{(11)}$$

instead of (9). Now (10) and (11) imply that $(1-L)^5\mu_{n+1} = 0$, and so μ_n is a fourth-order polynomial of n. (This is because $(1-L)^4\mu_{n+1}$ is constant, and so $(1-L)^3\mu_{n+1}$ is linear in n, and so on.) Thus D_n is also a fourth-order polynomial of n because of (10). The coefficients for the polynomial can be determined from $D_1, \ldots, D_5$. The detailed algebra is omitted. ■

Extension of the present method to higher order AR processes is conceptually possible. Consider the differenced AR(p) process $\prod_{k=1}^{p}(1 - \phi_k L)\Delta y_t = \Delta\varepsilon_t$. Note that multiple AR unit roots are not allowed because then Δy_t are not covariance stationary. Furthermore, if $\phi_i = 1$ for some i, then it leads to a pure AR$(p - 1)$ process $\prod_{k \neq i}(1 - \phi_k)\Delta y_t = \varepsilon_t$ with $|\phi_k| < 1$ for $k \neq i$, for which the determinant is widely available, including Galbraith and Galbraith (1974). So we assume that $|\phi_k| < 1$ for all k.

The covariance matrix for Δy_t in this case is explicitly calculated by Karanasos (1998), and its determinant can be expressed in terms of nonlinear simultaneous difference equations, which are not presented in this note. Solving them is quite challenging even for $p = 2$, and the double-differencing case seems still more complicated. Yet, derivation would possibly be attempted along this line if other methods (e.g., Galbraith and Galbraith, 1974) are overly complicated.

NOTES

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2. I thank an anonymous referee for suggesting simplifying the proofs, which gave me inspiration that eventually developed into this solution.

REFERENCES

Galbraith, R.F. & J.I. Galbraith (1974) On the inverses of some patterned matrices arising in the theory of stationary time series. *Journal of Applied Probability* 11, 63–71.

Grenander, U. & G. Szegö (1958) *Toeplitz Forms and Their Applications*. University of California Press.

Haddad, J.N. (2004) On the closed form of the covariance matrix and its inverse of the causal ARMA process. *Journal of Time Series Analysis* 25, 443–448.

Han, C. & P.C.B. Phillips (2006) GMM Estimation for Dynamic Panels with Fixed Effects and Strong Instruments at Unity. Cowles Foundation Discussion Paper no. 1599.

Hoy, M., J. Livernois, C. McKenna, R. Rees, & T. Stengos (1996) *Mathematics for Economics*. Addison-Wesley.

Hsiao, C., M.H. Pesaran, & A.K. Tahmiscioglu (2002) Maximum likelihood estimation of fixed effects dynamic panel data models covering short time periods. *Journal of Econometrics* 109, 107–150.

Karanasos, M. (1998) A new method for obtaining the autocovariance of an ARMA model: An exact form solution. *Econometric Theory* 14, 622–640.

McCabe, B.P.M. & S.J. Leybourne (1998) On estimating an ARMA model with an MA unit root. *Econometric Theory* 14, 326–338.

Ploberger, W. & P.C.B. Phillips (2002) Optimal Testing for Unit Roots in Panel Data. Mimeo, University of Rochester.

Zinde-Walsh, V. (1988) Some exact formulae for autoregressive moving average processes. *Econometric Theory* 4, 384–402.