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Scott's Induced Subdivision Conjecture for Maximal Triangle-Free Graphs

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Scott conjectured in [6] that the class of graphs with no induced subdivision of a given graph is χ-bounded. We verify his conjecture for maximal triangle-free graphs.

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Let F be a graph and let $\mathrm{Forb}^*(F)$ be the class of graphs with no induced subdivision of F. A class $\mathcal{G}$ of graphs is *χ-bounded* if there is a function f such that every graph G of $\mathcal{G}$ satisfies $\chi(G) \leqslant f(\omega(G))$, where χ and ω denote the chromatic number and the clique number of G. Gyárfás conjectured that $\mathrm{Forb}^*(F)$ is χ-bounded if F is a cycle [3]. Scott proved that for each tree T, $\mathrm{Forb}^*(T)$ is χ-bounded, and conjectured the following [6].

Conjecture 1. *For every graph F, $\mathrm{Forb}^*(F)$ is χ-bounded.*

This question is open for triangle-free graphs, which is probably the core of the problem. It also has nice corollaries, for instance it would imply that the triangle-free intersection graphs of segments in the plane have a bounded chromatic number. Indeed, by planarity, they cannot contain an induced subdivision of a K_5 with all the edges subdivided twice. This is a well-known question of Erdős, first cited in [3]. Our goal is to prove Scott's conjecture for triangle-free graphs with diameter two, *i.e.*, maximal triangle-free graphs.

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Theorem 1. *There is a constant c such that every maximal triangle-free graph G with $\chi(G) \geqslant \mathrm{e}^{c \cdot l^4}$ contains an induced subdivision of every graph F on l vertices.*

Proof. Let H be the *neighbourhood hypergraph* of G, *i.e.*, the hypergraph with vertex set V and with hyperedges the closed neighbourhoods of the vertices of G. Observe that H has packing number one, *i.e.*, its hyperedges pairwise intersect. Note also that if the transversality of H is t (minimum size of a set of vertices intersecting all hyperedges), then $\chi(G) \leqslant 2t$. Indeed, G can be covered by t closed neighbourhoods, hence by t induced stars since G is triangle-free. Since $\chi(G) \geqslant \mathrm{e}^{c \cdot l^4}$, the transversality of H is at least $1/2 \cdot \mathrm{e}^{c \cdot l^4}$.

Ding, Seymour and Winkler [2] proved that if a hypergraph H has packing number one and transversality greater than $11d^2(d+4)(d+1)^2$, it contains d hyperedges $e_1, \ldots, e_d$ and a set of vertices $Y = \{y_{i,j} : 1 \leqslant i < j \leqslant d\}$ such that $y_{i,j} \in e_i \cap e_j$ and $y_{i,j} \notin e_k$ for all $k \neq i, j$. Since the transversality of H is at least $1/2 \cdot \mathrm{e}^{c \cdot l^4}$, we have such a collection of hyperedges $e_1, \ldots, e_d$ with $d \geqslant \mathrm{e}^{c_2 \cdot l^4}$ where c_2 is a constant.

Each e_i corresponds to the closed neighbourhood of a vertex x_i of G. Let $X = \{x_1, \ldots, x_d\}$. Since G is triangle-free, there exists a stable set S in X of size at least $\sqrt{d}$ (which has size at least $\mathrm{e}^{c_3 \cdot l^4}$). Replacing X by S, we can assume that X is indeed a stable set, still denoting it by $\{x_1, \ldots, x_d\}$. Note that no $y_{i,j}$ belongs to X since $y_{i,j}$ would be a neighbour of some vertex of X, which is a stable set. Consequently, X is a stable set of G, the set Y is disjoint from X, and for every pair x_i, x_j there is a unique vertex $y_{i,j}$ of Y which is joined to exactly these two vertices of X.

By a theorem of Kim [4], every triangle-free graph on n vertices has a stable set of size $\Theta(\sqrt{n \log n})$. Since the restriction of G to Y is triangle-free and has size $\binom{d}{2}$, by Kim's theorem, it contains a stable set Y' of size $\Theta(d\sqrt{\log d})$. Consider the graph G' on vertex set X with an edge $x_i x_j$ if and only if $y_{i,j} \in Y'$. Note that if G'' is a subgraph of G' on vertex set X', then G'' appears as an induced subdivision in G. Indeed, the induced restriction of G to $X' \cup Y''$, where $y_{i,j} \in Y''$ whenever $x_i x_j$ is an edge of G'', is such a subdivision. So we just have to show that G' contains a subdivision of our original graph F as a subgraph.

A theorem due to Mader [5] and improved by Bollobás and Thomason [1] ensures that each graph with average degree $512 \cdot l^2$ contains a subdivision of K_l, hence of F. Since $d \geqslant \mathrm{e}^{c_3 \cdot l^4}$, there is a constant c_4 such that $\sqrt{\log d} \geqslant c_4 l^2$, hence G' contains a subdivision of F, and therefore G has an induced subdivision of F. □

Ding, Seymour and Winkler also bound the transversality of H in terms of the packing number. Hence the chromatic number of a triangle-free graph G is also bounded when the maximum packing of neighbourhoods is bounded. This is the case for instance if the minimum degree is $c \cdot n$ for some fixed constant $c > 0$.

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