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Roll waves and plugs in two-layer flows

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In this paper I consider a system of equations describing two stratified fluids flowing in closed, slightly inclined ducts. In the framework of the shallow water approximation with turbulent friction acting on the wall and at the interface, I investigate a special class of periodic travelling waves containing stable moving jumps, namely, roll waves and periodic slug flows. The modulation equations of roll waves and slugs are derived and a nonlinear stability criterion is obtained. As for plug flow, only a gas–liquid system is considered. The stability criterion is expressed in terms of an integro-differential relation. For self-similar cross-section, this criterion is simplified to a relation for a function of one variable only.

1 Introduction

Two-layer flows in closed channels are of considerable interest in many industrial processes and equipments. A variety of configurations may occur over a range of flow rates. Roll waves and slugs are among these flow patterns, which may exhibit quasi-periodic spatial structures. In the shallow water approximation these flows can be described by separated flow equations for stratified flow regimes.

This paper presents a nonlinear study of the stability of roll waves and slugs for two-layer flows in inclined channels of arbitrary cross-section. Throughout the paper, the averaged equations for a separated-flow model of instationary stratified flow regimes with turbulent friction on the wall and at the interface are considered. Starting from these equations standard roll waves, i.e. periodic patterns of bores separated by continuous profiles of the interface, and plug flow regimes for zero-mean pressure gradient gas–liquid flow system, which consist of successive cells made of liquid, separated from one another by a liquid layer topped by a gaseous bubble. The viscous Kelvin–Helmholtz (VKH) instability is generally associated with the transition from stratified to slug flow through the generation of nonlinear roll waves [1, 3]. The VKH instability is a necessary condition for roll waves to occur and for their transition into slug flow regime. The generation of roll waves driven by a pressure gradient in wide horizontal closed channels, and in inclined ducts with zero-mean pressure gradient have many common characteristics with the generation of roll waves in inclined open channels [15]. The structure of roll waves and slugs in two-layer flow is rather complicated. The relation between flow conditions and plug occurrence has been modelled on the basis of continuous one-dimensional conservative laws.

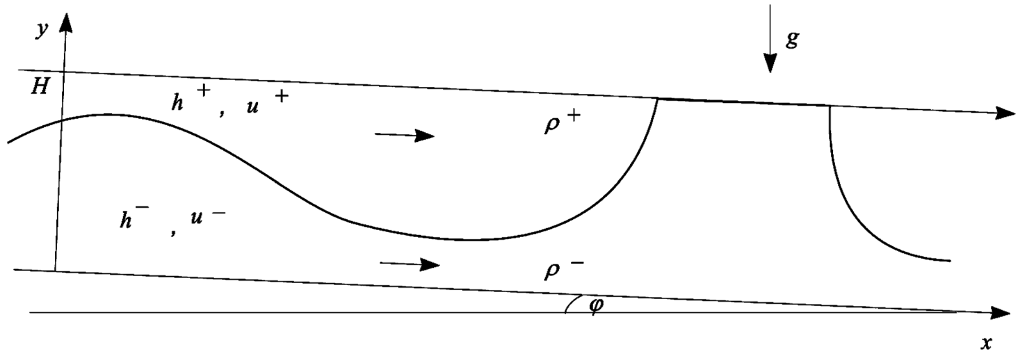

FIGURE 1. Sketch of flow.

The nonlinear stability problem of roll waves for flows in inclined open channel of arbitrary shape is solved by deriving the modulation equations [5]. This approach has been extended to the present investigation of a two-layer system, the stability criterion of roll waves is formulated in terms of hyperbolicity of the modulation equations, depending on two governing parameters. The main difficulty in obtaining the nonlinear stability domain on a roll wave diagram is due to singularities in the hyperbolicity condition of the modulation equation for the waves of infinitesimal and maximal amplitude. As for plug flow, only a gas–liquid system is considered, amounting to investigating, in fact, roll wave stability of the lower layer. For the class of self-similar cross-sections, the modulation equations take a rather simple form and the hyperbolicity criterion is reduced to a condition in one variable.

For gas–liquid flow, two different conservation law equations have been investigated; the plug flow model has been derived, and modulation equations and the stability criteria for roll wave and plug flow regimes have been derived. A simple form of hyperbolicity criterion for the modulation equations lets us show that infinitesimal waves and long waves with and without plugs are unstable and only roll waves of finite length are stable for the class of self-similar channels.

2 Governing equations

I assume that one-dimensional two-layer flows of immiscible fluids in long, inclined ducts of arbitrary cross-section are governed by the following system [1, 2]:

$$
\begin{aligned}
&(\rho^+A^+)_t + (\rho^+A^+u^+)_x = 0\\
&(\rho^-A^-)_t + (\rho^-A^-u^-)_x = 0\\
&\rho^+(u_t^+ + u^+u_x^+) + p_x^+ = -\tau^+/R^+ - \tau_i/R_i^+ + \rho^+ g\sin\varphi\\
&\rho^-(u_t^- + u^-u_x^-) + p_x^- = -\tau^-/R^- + \tau_i/R_i^- + \rho^- g\sin\varphi.
\end{aligned}
\tag{1}
$$

Here t is time, x is distance along the duct, g is acceleration due to gravity, φ is the angle of inclination of the channel, $\rho^\pm$, $A^\pm$, $u^\pm$ are the density, the cross-section area and the stream-wise velocity in the upper and lower layer, respectively (Figure 1). The vertical pressure distribution in the flow is supposed to be hydrostatic, therefore the pressures at the top and the bottom of the duct, $p^{\pm}$, are connected by the expression

$$p^- = p^+ + \rho^+ g \cos \varphi h^+ + \rho^- g \cos \varphi h^-, \tag{2.1}$$

where h^+ and h^- are the depths of the upper and lower layers of fluid. In the right-hand terms of (1), $\tau^{\pm}$ are the shear stresses at the wall applied to the corresponding wet perimeter $L_w^{\pm}$, τ_i is the shear stress at the interface with the perimeter L_i, $R^{\pm}$ and $R_i^{\pm}$ are the hydraulic radii, calculated by the formulae

$$R^{\pm} = A^{\pm}/L_w^{\pm}, \qquad R_i^{\pm} = A^{\pm}/L_i. \tag{2.2}$$

The shear stresses $\tau^{\pm}$, τ_i are assumed to be

$$\begin{aligned} &\tau^{\pm} = c^{\pm}\rho^{\pm}u^{\pm}|u^{\pm}|, \quad c^{\pm} \equiv \text{const}, \\ &\tau_i = c_i\rho^+(u^+ - u^-)|u^+ - u^-|, \quad c_i \equiv \text{const}. \end{aligned} \tag{2.3}$$

Consider the case where the two phases are incompressible ($\rho^{\pm} \equiv$ const) and filling the duct entirely, i.e.

$$A^+ + A^- = A_0 \equiv \text{const}, \quad h^- + h^+ = h_0 \equiv \text{const}. \tag{2.4}$$

By using the following dimensionless variables

$$\begin{aligned} &\widetilde{x} = c^- x/h_0, \qquad \widetilde{t} = c^- t/\sqrt{b/h_0}, \qquad \widetilde{h} = h^-/h_0, \qquad b = (1-\lambda) g \cos \varphi, \\ &\widetilde{A} = A^-/A_0, \qquad u = u^-/\sqrt{b \cdot h_0}, \qquad w = u^+/\sqrt{b \cdot h_0}, \qquad \widetilde{R}^{\pm} = R^{\pm}/h_0, \end{aligned}$$

where

$$\alpha = \tan \varphi, \quad \beta_i = \lambda c_i/c^-, \quad \gamma = \lambda c^+/c^-, \quad \lambda = \rho^+/\rho^-,$$

(2.1)–(2.4) can be simplified. For co-current flows ($u > 0$, $w > 0$) the equations take the following form: ("tilde" over variables is omitted)

$$\begin{aligned} &A_t + (Au)_x = 0, \\ &(1-A)_t + ((1-A)w)_x = 0, \\ &(u - \lambda w)_t + (u^2/2 - \lambda w^2/2 + h)_x = F, \\ &F = \alpha - u^2/R^- + \beta(w-u)^2(1/R_i^+ + 1/R_i^-) + \gamma w^2/R^+, \\ &\beta = \beta_i \operatorname{sgn}(w-u). \end{aligned} \tag{2.5}$$

The linear stability of steady-state flows for (2.5) has been investigated in [1, 3, 4]. To describe the transition from a stratified flow regime to flows with liquid plugs, in which one phase completely occupies the duct cross-section, we must consider flows with waves of finite amplitude at the interface.

There are different mechanisms of the transition from stratified to wavy flow regimes in two-layer flows. Depending on a solution the system (2.5) is of elliptic–hyperbolic type. In the elliptic domain long-wave perturbations of any steady-state flow are growing exponentially and the solution cannot be realized in long channels (the Kelvin–Helmholtz instability). Another mechanism of instability is due to friction terms in (2.5). A steady-state solution of (2.5) from the hyperbolic region is unstable when the velocity of the kinematic wave exceeds the velocity of long waves [17]. In this case the VKH instability is realized. It is well known that VKH instability results in roll wave generation in both open channel flow and in two-layer flows [1, 3, 4]. In the frame of the model (2.5), roll waves are travelling periodic waves containing stable discontinuities or jumps. Roll waves for the one-layer flow in an inclined channel with arbitrary cross-section have been investigated in [6]. It is shown that, depending on the channel form, the multijump configuration of roll waves can be found.

In the present study I consider roll waves of finite amplitude for two cases: (a) two-layer flows in a wide horizontal duct and (b) the plug flow regime in inclined channels.

To describe discontinuous solutions, jump conditions for (2.5) must be chosen. Note that (2.5) are represented in conservation form and the jump conditions are uniquely defined by the full set of conservation laws [16]. The corresponding jump conditions provide the mass and energy conservation in the phases. The main disadvantage of the jump conditions for the two-layer flow derived from (2.5) is that they are inconsistent with the total momentum conservation. Nevertheless, (2.5) and the jump conditions derived from this set of conservation laws are widely used to simulate two-layer flows with discontinuities.

Another set of conservation laws for two-layer flows have been used in [9, 18]. The main assumption of their approach is that the total momentum in two-layer flow and the energy in one of the layers are conserved. It means that the energy dissipation occurs completely in the other layer. Suppose that the dissipating layer is the lower one. The system of conservation laws takes the form

$$\begin{aligned}
&A_t + (Au)_x = 0,\\
&(1-A)_t + ((1-A)w)_x = 0, \qquad\qquad (2.6)\\
&(A(u-\lambda w))_t + (Au^2 + \lambda(1/2 - A)w^2 + P(A))_x = AF.
\end{aligned}$$

Here $P(A) = \int_0^A sh'(s)ds$ is the total pressure due to buoyancy effects. Note that (2.5) and (2.6) are equivalent for smooth solutions, but the jump conditions are different.

The set of conservation laws (2.6) is more appropriate for film flows in a duct. In this case all dissipation occurs in the liquid layer due to wave breaking.

In the present study I consider roll wave of finite amplitude mainly for (2.5). Roll waves governed by (2.6) will be considered only in Section 7 for the plug flow regime, when the parameter λ vanishes.

3 Travelling waves

A theoretical study of finite amplitude roll waves in two-layer flow between two parallel horizontal planes has been performed in [11] for two liquids of small density difference (the Boussinesq approximation). With model (2.5) the theory can be extended to the general case of any density difference between two layers.

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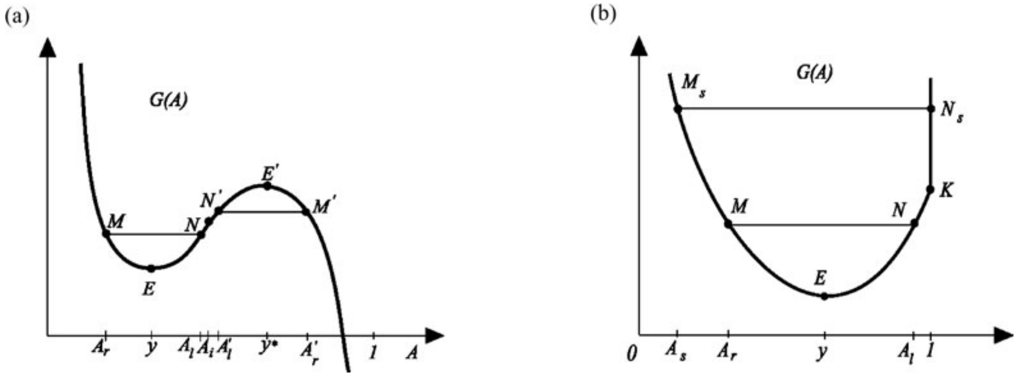

FIGURE 2. The function $G(A)$: (a) $m^+ > 0$; (b) $m^+ = 0$.

Consider first the structure of nonlinear roll waves travelling downstream the channel with a constant velocity $\mathscr{D}$, i.e.

$$0 < u \leqslant \mathscr{D}, \quad 0 < w \leqslant \mathscr{D}.$$

In the frame of reference moving with the wave velocity the flow is steady, therefore a solution of (2.5) depends on a single variable $\xi = x - \mathscr{D}t$. Equations (2.5) take the form

$$A(\mathscr{D} - u) = m, \qquad (1 - A)(\mathscr{D} - w) = m^+, \tag{3.1}$$

$$\frac{d}{d\xi}\left(\frac{1}{2}(u - \mathscr{D})^2 - \frac{\lambda}{2}(w - \mathscr{D})^2 + h\right) = F. \tag{3.2}$$

Note that for a given channel $h = h(A)$ and the system (3.1)–(3.2) can be reduced to one equation,

$$\frac{dA}{d\xi} = \frac{F(A)}{\Delta(A)}, \tag{3.3}$$

with

$$\Delta(A) = G'(A) = h'(A) - (m^2/A^3 + \lambda m^{+2}/(1 - A)^3), \tag{3.4}$$

$$G(A) = \frac{m^2}{2A^2} - \frac{\lambda m^{+2}}{2(1 - A)^2} + h(A). \tag{3.5}$$

The function $F = F(A)$ is found from (2.5) by eliminating u and w in view of (3.1).

The dependence $G = G(A)$ is shown for $m > 0$ in Figure 2. The asymptotic behaviour of $G(A)$ at $A = 1$ is different according to $m^+ > 0$ (Figure 2(a)) or $m^+ = 0$ (Figure 2(b)). A periodic travelling wave (roll wave) can be constructed as the transition MNEM (regular roll wave (RW)), Figures 2(a), 2(b) and 3(a) and as the transition M′N′E′M′ (inverted RW, Figures 2(a) and 3(b)). A regular RW consists of the smooth part MEN described by a solution of (3.3) and contains the hydraulic jump MN, since for discontinuous solutions of (2.5) the relations at the jumps moving with the velocity $\mathscr{D}$ can be expressed in the form

$$G(A_l) = G(A_r),$$

where A_l, A_r are the limits on the left and right of the jump. The flow is supercritical

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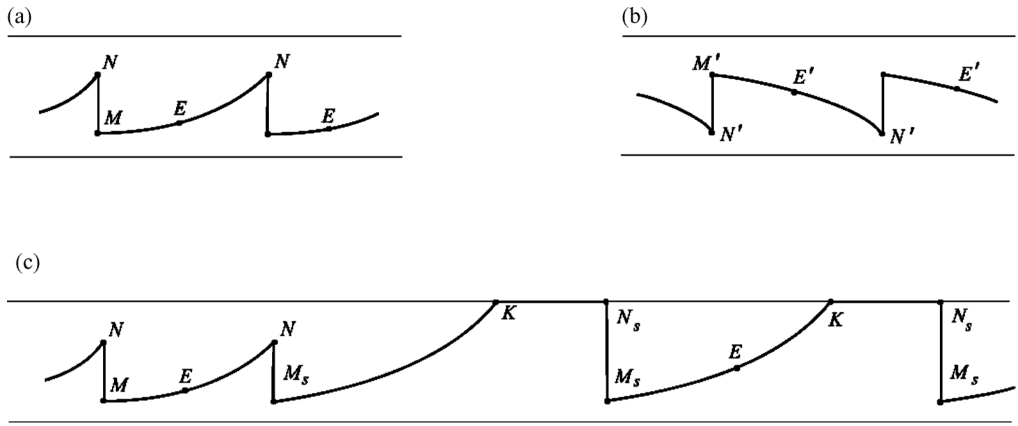

FIGURE 3. Profile of roll waves: (a) regular RW; (b) inverted RW; (c) $m^+ = 0$: RW with plugs.

($\Delta < 0$) at ME and subcritical ($\Delta > 0$) at EN. Therefore, it is necessary for RW existence that there is a critical cross-section y where

$$F(y) = 0, \qquad \Delta(y) = 0. \tag{3.6}$$

Moreover, to construct a regular roll wave MNEM it is necessary and sufficient that the following conditions are satisfied in the vicinity of a critical point y:

$$\begin{aligned} F(A) < 0 \quad &\text{for } A_r < A < y, \\ F(A) > 0 \quad &\text{for } y < A < A_l. \end{aligned} \tag{3.7}$$

In view of (3.7) the profile of wave $A(\xi)$ is the monotone increasing function for $A_r < A < A_l$ and the jump MN is stable, since the flow before the jump is supercritical ($\Delta < 0$) and it is subcritical behind ($\Delta > 0$). The function $G(A)$ has a local minimum at y. For $m^+ > 0$ there exists $y_* > y$, with $\Delta(y_*) = 0$, in which a local maximum of $G(A)$ is realized (Figure 2(a)), Therefore, regular RW are possible only for $y < A_l < y_*$.

In the vicinity of y_* it is possible to construct inverted roll waves M′N′E′M′ (Figure 2(a)). Now the jump M′N′ is stable with $A'_l < y_* < A'_r$ and the necessary and sufficient conditions for inverted roll wave existence take the form

$$\begin{aligned} F(A) < 0 \quad &\text{for } A'_l < A < y_*, \\ F(A) > 0 \quad &\text{for } y_* < A < A'_r. \end{aligned} \tag{3.8}$$

The profile of an inverted roll wave is shown in Figure 3(b). It is worthy of noting that for existence of regular and nonregular roll waves the necessary condition reads $F'(y_c) > 0$ with $y_c = y$ or $y_c = y_*$.

I have considered the simplest configurations of roll waves. The dependence $G = G(A)$ can have several local extrema even for one-layer flow ($\lambda = 0$) depending on the channel form. In this case the multijump structure of roll waves can be constructed, as has been shown in [8]. I pay further attention to two cases, which lead to the RW configuration shown in Figures 2 and 3, namely,

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(a) two-layer flow in a horizontal duct;
(b) slug flow regime in a slightly inclined duct.

4 Two-layer flow in a horizontal duct

Consider the flow in a wide horizontal duct ($\alpha = 0,\ A = h,\ R^+ = R_i^+ = 1 - h,\ R^- = R_i^- = h$). Suppose that the total flow rate is constant, i.e.

$$hu + (1 - h)w = u_m \equiv \text{const.} \tag{4.1}$$

Let us show that for given parameters β, γ of the model (2.5) and for given flow rate u_m, all roll waves may by described by two parameters, say, y_c and h_r. Here y_c is the critical depth with $\Delta(y_c) = 0$ and h_r is the minimal depth of the lower layer in a wave.

Let $0 < y_c < 1$. I am going to show that u_c, w_c, m, m^+, $\mathscr{D}$ depend only on y_c and that these parameters can be found in unique way for downward roll waves ($0 < u < \mathscr{D}, 0 < w < \mathscr{D}$). Here u_c and w_c are the critical velocities, i.e.

$$\begin{aligned} y_c u_c + (1 - y_c) w_c &= u_m, \\ F(y_c) = -\frac{u_c^2}{y_c} + \frac{\beta(w_c - u_c)^2}{y_c(1 - y_c)} + \frac{\gamma w_c^2}{1 - y_c} &= 0, \\ \Delta(y_c) = 1 - \frac{(\mathscr{D} - u_c)^2}{y_c} - \frac{\lambda(\mathscr{D} - w_c)^2}{1 - y_c} &= 0. \end{aligned} \tag{4.2}$$

The second equation in (4.2) can be reduced to the following equation for $\zeta = w_c/u_c$ and $\beta = \beta_i\ \mathrm{sgn}(w_c - u_c)$:

$$(\gamma y_c + \beta)\zeta^2 - 2\beta\zeta + \beta + y_c - 1 = 0 \tag{4.3}$$

The only admissible root of (4.3) is given by

$$\zeta = \frac{\beta + \sqrt{\beta^2 - (\gamma y_c + \beta)(\beta + y_c - 1)}}{\gamma y_c + \beta}. \tag{4.4}$$

For $0 < y_c < 1/(\gamma + 1)$ it follows from (4.4) that $\zeta > 1$ ($\beta > 0$). For the case $1/(\gamma + 1) < y_c < 1$ and $\beta_i > \gamma$ we have $\zeta < 1 (\beta < 0)$.

From the first equation in (4.2) the critical velocities u_c, w_c can be found in a unique way:

$$u_c = u_m/(y_c + \zeta(1 - y_c)), \quad w_c = \zeta u_c. \tag{4.5}$$

The wave velocity $\mathscr{D}$ may be found from the third equation of (4.2) as follows. The function

$$f(\mathscr{D}) = 1 - \frac{(\mathscr{D} - u_c)^2}{y_c} - \frac{\lambda(\mathscr{D} - w_c)^2}{1 - y_c}$$

has its maximum at

$$\mathscr{D} = \mathscr{D}_{\max} = \chi u_c + (1 - \chi)w_c, \qquad 0 < \chi = \frac{1}{y_c}\left(\frac{1}{y_c} + \frac{\lambda}{1 - y_c}\right)^{-1} < 1.$$

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Therefore, $(\mathscr{D}_{\max} - u_c)(\mathscr{D}_{\max} - w_c) \leqslant 0$. We are looking for a solution $f(\mathscr{D}) = 0$ with $\mathscr{D} > u_c$, $\mathscr{D} > w_c$, so the dependence $\mathscr{D} = \mathscr{D}(y_c)$ is given by the formula

$$\mathscr{D} = \frac{b + \sqrt{b^2 - ac}}{a}, \tag{4.6}$$

with

$$a = (1 - y_c - \lambda y_c), \qquad b = (1 - y_c)u_c + \lambda y_c w_c,$$
$$c = (1 - y_c)u_c^2 + \lambda y_c w_c^2 - y_c(1 - y_c).$$

The relation (4.6) presents a real value of $\mathscr{D}$ if the following restrictions on the total flow rate u_m are satisfied:

$$\begin{aligned} &f(w_c) \geqslant 0 \quad \text{or} \quad u_m^2 < y_c(y_c + \zeta(1 - y_c))^2/(1 - \zeta)^2 \quad \text{for } w_c > u_c, \\ &f(u_c) \geqslant 0 \quad \text{or} \quad u_m^2 < \lambda^{-1}(1 - y_c)(y_c + \zeta(1 - y_c))^2/(1 - \zeta)^2 \quad \text{for } u_c > w_c. \end{aligned} \tag{4.7}$$

Finally, under the conditions (4.7), we have found u_c, w_c, $\mathscr{D}$, $m = (\mathscr{D} - u_c)y_c$, $m^+ = (\mathscr{D} - w_c)(1 - y_c)$ as functions of the variable y_c.

The next step is to investigate the dependence of $G(h)$ on h for two-layer flows in a horizontal channel. Note that

$$\Delta'(h) = \frac{3m^2}{h^4} - \frac{3\lambda m^{+2}}{(1 - h)^4} \tag{4.8}$$

and there is only one point of inflection $h = h_i$ of the function $G(h)$, i.e. $\Delta'(h_i) = 0$, $0 < h_i < 1$. It follows from (4.8) that $\Delta'(h_i) > 0$ for $0 < h < h_i$ and $\Delta'(h_i) < 0$ for $h_i < h < 1$. Therefore, the function $G(h)$ has exactly two local extrema, as shown in Figure 2(a), if and only if $\Delta(h_i) > 0$, i.e.

$$(m^2)^{1/4} + (\lambda m^{+2})^{1/4} < 1.$$

Let $\Delta(h_i) > 0$ and there exist two extremum points of $G(h)$:

$$\Delta(y) = 0, \qquad \Delta(y_*) = 0, \qquad 0 < y < h_i < y_* < 1.$$

A regular roll wave MNEM exists if $y_c = y$ and (3.7) is satisfied (Figure 2(a)). For given minimal depth h_r in a wave the conjugate depth h_l can be calculated from

$$G(h_r) = G(h_l) \tag{4.9}$$

with

$$G(h) = h + \frac{m^2}{2h^2} - \frac{\lambda m^{+2}}{2(1 - h)^2}$$

and the wave profile can be found as a solution of (3.3) for $h_r < h < h_l$. Therefore, for a fixed value u_m satisfying (4.7) the diagram of admissible roll waves can be constructed on the (y, h_r) plane. To find all points (y_c, h_r) corresponding to admissible roll waves, we start from the diagonal (y_c, y_c) of the square $[0, 1] \times [0, 1]$. A point (y_c, y_c) describes a roll wave of infinitesimal amplitude if

$$F'(y_c) > 0. \tag{4.10}$$

Note that (4.10) is the exact linear instability condition of a steady-state flow [3, 4, 8].

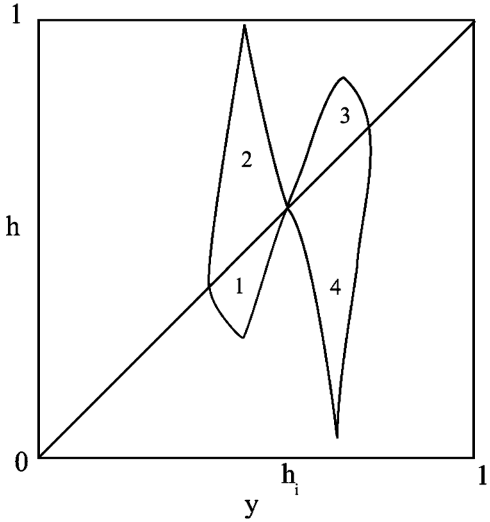

FIGURE 4. RW diagram for two-layer flows in a horizontal channel. 1, 2, regular RW; 3, 4, inverted RW.

Then, for every value of y_c satisfying (4.10), we choose a value of the second governing parameter h_r. Note that $h_r < y < h_l$ for $0 < y_c < h_i$ and $h_l < y_c < h_r$ for $h_i < y_c < 1$. Here h_l is the conjugate depth calculated from (4.9) and the point of inflection h_i is expressed by virtue of (4.8) as follows:

$$h_i = \frac{1}{1 + (\lambda m^{+2}/m^2)^{1/4}}.$$

It is clear that the set of admissible values of h_r and h_l satisfying (3.7) is an interval (h^-, h^+). At one of the boundaries of the interval $h^\pm$ the function $\varDelta(h)$ or $F(h)$ vanishes and we have the roll wave of limiting amplitude. All possible admissible parameters (y, h_r) (domains 1, 3) and (y, h_l) (domains 2, 4) are shown in Figure 4 for $u_m = 3$, $\lambda = \beta = \gamma = 0.7$. For $0 < y < h_i$ (domains 1, 2) the diagram describes the regular RW and for $h_i < y < 1$ (domains 3, 4) it gives the admissible parameters for the inverted RW.

5 Modulation equations

Consider the stability of roll waves of finite amplitude. Let the flow in an inclined duct of arbitrary cross-section be governed by (2.5) and y_c and z be the critical and the minimal section occupied by the lower fluid in a roll wave, respectively, and $v = v(z, y_c)$ be the conjugate section ($y_c = y$, $z = A_r$, $v = A_l$ for a regular roll wave and $y_c = y_*, z = A_l', v = A_r'$ for an inverted roll wave; Figure 2(a)). The problem of nonlinear stability of periodic wave trains with slowly varying parameters (z, y_c) can be solved by analysis of the modulation equations for such trains [17, 5, 12]. After averaging (2.5) over a fixed length scale, which is large enough compared to the length of roll waves, we have the following modulation

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equations:

$$\begin{aligned}&\overline{A}_t + (\overline{Au})_x = 0,\\&(\overline{u-\lambda w})_t + \left(\overline{\frac{1}{2}u^2 + h - \frac{\lambda}{2}w^2}\right)_x = 0.\end{aligned} \tag{5.1}$$

Here

$$\begin{aligned}
\overline{A} &= \frac{1}{L}\int_0^L A(\xi)d\xi = \frac{1}{L}\int_z^v sa(s,y_c)ds,\\
\overline{Au} &= \frac{1}{L}\int_0^L A(\xi)u(\xi)d\xi = \frac{1}{L}\int_z^v (s\mathscr{D} - m)a(s,y_c)ds = \mathscr{D}\overline{A} - m,\\
\overline{u-\lambda w} &= \frac{1}{L}\int_0^L (u(\xi) - \lambda w(\xi))d\xi = (1-\lambda)\mathscr{D} - \Phi(z,y_c),\\
\overline{\frac{1}{2}u^2 + h - \frac{\lambda}{2}w^2} &= \frac{1}{L}\int_0^L \left(\frac{1}{2}u^2(\xi) + h(\xi) - \frac{\lambda}{2}w^2(\xi)\right)d\xi\\
&= \frac{1}{2}(1-\lambda)\mathscr{D}^2 - \mathscr{D}\Phi(z,y_c) + \Psi(z,y_c),\\
L &= \int_0^L d\xi = \int_z^v a(s,y_c)ds\\
a(s,y_c) &= d\xi/dA = \Delta(s,y_c)/F(s,y_c),\\
\Phi(z,y_c) &= \frac{1}{L}\int_z^v \left(\frac{m}{s} - \frac{\lambda m^+}{1-s}\right) a(s,y_c)ds,\\
\Psi(z,y_c) &= \frac{1}{L}\int_z^v \left(\frac{m^2}{2s^2} - \frac{\lambda m^{+2}}{2(1-s)^2} + h(s)\right)a(s,y_c)ds.
\end{aligned} \tag{5.2}$$

Note that all averaged quantities in (5.2) are the functions of the two governing parameters z and y_c, since $m = m(y_c), m^+ = m^+(y_c), v = v(z, y_c)$. The right-hand side of (2.5) vanishes after averaging due to periodicity of a wave train. By virtue of (5.2) the modulation equations represent the following nonlinear systems of partial differential equations for two unknown variables z and y_c:

$$\begin{aligned}&\overline{A}_t + (\mathscr{D}\overline{A} - m)_x = 0,\\&((1-\lambda)\mathscr{D} - \Phi)_t + \left(\frac{1-\lambda}{2}\mathscr{D}^2 - \mathscr{D}\Phi + \Psi\right)_x = 0.\end{aligned} \tag{5.3}$$

In the general case, (5.3) have the mixed type. They can be hyperbolic or elliptic depending on the solution. We say that a periodic wave train is stable if the corresponding governing parameters y_c and z belong to the hyperbolicity domain of (5.3). Therefore, the problem of the stability of finite amplitude roll waves is reduced to the analysis of the hyperbolicity of the modulation equations.

The characteristics of (5.3) can be expressed as follows:

$$\frac{dx}{dt} = \frac{1}{2\theta}\left(2\theta\mathscr{D} + W \pm \sqrt{W^2 + 4\delta\theta\Psi_z/\overline{A}_z}\right),$$

$$W = \delta\Phi_z/\overline{A}_z - \Phi\frac{d\mathscr{D}}{dy_c} + \Psi_{y_c} - \Psi_z\overline{A}_{y_c}/\overline{A}_z,$$

$$\theta = (1-\lambda)\frac{d\mathcal{D}}{dy_c} - \Phi_{y_c} + \Phi_z \overline{A}_{y_c}/\overline{A}_z,$$

$$\delta = \overline{A}\frac{d\mathcal{D}}{dy_c} - \frac{dm}{dy_c}.$$

The hyperbolicity condition for (5.3) reads

$$W^2 + 4\delta\theta\Psi_z/\overline{A}_z > 0. \tag{5.4}$$

To apply the criterion (5.4) for the problem of roll wave stability, it is necessary to apply the formulae (5.2) taking into account the singularities for $L \to 0$ and $L \to \infty$.

6 Plug flows in an inclined duct

Let us consider two-layer flows in an inclined duct of arbitrary cross-section. The stratified flow regime transforms into a plug flow when one of the phases, say, the lower liquid, occupies the total cross-section of the duct due to the intensive wave motion at the interface (Figure 1). In this case, if the entrainment between the fluids is neglected, we have for a travelling wave moving with the velocity $\mathcal{D}$

$$m^+ = (1-A)(\mathcal{D} - w) = 0, \tag{6.1}$$

and $\mathcal{D} \equiv w$ for $A < 1$.

The function $G(A)$ takes the form

$$G(A) = \frac{m^2}{2A^2} + h(A), \quad 0 < A < 1.$$

In ducts of a relative simple form, such as the round tubes or the parallel plates, the behaviour of $G(A)$ is shown in Figure 2(b). It has one absolute minimum at $A = y$.

The flow in the liquid plug is governed by the equations ($R = R^-$)

$$\begin{gathered} A \equiv 1, \quad u \equiv u_* = \mathcal{D} - m, \\ \frac{dp^+}{d\xi} = \alpha - \frac{u_*^2}{R(1)}. \end{gathered} \tag{6.2}$$

We have $p^+ \equiv \text{const}$ outside of the liquid plug. Therefore, the length of the liquid plug L_s is proportional to the pressure drop Δp^+ in the plug, i.e.

$$L_s = \frac{\Delta p^+}{(\alpha - u_*^2/R(1))}, \quad \Delta p^+ = G(A_s) - G(1),$$

where A_s is the wet cross-section before the plug (Figures 2(b), 3(c)). Now we can construct roll waves with and without plugs for $m^+ = 0$ analogously to the case $m^+ \neq 0$. For a given parameters y_c and $A_r, 0 < A_r < y_c = y < 1$ satisfying (3.7), the transition MNEM in Figures 2(a) and 2(b) describes regular roll waves, if the conjugate state $A_l < 1$. If $A_l = 1$ and $\Delta p^+ > 0$, we have roll waves with plug, described by the diagram $\mathrm{M}_s\mathrm{N}_s\mathrm{KEM}_s$ in Figure 2(b). The possible profiles of the wave are shown in Figure 3(c).

If $\lambda \ll 1$ (gas–liquid flow system [14]), and the upper fluid moves between the successive plugs with the wave velocity $\mathscr{D}$, we can neglect the gas friction at the wall and at the interface. This results in the following relations for periodic travelling waves ($m^+ \equiv 0,\ \beta = 0,\ \gamma = 0$):

$$
\begin{aligned}
G(A) &= \frac{m^2}{2A^2} + h(A) + p^+, \quad p^+ = 0 \quad \text{for} \quad 0 < A < 1,\\
F(A) &= \begin{cases} \alpha - u^2(A)/R(A), & 0 < A < 1\\ \alpha - u_*^2/R(1), & A = 1. \end{cases}
\end{aligned}
\tag{6.3}
$$

Contrary to the two-layer flows in a horizontal channel, which are driven by the pressure gradient, the mean pressure gradient in the flow under consideration is assumed to be zero. For such kind of flows only the mean flow rate of the liquid can be given. The mean flow rate of the gas caused by the wavy motion of the liquid layer can be found after calculation of the roll wave profile by (3.3) and (6.3). Therefore, we will investigate, in fact, the roll wave stability of the lower layer.

Let us consider the case of the channel form:

$$
h(A) = \frac{1}{\varkappa}A^{\varkappa}, \qquad R(A) = A^{\varkappa}. \tag{6.4}
$$

The different cases of self-similar channels may be described by (6.4) after proper scaling of the dependent and independent variables. In particular, it corresponds to flows over an incline ($\varkappa = 1$) and in the channel of triangular shape ($\varkappa = 2$). It has been noted in [12] that for one-layer flows in the self-similar channels satisfying (6.4), the stability of periodic travelling waves depends only on the combination of governing parameters $\zeta = z/y$. Here we study (2.5), which lead to different set of conservation laws compared to the system considered in [12]. Nevertheless, for roll waves of moderate amplitude, which do not touch the upper lid of the duct ($A < 1$), the stability criterion depends on the variable ζ only, exactly like that in open channel flows. To see that, let us express the averaged quantities in (5.2) using the self-similar variable $\zeta = z/y$:

$$
\begin{aligned}
&\overline{A} = y\overline{\zeta}, \quad \overline{\zeta} = \frac{1}{l}\int_{\zeta}^{\omega} s\widetilde{a}(s)\,ds,\\
&\overline{Au} = \mathscr{D}\overline{A} - m = y^{\frac{\varkappa}{2}+1}(\mathscr{D}_1\overline{\zeta} - 1), \quad \mathscr{D}_1 = 1 + \sqrt{\alpha},\\
&\overline{u - \lambda w} = y^{\frac{\varkappa}{2}}((1-\lambda))\mathscr{D}_1 - \varphi),\\
&\overline{\frac{1}{2}u^2 + h - \frac{\lambda}{2}w^2} = y^{\varkappa}\left(\tfrac{1-\lambda}{2}\mathscr{D}_1^2 - \mathscr{D}_1\varphi + \psi\right),\\
&\varphi = \frac{1}{l}\int_{\zeta}^{\omega}\frac{\widetilde{a}(s)}{s}\,ds, \quad \psi = \frac{1}{l}\int_{\zeta}^{\omega}\left(\frac{1}{2s^2} + \frac{s^{\varkappa}}{\varkappa}\right)\widetilde{a}(s)\,ds,\\
&L = y^{\varkappa}l, \quad l = \int_{\zeta}^{\omega}\widetilde{a}(s)\,ds,\\
&\widetilde{a}(s) = (s^{\varkappa} - s^{-2})/(\alpha s - (\mathscr{D}_1 - 1/s)^2 s^{1-\varkappa}).
\end{aligned}
\tag{6.5}
$$

The jump condition for the conjugate depths z and v ($z < y < v$) with the variables

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$\zeta = z/y$ and $\omega = v/y$ takes the form

$$\frac{1}{2\zeta^2} + \frac{\zeta^\varkappa}{\varkappa} = \frac{1}{2\omega^2} + \frac{\omega^\varkappa}{\varkappa}. \tag{6.6}$$

For open channel flows ($v < 1$) relation (6.6) results in $\omega = \omega(\zeta)$ ($\zeta < 1 < \omega$) and the functions $\overline{\zeta}$, l, φ, ψ in (6.5) depend only on the variable ζ. The quantities in (5.4) can be rewritten as follows:

$$\begin{aligned}
&\overline{A}_z = \overline{\zeta}', \quad \Psi_z = y^{\varkappa-1}\psi', \\
&\delta = y^{\frac{\varkappa}{2}}\widetilde{\delta}, \quad \widetilde{\delta} = \widetilde{\delta}(\zeta) = \frac{1}{2}(\varkappa\mathscr{D}_1\overline{\zeta} - \varkappa - 2), \\
&\theta = y^{\frac{\varkappa}{2}-1}\widetilde{\theta}, \quad \widetilde{\theta} = \widetilde{\theta}(\zeta) = \frac{\varkappa}{2}((1-\lambda)\mathscr{D}_1 - \varphi) + \overline{\zeta}\varphi'/\overline{\zeta}'), \\
&W = y^{\varkappa-1}\widetilde{W}, \quad \widetilde{W} = \widetilde{W}(\zeta) = \frac{1}{2}(\varkappa\mathscr{D}_1\overline{\zeta} - \varkappa - 2)\varphi'/\overline{\zeta}' - \\
&\qquad\qquad - \frac{1}{2}\varkappa\varphi\mathscr{D}_1 + \varkappa\psi - \overline{\zeta}\psi'/\overline{\zeta}'.
\end{aligned}$$

Here $'$ denotes the derivative with respect to ζ. Therefore, the hyperbolicity condition for the modulation equations (5.1) is transformed to the relation for the function of the variable ζ:

$$\widetilde{W}^2 + 4\widetilde{\delta}\widetilde{\theta}\psi'/\overline{\zeta}' > 0.$$

The necessary condition of roll wave existence,

$$F'(y) > 0,$$

gives the restriction on the duct inclination angle

$$\alpha = \tan\ \varphi > 4/\varkappa^2. \tag{6.7}$$

For the flows over an incline ($\varkappa = 1$) condition (6.7) coincides with the well-known criterion $\alpha > 4$ of instability of a steady-state flow [13]. For ducts satisfying (6.7), roll waves exist for the governing parameters y and z belonging to the sector

$$\zeta_m < \zeta = z/y < 1.$$

The value ζ_m is the maximal root of the equation

$$\alpha\zeta - (\mathscr{D}_1 - 1/\zeta)^2\zeta^{1-\varkappa} = 0,$$

which satisfies the condition $\zeta_m < 1$.

The expressions for the averaged quantities in the modulation equations (5.3) are to be modified for the plug flow regime ($L_s > 0$). For the self-similar channels (6.4) they can be

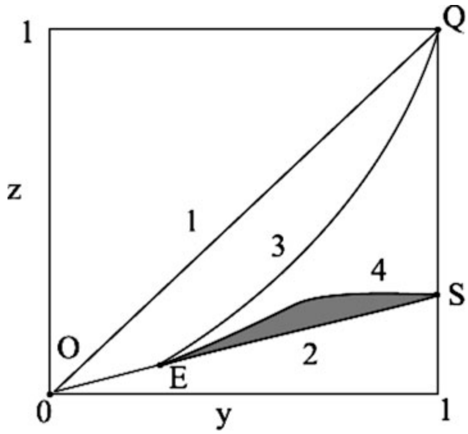

FIGURE 5. RW diagram for plug flows in an inclined channel. EQS: RW with plugs; RW are unstable in the shaded domain.

given as the functions of the variables y and z as follows:

$$\begin{aligned}
&L_i = \int_z^1 \widetilde{a}(s/y)\,ds, \qquad L_s = \frac{G(z)-G(1)}{\alpha - u_*^2},\\
&\overline{A} = \frac{1}{L}\left(\int_z^1 s\widetilde{a}(s/y)\,ds + L_s\right),\\
&\Phi = \frac{1}{L}\left(\int_z^1 \frac{m}{s}\widetilde{a}(s/y)\,ds + mL_s\right),\\
&\Psi = \frac{1}{L}\left(\int_z^1 \left(\frac{m^2}{2s^2} + \frac{s^\varkappa}{\varkappa}\right)\widetilde{a}(s/y)\,ds + \left(\frac{m^2}{2} + \frac{1}{\varkappa}\right)L_s + \frac{1}{2}(\alpha - u_*^2)L_s^2\right),\\
&\mathscr{D} = \mathscr{D}_1 y^{\varkappa/2}, \quad m = y^{\varkappa/2+1}, \quad L = L_i + L_s.
\end{aligned} \tag{6.8}$$

For the plug flow regime the hyperbolicity domain of (5.3), (6.8) can be found by (5.4). The RW diagram for plug flows between two parallel inclined plates ($\varkappa = 1$) is shown in Figure 5 for the Froude number $Fr = \alpha^{1/2} = 5$. All admissible parameters (y, z) of the roll waves with plug ($L_s > 0$) lie in the domain EQS. RW are unstable in the shaded domain.

7 Film flows with plugs

Now consider two-layer flows in an inclined duct described by (2.6). Equations (2.6) are more preferable for the stratified gas–liquid flow ($\lambda \ll 1$) with a low flow rate of the gas phase. It occurs for the plug flow regime in an inclined channel, when the gas is just trapped by the liquid and the mean pressure gradient in the flow equals zero. In this case the gas friction on the wall and at the interface can be neglected ($\beta = \gamma = 0$) and the function F in (2.6) with $R^- = R$ is expressed as follows ($R = R^-$):

$$F = \alpha - u^2/R(A). \tag{7.1}$$

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The stability of roll waves for (2.6), (7.1) can be investigated analogously to the case of the two-layer flow considered above. The smooth part of a travelling wave is described by (3.3). But the jump conditions for discontinuous solutions of (2.6) are expressed in the form

$$M(A_l) = M(A_r) \tag{7.2}$$

with the function

$$M(A) = \frac{m^2}{A} + \lambda\left(\frac{1}{2} - A\right)\frac{m^{+2}}{(1-A)^2} + P(A) \quad (0 < A < 1).$$

Since for the plug flow regime $w \equiv \mathscr{D}$ and, consequently, $m^+ \equiv 0$, the function $M(A) = m^2/A + P(A)$ replaces the function $G(A)$ in previous considerations. For the channels with the convex pressure term $P(A)$, the function $M(A)$ is convex too and it has the only critical cross-section y, $0 < y < 1$ with $m = m(y)$. The necessary and sufficient conditions for the roll wave existence coincide with (3.7) with the conjugate sections A_l, A_r satisfying (7.2). The stability of roll waves in the open channel flows for the regular case ($P'' > 0$) has been investigated in [12].

The flow with liquid plugs in a closed duct can be described in the way analogous to that considered above. When in a travelling wave the liquid occupies the total cross-section of the duct ($A = 1$), the velocity in the liquid plug is constant ($u \equiv u_*$) and the pressure distribution in a plug is linear, i.e.

$$\frac{dM}{d\xi} = \frac{dp^+}{d\xi} = \alpha - \frac{u_*^2}{R(1)}.$$

Therefore, the length of a plug L_s can be expressed as follows:

$$L_s = \frac{M(A_l) - M(1)}{\alpha - \dfrac{u_*^2}{R(1)}}.$$

Comparing the roll wave structure with and without liquid plugs one can see that a roll wave is defined completely by two parameters, and after averaging over the fixed-length scale, which is large enough compared with the length of roll wave, we have the following modulation equations (cf. (5.1)):

$$\begin{aligned} &\overline{A}_t + (\overline{Au})_x = 0 \\ &(\overline{A(u-\lambda w)})_t + \left(\overline{Au^2 + \lambda\left(\frac{1}{2} - A\right)w^2 + P}\right)_x = 0. \end{aligned} \tag{7.3}$$

To express the averaged quantities $\overline{A}$, $\overline{Au}$ as functions of y and z ($y = y_c$, $z = A_r$) we can

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use (5.2). The averaged momentum equation now takes on the form

$$
\begin{aligned}
&(\overline{A(u-\lambda w)}) = (1-\lambda)\mathscr{D}\overline{A} - m,\\
&\overline{Au^2 + \lambda\left(\frac{1}{2} - A\right)w^2 + P} = \mathscr{D}^2\overline{A} - 2\mathscr{D}m + \overline{M} + \lambda\left(\frac{1}{2} - \overline{A}\right)\mathscr{D},\\
&\overline{M} = \frac{1}{L}\int_0^L M(A(\xi))\,d\xi = \frac{1}{L}\int_z^v \left(\frac{m^2}{s} + P(s)\right)a(s,y)\,ds.
\end{aligned}
\tag{7.4}
$$

The hyperbolicity condition for (40) is

$$
\begin{aligned}
&\left(\overline{M}_y - \frac{\overline{M}_z\overline{A}_y}{\overline{A}_z} - 2m\frac{d\mathscr{D}}{dy} + \lambda\mathscr{D}\left(\frac{d\mathscr{D}}{dy} - \frac{dm}{dy}\right)\right)^2 + 4\delta\delta_\lambda\frac{\overline{M}_z}{\overline{A}_z} > 0,\\
&\delta_\lambda = (1-\lambda)\overline{A}\,\frac{d\mathscr{D}}{dy},\quad \delta = \overline{A}\,\frac{d\mathscr{D}}{dy} - \frac{dm}{dy}.
\end{aligned}
\tag{7.5}
$$

For the self-similar channels considered above,

$$
P = \frac{1}{\varkappa}A^{\varkappa+1},\quad R(A) = A^{\varkappa}. \tag{7.6}
$$

For $\lambda=0$, (2.6) and (7.6) describe the open channel flows over an incline ($\varkappa=1$), in the channel of triangular shape ($\varkappa=1/2$) and the film flows in tubes ($\varkappa=2/3$). The specific feature of (7.3), (7.6) is that the hyperbolicity condition (7.5) can be expressed by the function of one variable, say, $\zeta=z/y$ as it has been demonstrated in (6.5). It has been shown in [12] that the modulation equations (7.3) with $\lambda=0$ are hyperbolic for $\zeta_m \leqslant \zeta^- < \zeta < \zeta^+ < 1$. Note that the hyperbolicity interval (ζ^-,ζ^+) doesn't contain 1. It means that the roll waves of infinitesimal amplitude are always unstable and that for free surface flows the hyperbolicity domain of (7.3) is a sector on the (y,z) plane with boundaries $z=\zeta^{\pm}y$. It's worthy of noting that the values $\zeta^{\pm}$ are rather close to ζ_m. Thus, for $\varkappa=1$, $\alpha=9$ we have $\zeta^-=0.593$, $\zeta^+=0.604$, $\zeta_m=0.589$ and only roll waves, which are long enough, are stable for the shallow water equations (7.1).

Flows with plugs in an inclined closed duct can be considered in the similar way to the case of two-layer flows considered above. For the self-similar duct (7.6) with $A_0=1$, the formulae (6.5), (6.8) must be modified as follows:

$$
\begin{aligned}
&L_i = y^{\varkappa}\int_\zeta^{1/y}\widetilde{a}(s)\,ds,\qquad L_s = y^{\varkappa+1}(\mu(\zeta)-\mu(1/y))/q,\\
&L = L_i + L_s,\\
&\mu(s) = 1/s + s^{\varkappa+1}/(\varkappa+1),\quad q = \alpha - y^{\varkappa}(\mathscr{D}_1 - y)^2,\\
&\overline{A} = \frac{1}{L}\left(y^{\varkappa+1}\int_\zeta^{1/y}\widetilde{a}(s)\,ds + L_s\right),\\
&\overline{M} = \frac{1}{L}\left(y^{2\varkappa+1}\int_\zeta^{1/y}\mu(s)\widetilde{a}(s)\,ds + \left(y^{\varkappa+2} + \frac{1}{\varkappa+1}\right)L_s + \frac{1}{2}qL_s^2\right).
\end{aligned}
\tag{7.7}
$$

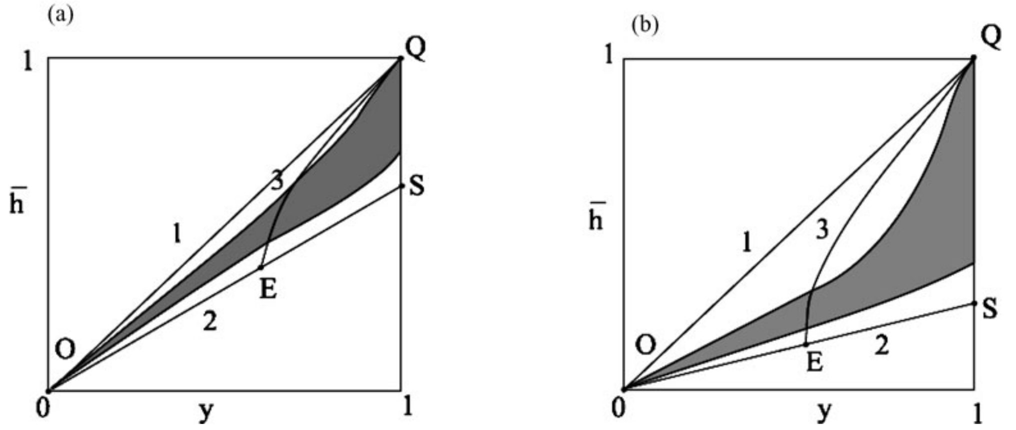

FIGURE 6. RW diagram for one-layer flows ($\lambda = 0$). OQE is for free surface RW, QES is for RW with plugs, stable RW are in the shaded domain: (a) $Fr = 3$; (b) $Fr = 5$.

Here the constant velocity $u_* = y^{\varkappa/2}(\mathscr{D}_1 - y)$ and the constant pressure gradient in a liquid plug

$$\frac{dp^+}{d\xi} \equiv q$$

are taken into account. The hyperbolicity criterion (7.5) can be applied to the stability analysis of the periodic solutions of (7.1) for the free surface flows as well as, in view of (7.7), for the flows with plugs.

As has been mentioned above, in free surface flows only roll waves of near maximal amplitude are stable. Therefore, the structure of the stability domain can be seen more clearly on the $(y, \overline{A})$ plane. In Figure 6 the stability diagram of roll waves over an incline ($\varkappa = 1, \lambda = 0$) is shown on the $(y, \overline{A})$ plane for two different values of the Froude number $Fr = \alpha^{1/2}$, (a) $Fr = 3$ and (b) $Fr = 5$. Roll waves exist between the diagonal OQ, which corresponds to the infinitesimal waves, and the straight line OS with the equation $\overline{A} = \zeta_m y$, which corresponds to the waves of maximal amplitude. Roll waves with the free surface exist in the domain OQE, roll waves with the liquid plugs exist in the domain QES. The hyperbolicity domains of (7.3) are shown as shaded areas. These stability domains are bounded for free surface waves (OQE) by the straight lines as it was explained above for the self-semilar channels. Note that for both the cases $Fr = 3$ and $Fr = 5$, infinitesimal waves and long waves with and without plugs are unstable. Only roll waves of finite length are stable.

In the gradientless two-layer flows, even a smalldensity fraction λ ($\lambda \ll, 1$) influences the RW stability, especially, in film flows. The RW stability diagram is shown in Figure 7 for two values of λ : $\lambda = 0.001$ (Figure 7(a)) and $\lambda = 0.1$ (Figure 7(b)). The Froude number $Fr = \alpha^{1/2} = 3$ in both cases. The notations are the same as in Figure 6. It can be seen in Figure 7 that the trapped gas phase results in the stability of RW for film flows ($y \ll 1$), but it does not significantly affect the stability of roll waves with plugs.

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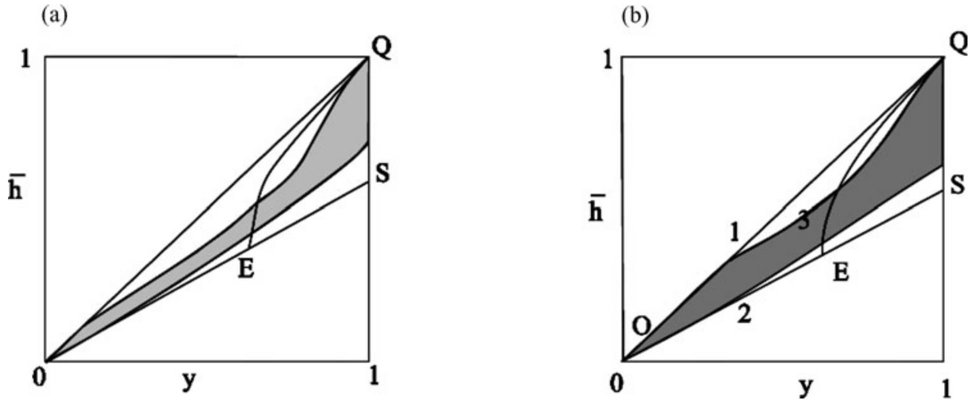

FIGURE 7. RW diagram for two-layer flows with trapped gas phase ($Fr = 3$): (a) $\lambda = 0.01$; (b) $\lambda = 0.1$. For other notations see Figure 6.

8 Conclusion

The basic mechanism of roll wave generation for one-layer flows in inclined open channel and for two-layer flows in tubes and ducts is the same, namely, the instability of steady-state shear flows. But the wave patterns in the two-layer flow are much more complicated compared to the open channel flow. Nevertheless, the periodic travelling waves or the roll waves are represented by a two-parameter family of solutions of the shallow water equations. This gives the possibility to extend the stability approach for roll waves of finite amplitude developed for one-layer flows and based on the modulation equations analysis.

The analysis can be applied also to the plug flow regime for pressure gradientless two-layer flows in an inclined channel. It is worthy of noting that the plug flows considered in the paper are the particular case of roll waves since such waves are defined completely by the critical and minimal sections as in the separated flow regime. Therefore, this analysis cannot be applied directly to the slug flow regime for two-layer flow in horizontal channels and tubes.

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