Bull. Aust. Math. Soc. **105** (2022), 19–30
doi:10.1017/S0004972721000381

SOLUTIONS TO A LEBESGUE–NAGELL EQUATION

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(Received 21 March 2021; accepted 20 April 2021; first published online 24 May 2021)

Abstract

We find all integer solutions to the equation $x^2 + 5^a \cdot 13^b \cdot 17^c = y^n$ with $a, b, c \geq 0$, $n \geq 3$, $x, y > 0$ and $\gcd(x, y) = 1$. Our proof uses a deep result about primitive divisors of Lucas sequences in combination with elementary number theory and computer search.

2020 *Mathematics subject classification*: primary 11D61; secondary 11D72.

Keywords and phrases: Diophantine equation, Lebesgue–Nagell equation, primitive divisor theorem.

1. Introduction

Let λ be a positive integer. The equation

$$x^2 + \lambda = y^n \tag{1.1}$$

in positive integers is called a Lebesgue–Nagell equation. Cohn [10] and Bugeaud *et al.* [7] studied (1.1) for λ in the range $1 \leq \lambda \leq 100$ and others studied (1.1) when λ is a perfect power (see [2–4, 17]). Many others studied (1.1) when the set of prime factors of λ is fixed (see [1, 8, 9, 11, 12, 14–16, 18]). For a comprehensive survey of equation (1.1) and its generalisations, see Le and Soydan [13] with over 350 references. In this paper, we study (1.1) when λ only has prime factors 5, 13, 17. Our result extends the work in [14, 16]. We will prove the following result.

THEOREM 1.1. *All integer solutions* (n, a, b, c, x, y) *to the equation*

$$x^2 + 5^a \cdot 13^b \cdot 17^c = y^n$$

with $n \geq 3$, $a, b, c \geq 0$, $x, y > 0$ *and* $\gcd(x, y) = 1$ *are listed in Table 1.*

A main tool in our paper is a deep result of Bilu *et al.* [5] on primitive divisors of Lucas sequences. All computations in the paper are done in MAGMA [6]. The MAGMA code is available from the author on request.

The author is partially supported by the Vietnam Institute for Advanced Study in Mathematics (VIASM) and the Vietnam National Foundation for Science and Technology Development (NAFOSTED) (grant number 101.04-2019.314).

TABLE 1. Solutions for Theorem 1.1.

(n, a, b, c, x, y)	(n, a, b, c, x, y)	(n, a, b, c, x, y)
(3, 0, 1, 0, 70, 17)	(3, 1, 1, 2, 716, 81)	(3, 2, 2, 0, 142, 29)
(3, 3, 0, 2, 2034, 161)	(4, 0, 0, 1, 8, 3)	(4, 1, 1, 0, 4, 3)
(4, 1, 1, 1, 36, 7)	(4, 5, 1, 1, 26556, 163)	(4, 1, 1, 2, 716, 27)
(6, 1, 1, 2, 716, 9)	(12, 1, 1, 1, 716, 3)	

2. Preliminaries

Let α and β be two algebraic integers such that $\alpha+\beta$ and $\alpha\beta$ are nonzero coprime integers and α/β is not a root of unity. The Lucas sequence $(L_n)_{n\geq 1}$ is defined by

$$L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all } n \geq 1.$$

A prime number p is called a primitive divisor of L_n if p divides L_n but p does not divide $(\alpha-\beta)^2 L_1 L_2 \ldots L_{n-1}$. From the theorems of Bilu *et al.* [5]:

(i) if L_n has a primitive divisor q, then n divides $q - \left(\frac{(\alpha-\beta)^2}{q}\right)$, where $\left(\frac{*}{*}\right)$ denotes the Legendre symbol;
(ii) if $n > 30$, then L_n has a primitive divisor;
(iii) for $4 < n \leq 30$, if L_n does not have a primitive divisor, then (n, α, β) can be found from Table 1 in [5].

3. Proof of Theorem 1.1

We fix the set $S = \{5, 13, 17\}$. We first assume that n is an odd prime in

$$x^2 + 5^a \cdot 13^b \cdot 17^c = y^n. \tag{3.1}$$

If $2 \mid y$, taking (3.1) mod 8 gives $1 + (-3)^{a+b} \equiv 0 \pmod 8$, which is impossible. Therefore, $2 \nmid y$ and $2 \mid x$. Write $5^a \cdot 13^b \cdot 17^c = DC^2$, where $C, D \in \mathbb{Z}^+$ and D is square-free. Let $K = \mathbb{Q}(\sqrt{-D})$. Let $h(K)$ and $\mathcal{O}_K$ be the class number and the ring of integers of K, respectively. Then $h(K) \in \{1, 2, 4, 8, 16\}$ and $\mathcal{O}_K = \mathbb{Z}[\sqrt{-D}]$. From (3.1),

$$(x + C\sqrt{-D})(x - C\sqrt{-D}) = (y)^n.$$

Since $2 \nmid y$ and $\gcd(x, y) = 1$, the two ideals $(x + C\sqrt{-d})$ and $(x - C\sqrt{-d})$ are coprime. Therefore,

$$(x + C\sqrt{-D}) = \mathcal{A}^n,$$

where $\mathcal{A}$ is an ideal in $\mathcal{O}_K$. Since $n \nmid h(K)$, the ideal $\mathcal{A}$ is principal. Since the units in $\mathcal{O}_K$ are $\{\pm 1\}$,

$$x + C\sqrt{-D} = (u + v\sqrt{-D})^n, \tag{3.2}$$

TABLE 2. Cases where the search in Lemma 3.1 did not succeed.

(0, 5, 5)	(1, 3, 4)	(1, 3, 5)	(1, 5, 4)	(2, 5, 4)	(2, 5, 5)
(3, 4, 4)	(4, 3, 5)	(5, 2, 4)	(5, 2, 5)	(5, 4, 4)	(5, 5, 1)
(5, 5, 2)	(5, 5, 3)	(5, 5, 4)	(5, 5, 5)		

where $u, v \in \mathbb{Z}$ and $y = u^2 + Dv^2$. Notice in (3.2) that $uv \neq 0$. Comparing the imaginary parts in (3.2) gives

$$x = \sum_{l=0}^{(n-1)/2} \binom{n}{k} u^{n-2l}(-D)^l v^{2l}. \tag{3.3}$$

Since $y = u^2 + Dv^2$ and $\gcd(x, y) = 1$, (3.3) forces $\gcd(u, v) = \gcd(u, D) = 1$.

LEMMA 3.1. *All solutions to (3.1) if $n = 3$ are listed in Table 1.*

PROOF. Let $a = 6a_1 + i$, $b = 6b_1 + j$ and $c = 6c_1 + k$, where $a_1, b_1, c_1, i, j, k \in \mathbb{N}$ and $i, j, k \leq 5$. Let $Y = x/(5^{3a_1} \cdot 13^{3b_1} \cdot 17^{3c_1})$ and $X = y/(5^{2a_1} \cdot 13^{2b_1} \cdot 17^{2c_1})$. From (3.1),

$$Y^2 = X^3 - 5^i \cdot 13^j \cdot 17^k. \tag{3.4}$$

We used the command SIntegralPoints() in MAGMA [6] to search for S-integral points on (3.4). MAGMA was able to find S-integral points on (3.4) and hence solutions to (1.1) as listed in Table 1 for all but the cases of (i, j, k) listed in Table 2, where the search did not succeed. The details of the computation are given in the appendix.

We show that (3.1) does not have solutions in these cases. With $u_1 = \lfloor i/2 \rfloor$, $u_2 = \lfloor j/2 \rfloor$ and $u_3 = \lfloor k/2 \rfloor$, we can write $C = 5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3}$. From (3.2),

$$5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3} = v(3u^2 - Dv^2). \tag{3.5}$$

From (3.5), $2 \nmid v$ and $2 \mid u$. Since D only has prime divisors $5, 13, 17$, taking (3.5) mod 4 gives $v \equiv -1 \pmod 4$. Since v only has prime divisors $5, 13, 17$, we have $v < 0$.

Case 1: $D = 5$. Here $(i, j, k) = (3, 4, 4)$, $(5, 2, 4)$ and $(5, 4, 4)$, so $k = 4$ and

$$5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+2} = v(3u^2 - 5v^2). \tag{3.6}$$

If $v = -5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3}$, then $1 = 5v^2 - 3u^2$, which is impossible mod 3 since $\left(\frac{5}{3}\right) = -1$. Hence, $v \neq -5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3}$. If $3a_1 + u_1 \geq 2$, from (3.6) we have $5 \nmid u$ because $5 \mid u$ forces $5 \nmid v$. Hence, $5^2 \nmid v(3u^2 - 5v^2)$, which is a contradiction. Since $\left(\frac{-15}{13}\right) = -1$ and $\gcd(u, v) = 1$, from (3.6) we have $13^{3b_1+u_2} \mid v$.

Case 1.1: $(i, j, k) = (3, 4, 4)$. Now

$$5^{3a_1+1} \cdot 13^{3b_1+2} \cdot 17^{3c_1+2} = v(3u^2 - 5v^2). \tag{3.7}$$

If $5 \mid v$, then $5^{3a_1+1} \mid v$ and $13^{3b_1+2} \mid v$. Therefore, $v = -5^{3a_1+1} \cdot 13^{3b_1+2}$. Then (3.7) reduces to

$$17^{3c_1+2} = 5^{6a_1+3} \cdot 13^{6b_1+4} - 3u^2.$$

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Taking this mod 3 gives $(-1)^{3c_1+2} \equiv -1 \pmod 3$, so $2 \nmid c_1$. Let $c_1 = 2c_2 + 1$, where $c_2 \in \mathbb{N}$. Then

$$17^{6c_2+5} = 5^{6a_1+3} \cdot 13^{6b_1+4} - 3u^2.$$

Taking this mod 9 gives $-1 \equiv 5 - 3u^2 \pmod 9$, so $u^2 \equiv 2 \pmod 3$, which is impossible.

If $5 \nmid v$, then $5 \mid u$. Let $u = 5u_1$, where $u_1 \in \mathbb{Z}$. Since $5^2 \nmid v(3u^2 - v^2)$, in (3.7) we have $3a_1 + 1 = 1$. Then $v = -13^{3b_1+2}$ or $v = -13^{3b_1+2} \cdot 17^{3c_1+2}$.

If $v = -13^{3b_1+2}$, then (3.7) reduces to

$$17^{3c_1+2} = 13^{6b_1+4} - 15u_1^2. \tag{3.8}$$

Taking this mod 3 gives $(-1)^{3c_1+2} \equiv 1 \pmod 3$, so $2 \mid 3c_1 + 2$. Therefore, (3.8) is impossible mod 13 since $\left(\frac{-15}{13}\right) = -1$.

If $v = -13^{3b_1+2} \cdot 17^{3c_1+2}$, then (3.7) reduces to

$$1 = 13^{6b_1+4} \cdot 17^{6c_1+4} - 15u_1^2,$$

which is again impossible mod 13 since $\left(\frac{-15}{13}\right) = -1$.

Case 1.2: $(i, j, k) = (5, 2, 4)$. In (3.6), $u_1 = 2$, so $5^{3a_1+2} \mid v$. Now $v = -5^{3a_1+2} \cdot 13^{3b_1+1}$ and (3.6) reduces to

$$17^{3c_1+2} = 5^{6a_1+5} \cdot 13^{6b_1+2} - 3u^2.$$

Taking this mod 3 gives $(-1)^{3c_1+2} \equiv -1 \pmod 3$, so $2 \nmid 3c_1 + 2$ and $2 \nmid c_1$. Let $c_1 = 2c_2 + 1$, where $c_2 \in \mathbb{N}$. Then

$$17^{6c_2+5} = 5^{6a_1+5} \cdot 13^{6b_1+2} - 3u^2.$$

Taking this mod 9 gives $-1 \equiv 5 - 3u^2 \pmod 9$, so $u^2 \equiv 2 \pmod 3$, which is impossible.

Case 1.3: $(i, j, k) = (5, 4, 4)$. In (3.6), $u_1 = 2$, so $5^{3a_1+2} \mid v$. Now $v = -5^{3a_1+2} \cdot 13^{3b_1+2}$ and (3.6) reduces to

$$17^{3c_1+2} = 5^{6a_1+5} \cdot 13^{6b_1+4} - 3u^2.$$

Taking this mod 3 gives $(-1)^{3c_1+2} \equiv -1 \pmod 3$, so $2 \nmid c_1$. Let $c_1 = 2c_2 + 1$, where $c_2 \in \mathbb{N}$. Then

$$17^{6c_2+5} = 5^{6a_1+5} \cdot 13^{6b_1+4} - 3u^2. \tag{3.9}$$

The mod 9 argument does not work for (3.9), so we use a different approach. Let

$$6a_1 + 5 = 4a_2 + i_1, \quad 6b_1 + 4 = 4b_2 + j_1 \quad \text{and} \quad 6c_2 + 5 = 4c_3 + k_1,$$

where $a_2, b_2, c_3 \in \mathbb{N}$, $i_1 \in \{1, 3\}$, $j_1 \in \{0, 2\}$ and $k_1 \in \{1, 3\}$. From (3.9),

$$3Y^2 = 5^{i_1} \cdot 13^{j_1} \cdot X^4 - 17^{k_1}, \quad \text{where } X = \frac{5^{a_2} \cdot 13^{b_2}}{17^{c_3}}, Y = \frac{u}{17^{2c_2}}. \tag{3.10}$$

We used the SIntegralLjunggrenPoints() command in MAGMA [6] to search for $\{17\}$-integral points on (3.10). The only solutions are $(X, Y) = (\pm 1, \pm 6)$ when $(i_1, j_1, k_1) = (3, 0, 1)$. This gives $a_2 = b_2 = c_3 = 0$. Therefore, $6c_2 + 5 = 4c_3 + k_1 = 1$, which is impossible.

https://doi.org/10.1017/S0004972721000381 Published online by Cambridge University Press

Case 2: $D = 5 \cdot 13$. Now $(i, j, k) = (1, 3, 4), (1, 5, 4), (5, 5, 2), (5, 5, 4)$ and

$$5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3} = v(3u^2 - 65v^2). \tag{3.11}$$

If $v = -5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3}$, then (3.11) reduces to $1 = 65v^2 - 3u^2$, which is impossible mod 5 since $\left(\frac{-3}{5}\right) = -1$. Hence, $v \neq -5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3}$.

Case 2.1: $(i, j, k) = (1, 3, 4)$. Then (3.11) reduces to

$$5^{3a_1} \cdot 13^{3b_1+1} \cdot 17^{3c_1+2} = v(3u^2 - 65v^2). \tag{3.12}$$

Since $3a_1 = 0$ or $3a_1 > 2$, we have $5^{3a_1} \mid v$.

If $17 \mid v$, then $17^{3c_1+2} \mid v$ and so $v = -5^{3a_1} \cdot 17^{3c_1+2}$. In (3.12), $13 \mid u$ and $3b_1 + 1 = 1$. Let $u = 13u_1$, where $u_1 \in \mathbb{Z}$. Then (3.12) reduces to

$$1 = 5^{6a_1+1} \cdot 17^{6c_1+4} - 39u_1^2,$$

which is impossible mod 3 since $\left(\frac{5}{3}\right) = -1$.

If $13 \mid v$, then $13^{3b_1+1} \mid v$. Therefore, $v = -5^{3a_1} \cdot 13^{3b_1+1}$. Then (3.12) reduces to

$$17^{3c_1+2} = 5^{6a_1+1} \cdot 13^{6b_1+3} - 3u^2.$$

Taking this mod 3 gives $(-1)^{3c_1+2} \equiv (-1) \pmod 3$, so $2 \nmid c_1$. Let $c_1 = 2c_2 + 1$, where $c_2 \in \mathbb{N}$. Then

$$17^{6c_2+5} = 5^{6c_1+1} \cdot 13^{6c_1+3} - 3u^2.$$

Taking this mod 9 gives $-1 \equiv 5 - 3u^2 \pmod 9$, so $u^2 \equiv 2 \pmod 3$, which is impossible.

If $13 \nmid v$ and $17 \nmid v$, then $v = -5^{3a_1}$. In (3.12), $13 \mid u$ and $3b_1 + 1 = 1$. Let $u = 13u_1$, where $u_1 \in \mathbb{N}$. Then (3.12) reduces to

$$17^{3c_1+2} = 5^{6a_1+1} - 39u_1^2.$$

Taking this mod 3 gives $(-1)^{3c_1+2} \equiv -1 \pmod 3$, so $2 \nmid c_1$. Let $c_1 = 2c_2 + 1$, where $c_2 \in \mathbb{N}$. Then

$$17^{6c_1+5} = 5^{6a_1+1} - 39u_1^2.$$

Taking this mod 9 gives $-1 \equiv 5 - 3u_1^2 \pmod 9$, so $u_1^2 \equiv 2 \pmod 3$, which again is impossible.

Case 2.2: $(i, j, k) = (1, 5, 4)$. Then (3.11) reduces to

$$5^{3a_1} \cdot 13^{3b_1+2} \cdot 17^{3c_1+2} = v(3u^2 - 65v^2). \tag{3.13}$$

Since $3a_1 = 0$ or $3a_1 > 2$, we have $5^{3a_1} \mid v$. From (3.13), we also have $13^{3b_1+2} \mid v$. Therefore, $v = -5^{3a_1} \cdot 13^{3b_1+2}$. Then (3.13) reduces to

$$17^{3c_1+2} = 5^{6a_1+1} \cdot 13^{6b_1+5} - 3u^2.$$

Taking this mod 3 gives $(-1)^{3c_1+2} \equiv -1 \pmod 3$, so $2 \nmid c_1$. Let $c_1 = 2c_2 + 1$, where $c_2 \in \mathbb{N}$. Then

$$17^{6c_2+5} = 5^{6a_1+1} \cdot 13^{6b_1+5} - 3u^2.$$

Hence,

$$Y^2 = 3(5 \cdot 13^5 \cdot X^6 + 17^5), \quad \text{where } Y = 3u, X = \frac{17^{c_2}}{5^{a_1} \cdot 13^{b_1}}. \tag{3.14}$$

Equation (3.14) is locally unsolvable at 3.

Case 2.3: $(i, j, k) = (5, 5, 2), (5, 5, 4)$. Now (3.11) reduces to

$$5^{3a_1+2} \cdot 13^{3b_1+2} \cdot 17^{3c_1+u_3} = v(3u^2 - 65v^2), \tag{3.15}$$

where $u_3 \in \{1, 2\}$. Then $5^{3a_1+2} \mid v$ and $13^{3b_1+2} \mid v$ and so $v = -5^{3a_1+2} \cdot 13^{3b_1+2}$. Hence, (3.15) reduces to

$$17^{3c_1+u_3} = 5^{6a_1+5} \cdot 13^{6b_1+5} - 3u^2.$$

Taking this mod 3 gives $(-1)^{3c_1+u_3} \equiv -1 \pmod 3$, so $2 \nmid 3c_1 + u_3$. Now we can write $3c_1 + u_3 = 6c_2 + \epsilon$, where $c_2 \in \mathbb{N}$ and $\epsilon \in \{1, 3, 5\}$. Then

$$17^{6c_2+\epsilon} = 5^{6c_1+5} \cdot 13^{6b_1+5} - 3u^2.$$

Taking this mod 9 gives $-1 \equiv 5 - 3u^2 \pmod 9$, so $u^2 \equiv 2 \pmod 3$, which again is impossible.

Case 3: $D = 5 \cdot 17$. Then $(i, j, k) = (5, 2, 5)$ and

$$5^{3a_1+2} \cdot 13^{3b_1+1} \cdot 17^{3c_1+2} = v(3u^2 - 5 \cdot 17 \cdot v^2). \tag{3.16}$$

Then $5^{3a_1+2} \mid v$ and $17^{3c_1+2} \mid v$. Since $(3 \cdot 5 \cdot 17/13) = -1$, we have $13 \nmid 3u^2 - 5 \cdot 7 \cdot v^2$. Hence, $13^{3b_1+1} \mid v$. Let $v = -5^{3a_1+2} \cdot 13^{3b_1+1} \cdot 17^{3c_1+2}$. Then (3.16) reduces to

$$1 = 85v^2 - 3u^2, \tag{3.17}$$

which is impossible mod 5 since $\left(\frac{-3}{5}\right) = -1$.

Case 4: $D = 13 \cdot 17$. Then $(i, j, k) = (0, 5, 5), (2, 5, 5), (4, 3, 5)$ and $k = 5$, so (3.5) reduces to

$$5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+2} = v(3u^2 - 221v^2).$$

Then $17^{3c_1+2} \mid v$. If $v = -5^{3a_1+u_1} \cdot 13^{3b_1+2} \cdot 17^{3c_1+2}$, then $1 = 221v^2 - 3u^2$, which is impossible mod 3 since $\left(\frac{221}{3}\right) = -1$. Therefore, $v \neq -5^{3a_1+u_1} \cdot 13^{3b_1+2} \cdot 17^{3c_1+2}$.

Case 4.1: $(i, j, k) = (0, 5, 5), (2, 5, 5)$. Then $j = k = 5$, so

$$5^{3a_1+u_1} \cdot 13^{3b_1+2} \cdot 17^{3c_1+2} = v(3u^2 - 221v^2), \tag{3.18}$$

where $u_1 \in \{0, 1\}$. Then $13^{3b_1+2} \mid v$ and $17^{3c_1+2} \mid v$, so $v = -13^{3b_1+2} \cdot 17^{3c_1+2}$ and

$$5^{3a_1+u_1} = 13^{6b_1+5} \cdot 17^{6c_2+5} - 3u^2. \tag{3.19}$$

Taking this mod 3 gives $(-1)^{3a_1+u_1} \equiv -1 \pmod 3$, so $2 \nmid 3a_1 + u_1$. Therefore, (3.19) is impossible mod 13 since $\left(\frac{-15}{13}\right) = -1$.

Case 4.2: $(i, j, k) = (4, 3, 5)$. Then

$$5^{3a_1+2} \cdot 13^{3b_1+1} \cdot 17^{3c_1+2} = v(3u^2 - 221v^2).$$

Therefore, $v = -13^{3b_1+1} \cdot 17^{3c_1+2}$ and

$$5^{3a_1+2} = 13^{6b_1+3} \cdot 17^{6c_1+5} - 3u^2.$$

Taking this mod 3 gives $(-1)^{3a_1+2} \equiv -1 \pmod 3$, so a_1 is odd. Let $a_1 = 2a_2 + 1$, where $a_2 \in \mathbb{N}$. Then

$$5^{6a_2+5} = 13^{6b_1+3} \cdot 17^{6c_1+5} - 3u^2,$$

which is impossible mod 13 since $\left(\frac{-15}{13}\right) = 1$.

Case 5: $D = 13$. Now $(i, j, k) = (2, 5, 4)$ and (3.5) reduces to

$$5^{3a_1+1} \cdot 13^{3b_1+2} \cdot 17^{3c_1+2} = v(3u^2 - 13v^2). \tag{3.20}$$

Then $13^{3b_1+2} \mid v$. Since $(3 \cdot 13/17) = -1$ and $\gcd(u, v) = 1$, we have $17 \nmid 3u^2 - 13v^2$ and so $17^{3c_1+2} \mid v$.

If $v = -13^{3b_1+2} \cdot 17^{2c_1+2}$, then (3.20) reduces to

$$5^{3a_1+1} = 13^{6b_1+5} \cdot 17^{6c_1+4} - 3u^2.$$

Taking this mod 3 gives $(-1)^{3a_1+1} \equiv 1 \pmod 3$ and hence $2 \mid 3a_1 + 1$. Taking it mod 17 gives $\left(\frac{-3}{17}\right) = 1$, which is impossible since $\left(\frac{-3}{17}\right) = -1$.

If $v = -5^{3a_1+1} \cdot 13^{3b_1+2} \cdot 17^{2c_1+2}$, then (3.20) reduces to

$$1 = 5^{6a_1+2} \cdot 13^{6b_1+5} \cdot 17^{6c_1+4} - 3u^2,$$

which is impossible mod 5 since $\left(\frac{-3}{5}\right) = -1$.

Case 6: $D = 5 \cdot 13 \cdot 17$. Now $(i, j, k) = (1, 3, 5), (5, 5, 1), (5, 5, 3), (5, 5, 5)$ and (3.5) reduces to

$$5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3} = v(3u^2 - 1105v^2). \tag{3.21}$$

If $v = -5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3}$, then $1 = 1105v^2 - 3u^2$, which is impossible mod 5 since $(\frac{-3}{5}) = -1$. Hence, $v \neq 5^{3a_1+u_1} \cdot 13^{3b_1+u_2} \cdot 17^{3c_1+u_3}$.

Case 6.1. $(i, j, k) = (1, 3, 5)$. Then (3.21) reduces to

$$5^{3a_1} \cdot 13^{3b_1+1} \cdot 17^{3c_1+2} = v(3u^2 - 1105v^2). \tag{3.22}$$

Since $3a_1 = 0$ or $3a_1 > 2$, we have $5^{3a_1} \mid v$. Also, $17^{3c_1+2} \mid v$. Hence, $v = -5^{3a_1} \cdot 17^{3c_1+2}$. Therefore, $13 \mid u$. Since $13^2 \nmid v(3u^2 - 1105v^2)$, in (3.22) we have $3b_1 + 1 = 1$. Let $u = 13u_1$, where $u_1 \in \mathbb{Z}$. Then (3.22) reduces to

$$1 = 5^{6a_1+1} \cdot 17^{6c_1+5} - 39u_1^2,$$

which is impossible mod 17 since $\left(\frac{-39}{17}\right) = -1$.

Case 6.2: $(i, j, k) = (5, 5, 1)$. Then (3.21) reduces to

$$5^{3a_1+2} \cdot 13^{3b_1+2} \cdot 17^{3c_1} = v(3u^2 - 1105v^2). \tag{3.23}$$

Hence, $5^{3a_1+2} \mid v$ and $13^{3b_1+2} \mid v$. Since $3c_1 = 0$ or $3c_1 > 2$, (3.23) implies that $17^{3c_1} \mid v$. Hence, $v = -5^{3a_1+2} \cdot 13^{3b_1+2} \cdot 17^{3c_1}$, which is impossible.

Case 6.3: $(i, j, k) = (5, 5, 3)$. Then (3.21) reduces to

$$5^{3a_1+2} \cdot 13^{3b_1+2} \cdot 17^{3c_1+1} = v(3u^2 - 1105v^2). \tag{3.24}$$

Therefore, $5^{3a_1+2} \mid v$ and $13^{3b_1+2} \mid v$. Thus, $v = -5^{3a_1+2} \cdot 13^{3b_1+2}$ and $17 \mid u$. Let $v = 17u_1$, where $u_1 \in \mathbb{N}$. Then (3.24) reduces to

$$17^{3c_1} = 5^{6a_1+5} \cdot 13^{6b_1+5} - 51u_1^2. \tag{3.25}$$

Therefore, $c_1 = 0$ and (3.25) is impossible mod 17 since $\left(\frac{5\cdot 13}{17}\right) = -1$.

Case 6.4: $(i, j, k) = (5, 5, 5)$. Then (3.21) reduces to

$$5^{3a_1+2} \cdot 13^{3b_1+2} \cdot 17^{3c_1+2} = v(3u^2 - 1105v^2).$$

Therefore, $5^{3a_1+2} \mid v$, $13^{3b_1+2} \mid v$ and $17^{3c_1+2} \mid v$. Hence, $v = -5^{3a_1+2} \cdot 13^{3b_1+2} \cdot 17^{3c_1+2}$, which is impossible. □

LEMMA 3.2. *All solutions to (3.1) if* $3 \mid n$ *are listed in Table 1.*

PROOF. Let $n = 3m$, where $m \in \mathbb{Z}^+$. Let $y_1 = y^m$. Applying Lemma 3.1 to the equation

$$x^2 + 5^a \cdot 13^b \cdot 17^c = y_1^3$$

gives Lemma 3.2. □

LEMMA 3.3. *All solutions to (3.1) if* $4 \mid n$ *are listed in Table 1.*

PROOF. Let $n = 4m$, where $m \in \mathbb{Z}^+$. Let $a = 4a_1 + i$, $b = 4b_1 + j$ and $c = 4c_1 + k$, where $a_1, b_1, c_1, i, j, k \in \mathbb{N}$ and $i, j, k \leq 3$. From (3.1),

$$Y^2 = X^4 - 5^i \cdot 13^j \cdot 17^k, \quad \text{where } Y = \frac{x}{5^{2a_1} \cdot 13^{2b_1} \cdot 17^{2c_1}},\ X = \frac{y^m}{5^{a_1} \cdot 13^{b_1} \cdot 17^{c_1}}. \tag{3.26}$$

We used the SIntegralLjunggrenPoints() command in MAGMA [6] to search for S-integral points on (3.26). We list the eight cases of (i, j, k) where (3.26) has S-integral points in Table 3. The symbol $\emptyset$ means that the resulting x, y do not satisfy the requirements $x, y > 0$ and $\gcd(x, y) = 1$. Equation (3.26) does not have S-integral points for other cases of (i, j, k), such as $(i, j, k) = (0, 0, 2), (0, 0, 3)$. □

LEMMA 3.4. *Equation (3.1) has no solutions if n simultaneously satisfies the conditions* $3 \nmid n$, $4 \nmid n$ *and* $n \geq 5$.

PROOF. We can assume that n is a prime number ≥ 5. From (3.2),

$$L_n = \frac{C}{v},$$

where $L_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, $\alpha = u + v\sqrt{-D}$ and $\beta = u - v\sqrt{-D}$. Since $uv \neq 0$, we have $|\mathbb{Q}(\alpha/\beta)\colon \mathbb{Q}| = 2$. If α/β is a root of unity, say $\alpha/\beta = \zeta_m$, a primitive m-root of unity, then $|\mathbb{Q}(\alpha/\beta) : \mathbb{Q}| = |\mathbb{Q}(\zeta_m)\colon \mathbb{Q}| = \phi(m)$. Consequently, $\phi(m) = 2$ and $m \in \{3, 4\}$,

https://doi.org/10.1017/S0004972721000381 Published online by Cambridge University Press

TABLE 3. Solutions for the proof of Lemma 3.3.

(i, j, k)	(X, Y)	$(n, a, b, c, x, y,)$
(0, 0, 0)	(1, 0)	$\emptyset$
(0, 0, 1)	(3, 8)	(4, 0, 0, 1, 8, 3)
(1, 1, 0)	(3, 4)	(4, 1, 1, 0, 4, 3)
(1, 1, 1)	(7, 36)	(4, 1, 1, 1, 36, 7)
(1, 1, 1)	(163/5, 26556/25)	(4, 5, 1, 1, 26556, 163)
(1, 1, 2)	(27, 716)	(4, 1, 1, 2, 716, 27)
(2, 2, 0)	(13, 156)	$\emptyset$
(2, 2, 2)	(85, 7140)	$\emptyset$

so that $\zeta_m \in \{\pm\sqrt{-1}, \frac{1}{2}(\pm 1 \pm \sqrt{-3})\}$. But this is impossible since

$$\mathbb{Q}(\zeta_m) = \mathbb{Q}\Big(\frac{\alpha}{\beta}\Big) = \mathbb{Q}(\sqrt{-D}) \neq \mathbb{Q}(\sqrt{-1}) \text{ or } \mathbb{Q}(\sqrt{-3}).$$

So, α/β is not a root of unity. Also, $\alpha + \beta$ and $\alpha\beta$ are coprime nonzero integers because $\gcd(u, v) = \gcd(u, D) = 1$.

From the work of Bilu *et al.* [5], if L_n has no primitive divisors, then $n \le 30$. Here n is a prime number and D is an odd positive integer. From Table 1 in [5], we conclude that $n \in \{5, 7, 13\}$ and $D \in \{7, 11, 15, 19, 341\}$. But this is impossible since D only has the prime divisors $5, 13, 17$.

Therefore, L_n has a primitive divisor q. Since $q \mid C$, we must have $q \in \{5, 13, 17\}$. Now n divides $q - \Big(\frac{(\alpha-\beta)^2}{q}\Big)$ and n is a prime ≥ 5. So, the only possibility is that $n = 7$, $q = 13$ and $\Big(\frac{(\alpha-\beta)^2}{q}\Big) = -1$. Since $(\alpha - \beta)^2 = -4v^2 D$, we have $D \in \{5, 5 \cdot 17\}$ and $13 \nmid v$. Let $C = 5^r \cdot 13^s \cdot 17^t$, where $r, s, t \in \mathbb{N}$.

Case 1: $D = 5$. Then

$$5^r \cdot 13^s \cdot 17^t = v(7u^6 - 175u^4v^2 + 525u^2v^4 - 125v^6). \tag{3.27}$$

Since v is odd, u is even. Taking (3.27) mod 4 gives $v \equiv -1 \pmod 4$. Since v could only have prime divisors 5 and 17, we must have $v < 0$. Therefore,

$$5^{r_1} \cdot 13^s \cdot 17^{t_1} = -7u^6 + 175u^4v^2 - 525u^2v^4 + 125v^6, \tag{3.28}$$

where $r_1, t_1 \in \mathbb{N}$. Let $5^{r_1} \cdot 13^s \cdot 17^{t_1} = BA^2$, where $A, B \in \mathbb{Z}^+$ and B is square-free. From (3.28),

$$Y^2 = X^3 + 175BX^2 + 3675B^2X + 6125B^3, \tag{3.29}$$

where $Y = 7AB^2/v^3$ and $X = -7Bu^2/v^2$. We used the SIntegralPoints() command in MAGMA to search for $\{5, 17\}$-integral points on (3.29). None of the solutions found gives a solution to (3.28) (see Table 4). The symbol $\emptyset$ means that (3.29) has no $\{5, 17\}$-integral points. Since $X = -7Bu^2/v^2 < 0$, several cases in Table 4 can be excluded immediately.

https://doi.org/10.1017/S0004972721000381 Published online by Cambridge University Press

TABLE 4. Solutions for the proof of Lemma 3.4, Case 1.

B	(X, Y)	(u, v)
1	∅	∅
5	(−749, 1624) , (−525, 7000) , (−125, 1000), (0, 875), (175, 7000) (975, 43000)	∅
13	(91, 9464), (679, 42392)	∅
17	∅	∅
65	(−9516, 149227), (−9425, 169000), (−2925, 169000), (100, 57875) (975, 169000)	∅
65	(1075, 181000), (3679, 512408),(47475, 11549000) (12236575, 42824431000)	∅
85	(−7325, 463000)	∅
221	(4251, 1266824)	∅
1105	(−45500, 10499125), (−25025, 1183000), (84175, 48503000) (1816675, 2577197000)	∅

TABLE 5. Solutions for the proof of Lemma 3.4, Case 2.

B	(X, Y)	(u, v)
1	∅	∅
5	(−7325, 463000)	∅
13	(4251, 1266824)	∅
17	∅	∅
65	(−45500, 10499125), (−25025, 1183000), (84175, 48503000) (1816675, 2577197000)	∅
85	(−216461, 7978712), (−151725, 34391000), (−36125, 4913000)	∅
85	(0, 4298875), (50575, 34391000), (281775, 211259000)	∅
221	(26299, 46496632), (196231, 208271896)	∅
1105	(−2750124, 733152251), (−2723825, 830297000) (−845325, 830297000)	∅
1105	(28900, 284339875), (281775, 830297000), (310675, 889253000)	∅
1105	(1063231, 2517460504), (13720275, 56740237000) (3536370175, 210396429503000)	∅

Case 2: $D = 5 \cdot 17$. Then

$$5^r \cdot 13^s \cdot 17^t = v(7u^6 - 2975u^4v^2 + 151725u^2v^4 - 614125v^6). \tag{3.30}$$

Therefore, $v < 0$ and v only has prime divisors $5, 17$. Dividing both sides of (3.30) by v gives

$$5^{r_2} \cdot 13^s \cdot 17^{t_2} = 7u^6 - 2975u^4v^2 + 151725u^2v^4 - 614125v^6, \tag{3.31}$$

TABLE 6. Solutions for the appendix.

(i, j, k)	(X, Y)	(n, a, b, c, x, y)	(i, j, k)	(X, Y)	$(n, \ldots)$
(0, 0, 0)	(1, 0)	∅	(3, 0, 3)	(85, 0)	∅
(0, 0, 2)	(17, 68)	∅	(3, 3, 0)	(65, 0)	∅
(0, 0, 3)	(17, 0)	∅	(3, 3, 3)	(1105, 0)	∅
(0, 1, 0)	(17, 70)	(3, 0, 1, 0, 70, 17)	(3, 4, 4)	NA	NA
(0, 1, 2)	(17, 34)	∅	(3, 4, 5)	NA	NA
(0, 3, 0)	(13, 0)	∅	(4, 2, 0)	(325, 5850)	∅
(0, 3, 3)	(221, 0)	∅	(4, 2, 2)	(325, 1950)	∅
(0, 5, 5)	NA	NA	(4, 2, 2)	(425, 6800)	∅
(1, 1, 2)	(81, 716)	(3, 1, 1, 2, 716, 81)	(4, 2, 4)	(7225, 606900)	∅
(1, 3, 4)	NA	NA	(4, 2, 4)	(19721, 2767856)	∅
(1, 3, 5)	NA	NA	(4, 3, 0)	(10829, 1126892)	∅
(1, 5, 4)	NA	NA	(4, 3, 3)	(2925, 135200)	∅
(2, 0, 0)	(5, 10)	∅	(4, 3, 5)	NA	NA
(2, 0, 1)	(21, 94)	∅	(4, 4, 2)	(71825, 19249100)	∅
(2, 0, 4)	(185, 2060)	∅	(5, 1, 4)	NA	NA
(2, 0, 4)	(1445, 54910)	∅	(5, 2, 4)	NA	NA
(2, 1, 3)	(425, 8670)	∅	(5, 2, 5)	NA	NA
(2, 2, 0)	(29, 142)	(3, 2, 2, 0, 142, 29)	(5, 4, 4)	NA	NA
(2, 2, 0)	(65, 520)	∅	(5, 5, 1)	NA	NA
(2, 2, 2)	(221, 3094)	∅	(5, 5, 2)	NA	NA
(2, 4, 0)	(169, 2028)	∅	(5, 5, 3)	NA	NA
(2, 5, 4)	NA	NA	(5, 5, 4)	NA	NA
(3, 0, 0)	(5, 0)	∅	(5, 5, 5)	NA	NA
(3, 0, 2)	(161, 2034)	(3, 3, 0, 2, 2034, 161)			

where $r_2, t_2 \in \mathbb{N}$. Write $5^{r_2} \cdot 13^s \cdot 17^{t_2} = B'A'^2$, where $B', A' \in \mathbb{Z}^+$ and B' is square-free. From (3.31),

$$Y^2 = X^3 + 2975BX^2 + 1062075B^2X + 30092125B^3, \tag{3.32}$$

where $Y = 7A'B'^2$ and $X = -7B'u^2/v^2$. We used MAGMA to search for $\{5, 17\}$-integral points on (3.32). None of the solutions found gives a solution to (3.30) (see Table 5). The symbol ∅ means that (3.32) has no $\{5, 17\}$-integral points. Since $X = -7B'u^2/v^2 < 0$, several cases in Table 5 can be excluded immediately. □

Appendix

We used the SIntegralPoints() command in MAGMA [6] to search for S-integral points on (3.4). All the cases of (i, j, k) where the set of S-integral points on (3.4) is not empty or could not be determined using MAGMA are listed in Table 6. The symbol NA means that S-integral points could not be determined. The symbol ∅ means that the resulting (x, y) does not satisfy the requirements $x, y > 0$ and $\gcd(x, y) = 1$.

Acknowledgement

The author would like to thank the anonymous referee for his valuable comments.

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