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A NOTE ON OPEN BOOK EMBEDDINGS OF 3-MANIFOLDS IN S^5

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Abstract

In this note, we show that given a closed connected oriented 3-manifold M, there exists a knot K in M such that the manifold M' obtained from M by performing an integer surgery admits an open book decomposition which embeds into the trivial open book of the 5-sphere S^5.

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1. Introduction

An open book decomposition of a closed connected oriented n-manifold M is a fibration $\pi : M \setminus B \to S^1$, where B is a codimension-two oriented submanifold of M with a trivial normal bundle. The submanifold B is called the binding of the open book. The closure of each fibre of π is called a page of the open book and each page is a codimension-one submanifold of M with boundary B. Alexander, in [1], showed that every closed oriented 3-manifold admits an open book. For more details on open books, we refer to the survey [2].

Open book decomposition on odd-dimensional manifolds is an important tool in studying contact structures on manifolds. By a contact structure on a closed oriented $(2n + 1)$-manifold M, we mean a maximally nowhere integrable hyperplane field on M. Giroux, in [4], showed that every co-oriented contact structure on a closed oriented $(2n + 1)$-manifold is supported by an open book. In [11], Thurston and Winkelnkemper constructed contact structures on closed oriented 3-manifolds using open books. In [4], Giroux showed that there is a one-to-one correspondence between the isotopy classes of co-oriented contact structures on a closed oriented 3-manifold M and the open book decompositions of M up to positive stabilisations. By a positive stabilisation operation on an open book of M, we mean a plumbing of a positive Hopf band to the page of the open book. Open book decomposition is a useful tool in studying 3-manifolds. For instance, an open book decomposition of a manifold naturally gives a Heegaard splitting of the manifold, where the Heegaard surface is the closure of the union of two pages.

Open book embeddings of closed oriented 3-manifolds into the open books of $S^2 \times S^3$ as well as into $S^2 \tilde{\times} S^3$ with pages a 2-disc bundle over S^2 and monodromy the identity map are studied in [9]. Results in [9] are extended for nonorientable closed 3-manifolds in [3]. We say that a smooth manifold M with a given open book decomposition admits an open book embedding in an open book decomposition of a smooth manifold N if there exists an embedding of M in N such that, as a submanifold of N, the given open book decomposition on M is compatible with the open book decomposition of N. For the precise definition of an open book embedding, we refer to Definition 2.3. However, the existence of an open book for a closed oriented 3-manifold, such that its open book embeds in the trivial open book of the 5-sphere S^5 with pages the 4-disk D^4 and monodromy the identity map of D^4, is not known.

In this note, we study open book embeddings of closed oriented 3-manifolds in the 5-sphere S^5. We prove the following result.

THEOREM 1.1. *Let M be a closed connected oriented 3-manifold. There exists a knot K in M such that the manifold M' obtained from M, by performing an integer surgery, admits an open book decomposition and this open book embeds into the trivial open book of the 5-sphere S^5.*

2. Preliminary

First, we recall the notions necessary for this note. All the manifolds and maps we consider are smooth.

DEFINITION 2.1. An open book decomposition of a closed connected oriented manifold M is a pair (B, π), with the following properties.

(1) B is an oriented codimension-two submanifold in M with a trivial normal bundle called the binding of the open book.
(2) $\pi : M \setminus B \to S^1$ is a locally trivial fibration such that the fibration π in a tubular neighbourhood of B looks like the trivial fibration of $(B \times D^2) \setminus B \times \{0\} \to S^1$ sending (x, r, θ) to θ, where $x \in B$ and (r, θ) are polar coordinates on D^2.
(3) For each $\theta \in S^1, \pi^{-1}(\theta)$ is the interior of a compact codimension-one submanifold $N_\theta \subset M$ and $\partial N_\theta = B$. The submanifold $N = N_\theta$, for any θ, is called the page of the open book.

The fibration $\pi : M \setminus B \to S^1$ with fibre N is determined by N and the monodromy ϕ of the fibration up to conjugation by an orientation preserving diffeomorphism, that is, the fibre bundle $M \setminus B$ is canonically isomorphic to the mapping torus

$$\mathcal{MT}(N, \phi) = N \times [0, 1]/ \sim,$$

where $\sim$ is the equivalence relation identifying $(x, 1)$ with $(\varphi(x), 0)$. From the above definition, we can see that M is diffeomorphic to

$$\mathcal{MT}(N, \phi) \cup_{id} B \times D^2.$$

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Thus, an open book decomposition of M is determined, up to diffeomorphism of M, by the topological type of the page N and the isotopy class of the monodromy which is an element of the mapping class group of N. The mapping class group of a manifold N with nonempty boundary is the group of isotopy classes of orientation-preserving diffeomorphisms of N which are the identity in a collar neighbourhood of the boundary.

This naturally leads to the notion called an *abstract open book* defined as follows.

DEFINITION 2.2. An abstract open book associated with an n-manifold M is a pair (Σ, ϕ), where Σ is a compact connected oriented $(n-1)$-manifold with nonempty boundary and ϕ is an orientation-preserving diffeomorphism of Σ such that M is diffeomorphic to

$$M_{(\Sigma,\phi)} = \mathcal{MT}(\Sigma, \phi) \cup_{id} \partial\Sigma \times D^2,$$

where id denotes the identity map of $\partial\Sigma \times S^1$.

The map ϕ in the above definition is called the monodromy of the abstract open book. Note that the mapping class of ϕ determines M uniquely up to diffeomorphism. We will denote the manifold M with an abstract open book decomposition (Σ, ϕ) by $\mathcal{A}ob(\Sigma, \phi)$. One can easily see that given an abstract open book decomposition of M, we can clearly associate an open book decomposition of M with pages Σ and *vice versa*. We will not generally distinguish between open books and abstract open books. For more details on open books, see [2].

Let us recall the notion of an open book embedding.

DEFINITION 2.3. Let M^k and N^l be manifolds with open book decompositions (B_1, π_1) and (B_2, π_2), respectively. We say an embedding $f : M \hookrightarrow N$ is an open book embedding of (B_1, π_1) into (B_2, π_2) if f embeds B_1 into B_2 such that $\pi_2 \circ f = \pi_1$.

Similarly, we can also define an abstract open book embedding.

DEFINITION 2.4. Let $M = \mathcal{A}ob(\Sigma_1, \phi_1)$ and $N = \mathcal{A}ob(\Sigma_2, \phi_2)$ be two abstract open books. We say that there exists an abstract open book embedding of M into N if there exists a proper embedding f of Σ_1 into Σ_2 such that ϕ_1 is isotopic to $f^{-1} \circ \phi_2 \circ f$.

It is clear from the definition that an abstract open book embedding produces an embedding for the associated open book and *vice versa*.

3. Proof of Theorem 1.1

In this section, we discuss our proof of Theorem 1.1. Recall that given an embedded circle c in the interior of a surface Σ, a Dehn twist d_c (or d_c^{-1}) on Σ along the circle c is a diffeomorphism which is the identity outside a neighbourhood of c and is a full twist on an annular neighbourhood of c. We begin by stating the following lemma, which is proved in [2].

LEMMA 3.1. *Let M be a closed oriented 3-manifold. Let $M = \mathcal{A}ob(\Sigma, \phi)$. Suppose L is a knot sitting on a page Σ of the open book. Then, the new manifold M' obtained*

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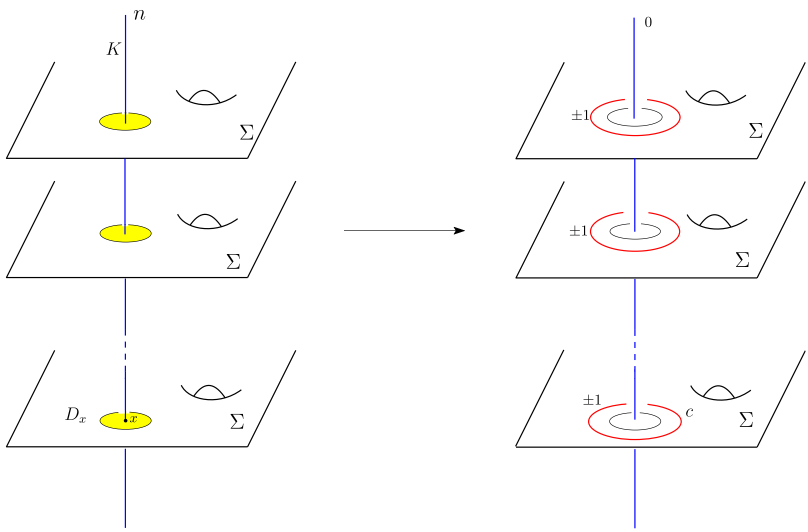

FIGURE 1. Blowing up operations preformed on the knot K.

by ± 1 *surgery along* L*, with respect to the page framing, admits an open book* $\mathcal{A}ob(\Sigma, \phi \circ d_L^{\mp 1})$ *with pages* Σ *and monodromy* $\phi \circ d_L^{\mp 1}$*.*

The following lemma will be needed in the proof of Theorem 1.1.

LEMMA 3.2. *Let* M *be a closed oriented* 3*-manifold. Let* $M = \mathcal{A}ob(\Sigma, \phi)$ *be an abstract open book of* M*. Suppose that* K *is a knot in* M *such that* $K = \{x\} \times [0, 1]/(x, 1) \sim (\phi(x), 0)$ *in* M *for some interior point* $x \in \Sigma$*. Let* D_x *be an open disc neighbourhood of* x *in* Σ *such that* $\phi|_{D_x} = id$ *and let* c *be the curve parallel to* ∂D_x *in* $\Sigma \setminus D_x$*. Then, an integer* n *surgery along* K *gives a new manifold* $M' = \mathcal{A}ob(\Sigma', \phi')$ *with an open book decomposition having the surgery dual of* K *as one of the binding components, where the surface* $\Sigma' = \Sigma \setminus D_x$ *is the page and the map* $\phi' = \phi|_{\Sigma'} \circ d_c^{-n}$ *is the monodromy of the open book of* M'*.*

PROOF. The knot K is transverse to each page of the open book $\mathcal{A}ob(\Sigma, \phi)$ and intersects each page exactly once (see Figure 1). Suppose $n = 0$. Then, performing 0 surgery along K is equivalent to removing D_x from each page and filling the resulting boundary $\partial D_x \times S^1$ of $M \setminus \mathcal{N}(K)$ by $\partial D_x \times D^2$ using the identity map. Here, $\mathcal{N}(K) = D_x \times K$ denotes a tubular neighbourhood of K in M. From this, one can easily see that the manifold M', obtained from M by performing 0 surgery along K, admits an open book with pages $\Sigma' = \Sigma \setminus D_x$ and monodromy $\phi' = \phi|_{\Sigma'}$.

Suppose that n is a positive integer. From blowing up operations (see Figure 1), one can see that performing integer n surgery along K is equivalent to performing -1 surgery along n copies of a circle c parallel to ∂D_x lying in n distinct pages of the open book $\mathcal{A}ob(\Sigma, \phi)$ and performing 0 surgery along K. Using Lemma 3.1, one can see that the manifold M' obtained from M by performing a positive integer

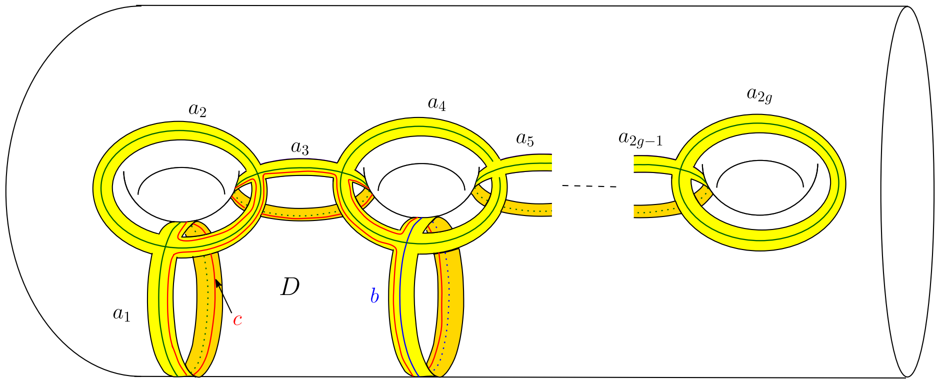

FIGURE 2. The Humphries generating curves $b, a_1, \ldots, a_{2g}$ on the surface Σ of genus g with connected boundary. The Dehn twist along these curves generates the mapping class group of Σ. The union of the regular neighbourhoods of the Humphries generating curves is a subsurface S of Σ with two boundary components. One of the boundary components of S is parallel to $\partial\Sigma$ and the other boundary component bounds a disc D in Σ.

n surgery along K admits an open book with pages $\Sigma' = \Sigma \setminus D_x$ and monodromy $\phi' = \phi|_{\Sigma'} \circ d_c^{-n}$.

Similarly, when n is a negative integer, we can see that the surgered manifold M' admits an open book with pages $\Sigma' = \Sigma \setminus D_x$ and monodromy $\phi' = \phi|_{\Sigma'} \circ d_c^{n}$. □

REMARK 3.3. Suppose that Σ' has an arc α joining the boundary component ∂D_x to some component of $\partial\Sigma$ such that ϕ fixes α pointwise. Then, the open book $\mathcal{A}ob(\Sigma', \phi')$ of M', obtained from M by ± 1 surgery along K, is just a stabilisation of $\mathcal{A}ob(\Sigma, \phi)$. In this case, M' is diffeomorphic to M.

Let M be a closed connected oriented 3-manifold. Recall that M admits an open book decomposition with connected binding (see [8]). Let $M = \mathcal{A}ob(\Sigma, \phi)$, where Σ is a compact connected oriented surface with connected boundary. By considering certain stabilisations of the abstract open book of $\mathcal{A}ob(\Sigma, \phi)$ if required, we can assume that Σ has the genus $g \geq 3$. The monodromy ϕ of the open book is an element in the mapping class group of Σ. Humphries [6] showed that the mapping class group of a closed surface $\tilde{\Sigma}$ of genus $g \geq 3$ is generated by the Dehn twist along $2g + 1$ closed curves $b, a_1, a_2, \ldots, a_{2g}$, as shown in Figure 2, where the closed surface $\tilde{\Sigma}$ is obtained from Σ by gluing a disc along the boundary. We call these curves $b, a_1, a_2, \ldots, a_{2g}$ in Σ the Humphries generating curves and the corresponding Dehn twists the Humphries generators of the mapping class group of Σ. Johnson [7] showed that the $2g + 1$ Dehn twists about $b, a_1, a_2, \ldots, a_{2g}$ on Σ also generate the mapping class group of Σ.

Consider a regular neighbourhood of each Humphries generating curve in Σ. The union S of the regular neighbourhoods of the Humphries generating curves can be considered as a surface obtained by plumbing $2g + 1$ annuli, as shown in Figure 2. The surface S has two boundary components. One of the boundary components of

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S is parallel to $\partial\Sigma$, that is, they bound an annulus A in Σ, and the other boundary component of S bounds a disc $D = \Sigma \setminus (S \cup A)$ in Σ, as shown in Figure 2. Note that $\Sigma \setminus D$ is diffeomorphic to S. The monodromy ϕ is supported in S and hence it is the identity on the disc D.

Let $x \in D$ be an interior point in $D \subset \Sigma$. Consider the knot $K = \{x\} \times [0,1]/(x,1) \sim (\phi(x), 0)$ in M. As discussed in Lemma 3.2, we get an open book $\mathcal{A}ob(\Sigma', \phi')$ of M' obtained by an integer n surgery along K in M. Here, $\Sigma' = S$ and $\phi' = \phi \circ d_c^{-n}$, where c is a curve in Σ' parallel to ∂D.

Now, we shall show that the open book $M' = \mathcal{A}ob(\Sigma', \phi')$ embeds into the trivial open book of S^5. We need to recall the following notion.

Definition 3.4. A diffeomorphism ϕ of a surface F is said to be a flexible diffeomorphism, with respect to a proper embedding f of Σ in a 4-manifold X, if there exists a diffeomorphism Φ of X isotopic to the identity map of X (also the identity near the boundary if X has nonempty boundary) such that ϕ is isotopic to $f^{-1} \circ \Phi \circ f$.

One can easily observe that if a diffeomorphism ϕ of F is isotopic to a diffeomorphism ψ of F and ϕ is flexible with respect to an embedding f of F in X, then ψ is also flexible in X with respect to f. To prove the existence of an (abstract) open book embedding of $\mathcal{A}ob(\Sigma', \phi')$ in the trivial open book of S^5, we need to find an embedding f of Σ' in D^4 such that ϕ' is flexible in D^4 with respect to f. As ϕ' is isotopic to a product of powers of Humphries generators and the Dehn twist d_c, it is enough to construct a proper embedding f of Σ' in D^4 such that the Humphries generators and the Dehn twist d_c are flexible in D^4 with respect to f.

3.1. Construction of an embedding of Σ' in D^4. A Dehn twist d_γ along an embedded circle γ in Σ' is supported in an annular neighbourhood of γ in Σ'. In [5], it is shown that there exists a (proper) embedding of an annulus $A = S^1 \times [0,1]$ in D^4 such that the Dehn twist along the central curve of A is flexible and hence so is each power of this Dehn twist. This follows from the fact that there exists a flow Φ_t on the 3-sphere S^3 associated to the open book decomposition of S^3 with pages a Hopf annulus and monodromy the Dehn twist along the central curve of the Hopf annulus such that the time 1 map Φ_1 on S^3 induces the Dehn twist along the central curve on each page of the open book of S^3. We can choose any Hopf annulus page $\mathcal{A}$ as an embedding of the annulus A in $S^3 = \partial D^4 \times \frac{1}{2} \subset \partial D^4 \times [0,1]$, where $\partial D^4 \times [0,1]$ is a collar of ∂D^4 in D^4. Using the flow Φ_t, we can construct a diffeomorphism of D^4 which is isotopic to the identity and induces the Dehn twist along the central curve on the embedded Hopf annulus $\mathcal{A}$. For more details, see [5, 9]. Note that S^3 admits an open book with pages a positive Hopf annulus as well as an open book with pages a negative Hopf annulus. So, we will construct a proper embedding of the surface Σ' in D^4 such that each of the curves $b, a_1, \ldots, a_{2g}$ and the boundary parallel curve c admits either a positive or a negative Hopf annulus regular neighbourhood in the embedded $\Sigma' \subset D^4$.

Consider a collar $S^3 \times [0,1]$ of the 4-ball D^4 with $\partial D^4 = S^3 \times 0$. Consider an embedding S_1 of the surface Σ', which is obtained by plumbing $2g$ positive Hopf

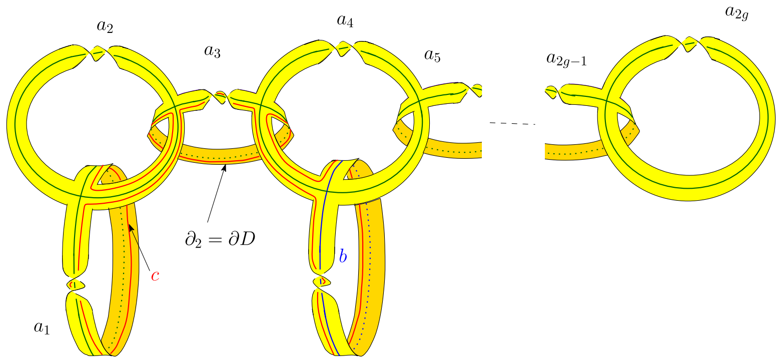

FIGURE 3. An embedding of the surface $\Sigma' = S$ in S^3, in which each Humphries generating curve as well as the boundary parallel curve c admits a Hopf annulus regular neighbourhood in S^3.

annuli $A_1, \ldots, A_{2g}$ with central curves $a_1, a_2, \ldots, a_{2g}$ and one negative Hopf annulus with central curve b in $S^3 \times \frac{1}{2}$, as shown in Figure 3. Recall that Σ' has two boundary components. We denote these components by $\partial_1 = \partial\Sigma$ and $\partial_2 = \partial D$. We make this embedding a proper embedding f of Σ' by attaching cylinders $\partial_i \times [0, \frac{1}{2}]$ to S_1 along ∂_i, for $i = 1, 2$, and smoothing out the corners to get a smooth embedding. By construction, we can see that a regular neighbourhood of each of the Humphries generating curves $a_1, \ldots, a_{2g}$ in $f(\Sigma')$ is a positive Hopf annulus in $S^3 \times \frac{1}{2} \subset D^4$ and a regular neighbourhood of the Humphries generating curve b in $f(\Sigma')$ is a negative Hopf annulus $S^3 \times \frac{1}{2}$. A regular neighbourhood of the curve c parallel to the boundary ∂_2 in $f(\Sigma')$ has two positive full twists and one negative full twist. Two of the twists cancel each other to get a positive Hopf annulus neighbourhood for the curve c in $S^3 \times \frac{1}{2}$ (see Figure 3). From this, one can see that the Dehn twists along the curves $c, b, a_1, \ldots, a_{2g}$ are flexible in D^4 with respect to the embedding f of Σ'. For more details regarding the flexibility of the Dehn twists, see [9]. An alternate argument for the flexibility of the Dehn twist along a simple closed curve in Σ' admitting a Hopf annulus regular neighbourhood in $S^3 \times \frac{1}{2}$ is given in [10]. For the sake of completeness, we summarise it here.

Observe that each curve $\alpha \in \{c, a_1, \ldots, a_{2g}\}$ bounds a disc D_α in D^4 which intersects $f(\Sigma')$ in α and a tubular neighbourhood $\mathcal{N}(D_\alpha) = D^2 \times D^2$ intersects $f(\Sigma')$ in a regular neighbourhood $\nu(\alpha)$ of α in $f(\Sigma')$. We can choose coordinates (z_1, z_2) on $\mathcal{N}(D_\alpha)$ such that $\nu(\alpha) = g^{-1}(1) \cap f(\Sigma')$, where $g : \mathbb{C}^2 \to \mathbb{C}$ is the map defined by $g(z_1, z_2) = z_1 z_2$. The monodromy of the singular fibration $g : \mathcal{N}(D_\alpha) = \mathbb{C}^2 \to \mathbb{C} = D^2$ with the singular point $(0, 0)$ is the positive Dehn twist d_α along the curve α which lies in the regular fibre $g^{-1}(1) \cap \Sigma' = \nu(\alpha)$. The isotopy of $\mathcal{N}(D_\alpha)$, which produces the positive Dehn twist d_α on $g^{-1}(1) \cap \Sigma'$, can be extended to an isotopy Ψ_s, $s \in [0, 1]$, of D^4 such that the isotopy is supported in the tubular neighbourhood $\mathcal{N}(D_\alpha)$. Then, we have $f^{-1} \circ \Psi_1 \circ f = d_\alpha$ up to isotopy. We have an isotopy of D^4 supported in the tubular neighbourhood $\mathcal{N}(D_\alpha)$ of D_α which produces the Dehn twist d_α on Σ'. From this, it follows that the Dehn twist d_α is flexible in D^4 with respect to the embedding f.

The curve b bounds a disc D_b in D^4 which intersects $f(\Sigma')$ in b and a tubular neighbourhood $\mathcal{N}(D_b) = D^2 \times D^2$ intersects $f(\Sigma')$ in a regular neighbourhood $\nu(b)$ of b in $f(\Sigma')$. We can choose coordinates (z_1, z_2) on $\mathcal{N}(D_b)$ such that $\nu(b) = g^{-1}(1) \cap f(\Sigma')$, where $g : \mathbb{C}^2 \to \mathbb{C}$ is the map defined by $g(z_1, z_2) = z_1\bar{z}_2$. The monodromy of the singular fibration $g : \mathcal{N}(D_b) = \mathbb{C}^2 \to \mathbb{C} = D^2$ with the singular point $(0, 0)$ is the negative Dehn twist d_b^{-1} along the curve b which lies in the regular fibre $g^{-1}(1) \cap \Sigma' = \nu(b)$. By arguments similar to those above, there is an isotopy of D^4 supported in the tubular neighbourhood $\mathcal{N}(D_b)$ of D_b which produces the Dehn twist d_b^{-1} on Σ'. Both d_b and d_b^{-1} are flexible in D^4 with respect to the embedding f.

It follows that the monodromy ϕ' is flexible in D^4 with respect to the embedding f of Σ', that is, there exists a diffeomorphism Φ of D^4 which is isotopic to the identity map of D^4 and $f^{-1} \circ \Phi \circ f$ is isotopic to ϕ'. This gives an abstract open book embedding of $M' = \mathcal{A}ob(\Sigma', \phi')$ into $S^5 = \mathcal{A}ob(D^4, \Phi)$. This completes the proof of Theorem 1.1.

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