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DOMAINS OF ATTRACTION FOR POSITIVE AND DISCRETE TEMPERED STABLE DISTRIBUTIONS

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Abstract

We introduce a large and flexible class of discrete tempered stable distributions, and analyze the domains of attraction for both this class and the related class of positive tempered stable distributions. Our results suggest that these are natural models for sums of independent and identically distributed random variables with tempered heavy tails, i.e. tails that appear to be heavy up to a point, but ultimately decay faster.

Keywords: Discrete tempered stable distribution; positive tempered stable distribution; domain of attraction

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1. Introduction

Stable distributions play a central role in many applications. However, their use is limited by the fact that they have an infinite variance, which is not realistic for most real-world applications. This has led to the development of tempered stable distributions, which is a class of models obtained by modifying the tails of stable distributions to make them lighter, while leaving their central portions, essentially, unchanged. Perhaps the earliest models of this type are Tweedie distributions, which were introduced in the seminal paper of Tweedie [35]. A more general approach, allowing for a wide variety of tail behavior, was given in Rosiński [28]. That approach was further generalized in several directions in [4], [12], and [29]. A survey, along with a historical overview and many references can be found in [13]. We will focus on the class of positive tempered stable (PTS) distributions. This class is important for many applications including actuarial science [16], biostatistics [26], mathematical finance [36], and computer science [6].

In a different direction, stable distributions have been modified to deal with over-dispersion when modelling count data. Specifically, the class of discrete stable distributions was introduced by Steutel and van Harn [32]; see also [9], [22], [23], and [33]. As with continuous stable distributions, these models have an infinite variance, which has led to the development of a tempered modification. In particular, Hougaard [18] introduced a class of models that has come to be known as Poisson–Tweedie. The name comes from the fact that these can be represented as a Poisson process subordinated by a Tweedie distribution. Many results along with applications to a variety of areas including economics, biostatistics, bibliometrics, and ecology can be found in, e.g. [1], [3], [10], [19], [20], [37], and the references therein.

In this paper we introduce a large class of discrete tempered stable (DTS) distributions, which generalize the class of Poisson–Tweedie models. We then prove limit theorems for PTS and

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DTS distributions, which can be thought of as characterizing their domains of attraction. Just as generalizations of the central limit theorem explain how stable and discrete stable distributions approximate sums of independent and identically distributed (i.i.d.) random variables with heavy tails, in our theorems we aim to provide a theoretical justification for the use of PTS and DTS models in approximating sums of i.i.d. random variables with tempered heavy tails, i.e. tails that appear to be heavy up to a point, but have been modified to, ultimately, decay faster. For a discussion of how such models occur in practice, see [6] and [15]. Related limit theorems for Poisson–Tweedie distributions can be found in [20]. In the continuous case, similar results for Tweedie distributions were studied in [14] and, from a different perspective, convergence of certain random walks to tempered stable distributions were studied in [7].

Before proceeding we introduce some notation. We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$, and we write $\mathcal{B}(\mathbb{R}_+)$ to denote the Borel sets on $\mathbb{R}_+$. For a probability measure μ with support contained in $\mathbb{R}_+$, we write $\hat{\mu}(z) = \int_{\mathbb{R}_+} \mathrm{e}^{-zx} \mu(\mathrm{d}x)$ to denote its Laplace transform and $X \sim \mu$ to denote that X is a random variable with distribution μ. For a function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $\beta \in \mathbb{R}$, we write $f \in \mathrm{RV}_\beta$ to denote that f is regularly varying with index β, i.e. that

$$\lim_{t\to\infty} \frac{f(xt)}{f(t)} = x^\beta \quad \text{for any } x > 0.$$

We write $\mathbf{1}_A$ to denote the indicator function on set A, for $x > 0$ we write $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1}\, \mathrm{d}t$ to denote the gamma function, and we write '$\xrightarrow{\mathbb{P}}$', '$\xrightarrow{\mathrm{D}}$', '$\xrightarrow{\mathrm{W}}$', and '$\stackrel{\mathrm{D}}{=}$' to denote, respectively, convergence in probability, convergence in distribution, weak convergence, and equality in distribution. For $c \in [-\infty, \infty]$, we write $f(t) \sim g(t)$ as $t \to c$ to denote $\lim_{t\to c} f(x)/g(x) = 1$.

2. Positive stable and PTS distributions

In this section we formally introduce positive stable and PTS distributions. We begin by recalling some basic facts about positive infinitely divisible distributions. An infinitely divisible distribution μ, with support contained in $\mathbb{R}_+$, has a Laplace transform of the form

$$\hat{\mu}(z) = \exp\left\{-bz - \int_{(0,\infty)} (1 - \mathrm{e}^{-zx}) M(\mathrm{d}x)\right\}, \qquad z \ge 0,$$

where $b \ge 0$ and M is a Borel measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (x \wedge 1) M(\mathrm{d}x) < \infty. \tag{1}$$

Here, b is called the drift and M is called the Lévy measure. These parameters uniquely determine the distribution and we write $\mu = \mathrm{ID}_+(M, b)$. For a general reference on infinitely divisible distributions, see [30].

A probability measure μ on $\mathbb{R}_+$ is said to be strictly α-stable if, for any $n \in \mathbb{N}$ and $X_1, X_2, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \mu$, we have

$$X_1 \stackrel{\mathrm{D}}{=} n^{-1/\alpha}(X_1 + X_2 + \cdots + X_n). \tag{2}$$

By positivity, we necessarily have $\alpha \in (0, 1]$. If $\alpha \in (0, 1)$ then $\mu = \mathrm{ID}_+(M_\alpha, 0)$, where

$$M_\alpha(\mathrm{d}x) = \eta x^{-1-\alpha}\, \mathbf{1}_{\{x>0\}}\, \mathrm{d}x \quad \text{for some } \eta \ge 0.$$

If $\alpha = 1$ then $\mu = \text{ID}_+(0, \eta)$ for some $\eta \geq 0$, and, thus, μ is a point mass at η. For $\alpha \in (0, 1)$, the Laplace transform is of the form

$$\hat{\mu}(z) = \exp\bigg(-\eta \frac{\Gamma(1-\alpha)}{\alpha} z^\alpha\bigg), \qquad z \geq 0.$$

We denote this distribution by $\text{PS}_\alpha(\eta)$. Note that $\text{PS}_\alpha(0)$ is a point mass at 0 for all $\alpha \in (0, 1)$. For more about stable distributions on $\mathbb{R}_+$, see [33].

It is well known that, for $\alpha \in (0, 1)$, stable distributions have an infinite mean, which is not realistic for many applications. This has lead to the development of distributions that look stable-like in some large central region, but with lighter tails. Following [29], we define PTS distributions as follows.

Definition 1. A distribution $\mu = \text{ID}_+(M, b)$ is called a PTS distribution if $b \geq 0$ and

$$M(\mathrm{d}x) = \eta q(x) x^{-1-\alpha} \mathbf{1}_{\{x>0\}}\, \mathrm{d}x,$$

where $\alpha \in (0, 1), \eta \geq 0$, and $q\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a bounded, nonnegative, Borel function satisfying

$$\lim_{x\downarrow 0} q(x) = 1, \tag{3}$$

and

$$\int_0^\infty (1 \wedge x) q(x) x^{-1-\alpha}\, \mathrm{d}x < \infty. \tag{4}$$

We call q the tempering function and we write $\mu = \text{PTS}_\alpha(q, \eta, b)$. When $b = 0$, we write $\text{PTS}_\alpha(q, \eta) = \text{PTS}_\alpha(q, \eta, 0)$.

Remark 1. In the definition, we require (3) for concreteness. In principle, it can be replaced by the condition $\lim_{x\downarrow 0} q(x) = c \in (0, \infty)$. However, in this case, we can reparametrize by incorporating c into η. On the other hand, if we allow the limit to equal 0 or ∞, then we would lose any relation to the original stable distribution. The requirement that $\alpha < 1$ follows from the fact that, under (3), these are the only possible values for which (4) can hold.

We are motivated by the case where the tempering function q satisfies the additional condition that $\lim_{x\to\infty} q(x) = 0$. In this case, $\text{PTS}_\alpha(q, \eta)$ is similar to $\text{PS}_\alpha(\eta)$ in some central region, but with lighter tails. In this sense, q 'tempers' the tails of the stable distribution. Despite this motivation, none of our results require this additional condition. We now provide several examples of tempering functions, others can be found in, e.g. [13] and [34].

Example 1. (i) When $q \equiv 1$, there is no tempering and $\text{PTS}_\alpha(q, \eta) = \text{PS}_\alpha(\eta)$.

(ii) When $q(x) = \mathrm{e}^{-ax}$ for some $a > 0$, we have the class of Tweedie distributions, which were introduced in [35]. When $\alpha = 0.5$, these correspond to inverse Gaussian distributions; see, e.g. [31].

(iii) When $q(x) = \mathbf{1}_{\{0\leq x<a\}}$ for some $a > 0$, we call this truncation. Such distributions are important for certain limit theorems; see [8].

3. Discrete stable and DTS distributions

A discrete analogue of stable distributions was introduced in [32]. Here (2) is modified to ensure that the right-hand side remains an integer. Specifically, [32] introduced the so-called

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'thinning' operation '$\circ$', which is defined as follows. If $\gamma \in (0, 1]$ and X is a random variable with support contained in $\mathbb{Z}_+$, then $\gamma \circ X$ is a random variable with distribution

$$\gamma \circ X \stackrel{\mathrm{D}}{=} \sum_{i=1}^{X} \varepsilon_i,$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. random variables independent of X having a Bernoulli distribution with $\mathbb{P}(\varepsilon_i = 1) = 1 - \mathbb{P}(\varepsilon_i = 0) = \gamma$. Here and throughout, we set $\sum_{i=1}^{0} \varepsilon_i = 0$. Note that, if $P(s)$ is the probability generating function (PGF) of X, i.e. $P(s) = \mathbb{E}[s^X]$, then the PGF of $\gamma \circ X$ is $P(1 - \gamma + \gamma s)$.

For $\alpha \in (0, 1]$, a distribution μ on $\mathbb{Z}_+$ is called discrete α-stable if, for any $n \in \mathbb{N}$, we have

$$X_1 \stackrel{\mathrm{D}}{=} n^{-1/\alpha} \circ (X_1 + X_2 + \cdots + X_n), \tag{5}$$

where $X_1, X_2, \ldots \stackrel{\text{i.i.d.}}{\sim} \mu$. The class of discrete 1-stable distributions coincides with the class of Poisson distributions. For $\alpha \in (0, 1)$, the PGF of a discrete stable distribution is of the form

$$\int_{\mathbb{Z}_+} s^x \mu(\mathrm{d}x) = \exp\biggl(-\eta \frac{\Gamma(1-\alpha)}{\alpha}(1-s)^\alpha\biggr), \qquad |s| \le 1,$$

where $\eta \ge 0$ is a parameter. We denote this distribution by $\mathrm{DS}_\alpha(\eta)$. A useful representation of discrete stable distributions is given in Theorem 6.7 of [33, p. 371]. It is as follows.

Proposition 1. *Fix* $\alpha \in (0, 1)$ *and* $\eta \ge 0$. *If* $\{N_t : t \ge 0\}$ *is a Poisson process with rate* 1 *and* $T \sim \mathrm{PS}_\alpha(\eta)$ *is independent of this process, then* $N_T \sim \mathrm{DS}_\alpha(\eta)$.

By analogy, we define DTS distributions as follows.

Definition 2. Fix $\alpha \in (0, 1)$ and $\eta \ge 0$. Let $T \sim \mathrm{PTS}_\alpha(q, \eta)$ and let $\{N_t : t \ge 0\}$ be a Poisson process with rate 1 independent of T. The distribution of N_T is called a DTS distribution. We denote this distribution by $\mathrm{DTS}_\alpha(q, \eta)$.

By a simple conditioning argument, the PGF of $\mathrm{DTS}_\alpha(q, \eta)$ is, for $s \in (0, 1]$,

$$\mathbb{E}[s^{N_T}] = \mathbb{E}[\mathrm{e}^{-(1-s)T}] = \exp\biggl\{-\eta \int_{(0,\infty)} (1 - \mathrm{e}^{-(1-s)x}) q(x) x^{-1-\alpha}\,\mathrm{d}x\biggr\}. \tag{6}$$

Remark 2. There are two simple ways to generalize Definition 2. The first is to allow the rate of the Poisson process to be $r > 0$ not necessarily 1. However, in this case, the distribution of N_T is $\mathrm{DTS}_\alpha(q_r, r^\alpha \eta)$, where $q_r(x) = q(x/r)$. The second is to allow $T \sim \mathrm{PTS}_\alpha(q, \eta, b)$ with $b > 0$. In this case, the distribution of N_T is the convolution of $\mathrm{DTS}_\alpha(q, \eta)$ and a Poisson distribution with mean b.

We can consider the same tempering functions as for PTS distributions. This leads to the following examples.

Example 2. (i) When $q \equiv 1$, we have $\mathrm{DTS}_\alpha(q, \eta) = \mathrm{DS}_\alpha(\eta)$.

(ii) When $q(x) = \mathrm{e}^{-ax}$ for $a > 0$, the corresponding distributions are Poisson–Tweedie. When $\alpha = 0.5$, these correspond to Poisson inverse Gaussian distributions, which were introduced in [17].

(iii) When $q(x) = \mathbf{1}_{\{0 \le x < a\}}$ for $a > 0$, we are in the case of truncation.

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We conclude this section by showing that we can approximate certain rescaled versions of DTS distributions by PTS distributions. The idea is motivated by [22], where similar results for certain generalizations of discrete stable distributions were presented. Let q be a tempering function. For any $a > 0$, define $X_a \sim \mathrm{DTS}_\alpha(q_{1/a}, a^{-\alpha}\eta)$, where $q_{1/a}(x) = q(ax)$. Since X_a is defined on $\mathbb{Z}_+$, aX_a is defined on $a\mathbb{Z}_+ = \{0, a, 2a, \dots\}$.

Proposition 2. *We have*

$$aX_a \xrightarrow{\mathrm{D}} \mathrm{PTS}_\alpha(q, \eta) \quad \text{as } a \downarrow 0.$$

Proof. From (6), it follows that the Laplace transform of aX_a is given, for $z \geq 0$, by

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-zaX_a}] &= \exp\bigg\{-a^{-\alpha}\eta \int_{(0,\infty)} (1 - \mathrm{e}^{-(1-\mathrm{e}^{-za})x}) q(ax) x^{-1-\alpha}\,\mathrm{d}x\bigg\} \\ &= \exp\bigg\{-\eta \int_{(0,\infty)} \bigg(1 - \exp\bigg(-\frac{(1-\mathrm{e}^{-za})}{a}x\bigg)\bigg) q(x) x^{-1-\alpha}\,\mathrm{d}x\bigg\} \\ &\to \exp\bigg\{-\eta \int_{(0,\infty)} (1 - \mathrm{e}^{-zx}) q(x) x^{-1-\alpha}\,\mathrm{d}x\bigg\} \quad \text{as } a \downarrow 0.\end{aligned}$$

Here the convergence follows by the facts that $(1 - \mathrm{e}^{-za})/a \to z$,

$$\bigg(1 - \exp\bigg(-\frac{(1-\mathrm{e}^{-za})}{a}x\bigg)\bigg) \leq 1 \wedge \frac{(1-\mathrm{e}^{-za})x}{a} \leq 1 \wedge (zx) \leq (z+1)(1 \wedge x),$$

and dominated convergence. □

4. Main results

Let μ be a probability measure on $\mathbb{R}_+$ such that, for $t > 0$,

$$\mu(\{x : x > t\}) = t^{-\alpha} L(t) \quad \text{for some } \alpha \in (0,1) \text{ and } L \in \mathrm{RV}_0.$$

Let

$$V(t) = \frac{t^\alpha}{L(t)} \quad \text{and} \quad a_t = \frac{1}{V^{\leftarrow}(t)},$$

where $V^{\leftarrow}(t) = \inf\{s : V(s) > t\}$ is the generalized inverse of V satisfying

$$V(V^{\leftarrow}(t)) \sim V^{\leftarrow}(V(t)) \sim t \quad \text{as } t \to \infty;$$

see [5]. Note that $a_t \in \mathrm{RV}_{-1/\alpha}$ and, thus, $a_n \to 0$ as $n \to \infty$. The following lemma is well known, but, for completeness, its proof is given in Section 6.

Lemma 1. *If* $X_1, X_2, \dots \overset{\text{i.i.d.}}{\sim} \mu$ *then*

$$a_n \sum_{i=1}^n X_i \xrightarrow{\mathrm{D}} \mathrm{PS}_\alpha(\alpha).$$

We now consider the effect of tempering on this result. Let q be a tempering function and, for $\ell > 0$, define

$$q_\ell(x) = q\bigg(\frac{x}{\ell}\bigg) \quad \text{and} \quad \mu_\ell(\mathrm{d}x) = c_\ell q_\ell(x) \mu(\mathrm{d}x),$$

where

$$c_\ell = \left[\int_{[0,\infty)} q_\ell(x)\mu(\mathrm{d}x)\right]^{-1}$$

is a normalizing constant. Note that, as $\ell \to \infty$, we have $c_\ell \to 1$ and $\mu_\ell \xrightarrow{\mathrm{w}} \mu$. Thus, for large ℓ, μ_ℓ is close to μ in some central region, but, if $\lim_{x\to\infty} q_\ell(x) = 0$ then it has lighter tails. In this sense, we interpret μ_ℓ as a tempered version of μ.

Example 3. (i) When $q \equiv 1$, there is no tempering and $\mu_\ell = \mu$ for each $\ell > 0$.

(ii) When $q(x) = \mathrm{e}^{-ax}$ for some $a > 0$, we have $q_\ell(x) = \mathrm{e}^{-ax/\ell}$. Thus, μ_ℓ is an Esscher transform of μ.

(iii) When $q(x) = \mathbf{1}_{\{0\le x<a\}}$ for some $a > 0$, we have $q_\ell(x) = \mathbf{1}_{\{0\le x<a\ell\}}$. Thus, μ_ℓ is μ truncated at $a\ell$. This means that if $X \sim \mu$ then μ_ℓ is the conditional distribution of X given the event $[X < a\ell]$.

Examples 3(ii) and 3(iii) lead to different modifications of μ which, for large values of ℓ, are similar to μ in some central portion, but have lighter tails. We now state our main result for convergence to PTS distributions. It characterizes the corresponding domain of attraction.

Theorem 1. *Let $\{\ell_n\}$ be a sequence of positive numbers with $\ell_n \to \infty$, let $X_{n1}, X_{n2}, \dots, X_{nn} \overset{\text{i.i.d.}}{\sim} \mu_{\ell_n}$ for each $n \in \mathbb{N}$, and let D be the set of discontinuities of q. Assume that the Lebesgue measure of D is* 0. *If $a_n\ell_n \to c \in (0,\infty)$ then*

$$a_n \sum_{i=1}^{n} X_{ni} \xrightarrow{\mathbb{P}} \mathrm{PTS}_\alpha(q_c, \alpha),$$

where $q_c(x) = q(x/c)$ for $x \ge 0$. If $a_n\ell_n \to \infty$ then

$$a_n \sum_{i=1}^{n} X_{ni} \xrightarrow{\mathbb{P}} \mathrm{PS}_\alpha(\beta) \tag{7}$$

with $\beta = \alpha$. If $a_n\ell_n \to 0$ and $\lim_{x\to\infty} q(x) = \zeta < \infty$, then (7) holds with $\beta = \alpha\zeta$.

Proof. The proof can be found in Section 6. □

Remark 3. For most applications, the parameter ℓ is not actually approaching ∞. Instead, it is some fixed but (very) large constant. Since $a_t \in \mathrm{RV}_{-1/\alpha}$, we can write $a_n\ell = [n^{-1}\ell^\alpha]^{1/\alpha}L'(n)$ for some $L' \in \mathrm{RV}_0$. Now, consider the sum of n i.i.d. random variables from μ_ℓ, and assume that the tempering function q is such that μ_ℓ has a finite variance. Theorem 1 can be interpreted as follows. When n is of the order of ℓ^α, the distribution of the sum is close to $\mathrm{PTS}_\alpha(q_c, \alpha)$. However, once n is much larger than ℓ^α, the central limit theorem will take effect and the distribution of the sum will be well approximated by the Gaussian. A constant that determines when such regimes occur is called the 'natural scale' in [15]. Thus, in this case, the natural scale is ℓ^α. Using slightly different perspectives, this was previously found to be the natural scale for Tweedie distributions in [14] and [15]. In particular, the result of [15] is based on the so-called pre-limit theorem, which was first introduced in [24].

The following transfer lemma allows us to transfer convergence results from the case of multiplicative scaling to that of scaling using the thinning operation '$\circ$'. It is an extension of a remark in [32].

Lemma 2. *Let* $\{X_n\}$ *be a sequence of random variables on* $\mathbb{Z}_+$ *and assume that* $\{\gamma_n\}$ *is a deterministic sequence in* $(0, 1]$ *with* $\gamma_n \to 0$*. If* $\gamma_n X_n \stackrel{\mathrm{D}}{\to} X$ *for some random variable* X *then*

$$\gamma_n \circ X_n \stackrel{\mathrm{D}}{\to} N_X,$$

where $\{N_t : t \geq 0\}$ *is a Poisson process with rate* 1 *and independent of* X*.*

Proof. This is a special case of Lemma 4, which is given below. □

Combining this with Lemma 1 provides the following lemma.

Lemma 3. *Assume that the support of* μ *is contained in* $\mathbb{Z}_+$*. If* $X_1, X_2, \ldots \stackrel{\text{i.i.d.}}{\sim} \mu$ *then*

$$a_n \circ \sum_{i=1}^{n} X_i \stackrel{\mathrm{D}}{\to} \mathrm{DS}_\alpha(\alpha).$$

Now combining Theorem 1 with Lemma 2, we have our main result for convergence to DTS distributions. It characterizes the corresponding domain of attraction.

Theorem 2. *Assume that the support of* μ *is contained in* $\mathbb{Z}_+$*. Let* $\{\ell_n\}$ *be a sequence of positive numbers with* $\ell_n \to \infty$*, let* $X_{n1}, X_{n2}, \ldots, X_{nn} \stackrel{\text{i.i.d.}}{\sim} \mu_{\ell_n}$ *for each* $n \in \mathbb{N}$*, and let* D *be the set of discontinuities of* q*. Assume that the Lebesgue measure of* D *is* 0*. If* $a_n \ell_n \to c \in (0, \infty)$ *then*

$$a_n \circ \sum_{i=1}^{n} X_{ni} \stackrel{\mathbb{P}}{\to} \mathrm{DTS}_\alpha(q_c, \alpha),$$

where $q_c(x) = q(x/c)$ *for* $x \geq 0$*. If* $a_n \ell_n \to \infty$ *then*

$$a_n \circ \sum_{i=1}^{n} X_{ni} \stackrel{\mathbb{P}}{\to} \mathrm{DS}_\alpha(\beta) \tag{8}$$

with $\beta = \alpha$*. If* $a_n \ell_n \to 0$ *and* $\lim_{x\to\infty} q(x) = \zeta < \infty$*, then (8) holds with* $\beta = \alpha\zeta$*.*

5. Generalization

In this section we extend our results to a more general framework. Let $\mathcal{P} = \{\mu_\theta : \theta \in \Theta\}$ be a parametric family of nonnegative random variables. If X is a random variable on $\mathbb{Z}_+$ and $\theta \in \Theta$, then we define $\theta \circ_{\mathcal{P}} X$ to be a random variable with distribution

$$\theta \circ_{\mathcal{P}} X \stackrel{\mathrm{D}}{=} \sum_{i=1}^{X} \varepsilon_{i,\theta},$$

where $\varepsilon_{1,\theta}, \varepsilon_{2,\theta}, \ldots \stackrel{\text{i.i.d.}}{\sim} \mu_\theta$ are independent of X. By a conditioning argument, it is not difficult to show that the Laplace transform of $\theta \circ_{\mathcal{P}} X$ is given by

$$L_X(-\log \hat{\mu}_\theta(z)) \quad \text{for } z \geq 0, \tag{9}$$

where L_X is the Laplace transform of X. Now assume that $\Theta \subset (0, \infty)$ and there is a sequence $\{\theta_n\}$ in Θ with $\theta_n \to 0$ as $n \to \infty$. We write $\theta \to 0$ to mean convergence along any such subsequence. Assume further that there is an infinitely divisible distribution μ with support contained in $\mathbb{R}_+$ such that

$$\lim_{\theta\to 0} \frac{\log(\hat{\mu}_\theta(z))}{\theta} \to \log(\hat{\mu}(z)) \quad \text{for } z \geq 0.$$

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Note that the operator '$\circ$' corresponds to the case where $\Theta = (0, 1]$, $\hat{\mu}_\theta(s) = (1 - \theta + \theta \mathrm{e}^{-z})$, and $\hat{\mu}(x) = \mathrm{e}^{\mathrm{e}^{-z}-1}$.

Definition 3. Fix $\alpha \in (0, 1)$, $\eta \geq 0$, and let $Z = \{Z_t : t \geq 0\}$ be a Lévy process, where $Z_1 \sim \mu$. If $T_1 \sim \mathrm{PS}_\alpha(\eta)$ is independent of Z then the distribution of Z_{T_1} is denoted by $S_\alpha^{\mathcal{P}}(\eta)$. Similarly, if $T_2 \sim \mathrm{PTS}_\alpha(q, \eta)$ is independent of Z then the distribution of Z_{T_2} is denoted by $\mathrm{TS}_\alpha^{\mathcal{P}}(q, \eta)$.

We now state the main result of this section.

Lemma 4. *Let $\{X_n\}$ be a sequence of random variables on $\mathbb{Z}_+$ and assume that $\{\theta_n\}$ is a deterministic sequence in Θ with $\theta_n \to 0$. If $\theta_n X_n \xrightarrow{\mathrm{D}} X$ for some random variable X then*

$$\theta_n \circ_{\mathcal{P}} X_n \xrightarrow{\mathrm{D}} Z_X,$$

where $\{Z_t : t \geq 0\}$ is a Lévy process with $Z_1 \sim \mu$ and independent of X.

Proof. The proof can be found in Section 6. □

From here we can immediately obtain analogues of Lemma 3 and Theorem 2 for this situation, i.e. in those results we can replace the operator '$\circ$' with '$\circ_{\mathcal{P}}$' and the distributions $\mathrm{DS}_\alpha(\eta)$ and $\mathrm{DTS}_\alpha(q, \eta)$ with $S_\alpha^{\mathcal{P}}(\eta)$ and $\mathrm{TS}_\alpha^{\mathcal{P}}(q, \eta)$, respectively. Note that, if $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} S_\alpha^{\mathcal{P}}(\eta)$ then

$$n^{-1/\alpha} \circ_{\mathcal{P}} (X_1 + X_2 + \cdots + X_n) \xrightarrow{\mathrm{D}} X_1 \quad \text{as } n \to \infty.$$

However, the distributions $S_\alpha^{\mathcal{P}}(\eta)$ do not, in general, satisfy a stability relation of the form (5). For such a relation to hold, we must place stronger assumptions of the parametric family $\mathcal{P}$. Several interesting examples, where such assumptions hold, are discussed in [23].

Remark 4. If μ_θ is infinitely divisible for every $\theta \in \Theta$ then we can extend the operation '$\circ_{\mathcal{P}}$' to every nonnegative random variable X. Specifically, we let $\theta \circ_{\mathcal{P}} X$ be a random variable with distribution having Laplace transform as in (9). In this case, the result of Lemma 4 still holds. However, if μ_θ is not infinitely divisible then we can always find a nonnegative random variable X for which (9) does not define a valid Laplace transform. A simple example is to let $\mathbb{P}(X = t) = 1$ for any $t > 0$ for which $[\hat{\mu}_\theta(z)]^t$ is not a valid Laplace transform.

6. Proofs

The proofs of Lemma 1 and Theorem 1 are based on verifying conditions for the convergence of sums of a triangular array. The general theory can be found in, e.g. [21] or [25]. However, when all random variables are positive, the conditions can be simplified. In this case, they can be stated as follows.

Proposition 3. *Let k_n be a sequence of positive integers with $k_n \to \infty$, let M be a Borel measure on $(0, \infty)$ satisfying (1), and let $\{X_{nm} : n = 1, 2, \ldots, m = 1, 2, \ldots, k_n\}$ be nonnegative random variables such that, for every n, the random variables $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ are i.i.d. and $X_{n1} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. If, for every $s > 0$ with $M(\{s\}) = 0$, we have*

$$\lim_{n\to\infty} k_n \mathbb{P}(X_{n1} > s) = M((s, \infty)) \tag{10}$$

and

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} k_n \mathbb{E}[X_{n1} \mathbf{1}_{\{X_{n1} < \varepsilon\}}] = 0, \tag{11}$$

then

$$\sum_{m=1}^{k_n} X_{nm} \xrightarrow{\mathrm{D}} \mathrm{ID}_+(M,0).$$

Proof. Let ν_n be the distribution of X_{n1}, let $\nu = \mathrm{ID}_+(M,0)$, and let $\hat{\nu}_n(u) = \int_{\mathbb{R}_+} \mathrm{e}^{-ux}\nu_n(\mathrm{d}x)$ and $\hat{\nu}(u) = \int_{\mathbb{R}_+} \mathrm{e}^{-ux}\nu(\mathrm{d}x)$ be the Laplace transforms of ν_n and ν, respectively. The Laplace transform of the distribution of $\sum_{m=1}^{k_n} X_{nm}$ is $[\hat{\nu}_n(u)]^{k_n}$. We need to show that

$$\lim_{n\to\infty}[\hat{\nu}_n(u)]^{k_n} = \hat{\nu}(u), \qquad u \ge 0.$$

We will write the left-hand side in a simpler form. Specifically, we have

$$\begin{aligned}\lim_{n\to\infty}[\hat{\nu}_n(u)]^{k_n} &= \lim_{n\to\infty}\exp\{k_n\log[\hat{\nu}_n(u)]\}\\ &= \lim_{n\to\infty}\exp\{k_n[\hat{\nu}_n(u)-1]\}\\ &= \lim_{n\to\infty}\exp\biggl(-k_n\int_{[0,\infty)}(1-\mathrm{e}^{-ux})\nu_n(\mathrm{d}x)\biggr),\end{aligned}$$

where the second equality follows from the facts that $\log(x) \sim (x-1)$ as $x \to 1$ and $\lim_{n\to\infty}\hat{\nu}_n(u) = 1$ for each $u \ge 0$ since $X_{n1} \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$.

Since, for fixed u, $f_u(x) = (1-\mathrm{e}^{-ux})$ is a bounded and continuous function of x, by the Portmanteau theorem for vague convergence (see Theorem 1 of [2]), (10) implies that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} k_n\int_{[\varepsilon,\infty)}(1-\mathrm{e}^{-ux})\nu_n(\mathrm{d}x) = \int_{[\varepsilon,\infty)}(1-\mathrm{e}^{-ux})M(\mathrm{d}x).$$

By (11) and well-known facts about the exponential function, we have

$$\begin{aligned}0 &\le \lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty} k_n\int_{[0,\varepsilon)}(1-\mathrm{e}^{-ux})\nu_n(\mathrm{d}x)\\ &\le \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} k_n\int_{[0,\varepsilon)}(1-\mathrm{e}^{-ux})\nu_n(\mathrm{d}x)\\ &\le \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} uk_n\int_{[0,\varepsilon)}x\nu_n(\mathrm{d}x)\\ &= 0.\end{aligned}$$

Combining the above with Lebesgue's dominated convergence theorem yields

$$\begin{aligned}&\liminf_{n\to\infty} k_n\int_{[0,\infty)}(1-\mathrm{e}^{-ux})\nu_n(\mathrm{d}x)\\ &\quad= \lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty} k_n\int_{[0,\varepsilon)}(1-\mathrm{e}^{-ux})\nu_n(\mathrm{d}x) + \lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty} k_n\int_{[\varepsilon,\infty)}(1-\mathrm{e}^{-ux})\nu_n(\mathrm{d}x)\\ &\quad= \lim_{\varepsilon\downarrow 0}\int_{[\varepsilon,\infty)}(1-\mathrm{e}^{-ux})M(\mathrm{d}x)\\ &\quad= \int_{(0,\infty)}(1-\mathrm{e}^{-ux})M(\mathrm{d}x).\end{aligned}$$

Similarly, we can repeat the above with lim sup in place of lim inf. Then, combining these results, we have

$$\lim_{n\to\infty}[\hat{\nu}_n(u)]^{k_n} = \lim_{n\to\infty}\exp\biggl(-k_n\int_{[0,\infty)}(1-\mathrm{e}^{-ux})\nu_n(\mathrm{d}x)\biggr)$$
$$= \exp\biggl(-\int_{(0,\infty)}(1-\mathrm{e}^{-ux})M(\mathrm{d}x)\biggr),$$

which is the Laplace transform of $\mathrm{ID}_+(M, 0)$, as required. □

Before proceeding, we define the Borel measures

$$M_n(A) = n\int_{[0,\infty)}\mathbf{1}_A(a_n x)\mu(\mathrm{d}x), \qquad A \in \mathcal{B}(\mathbb{R}_+),$$

and

$$M_\infty(A) = \int_{[0,\infty)}\mathbf{1}_A(x)\alpha x^{-\alpha-1}\mathrm{d}x, \qquad A \in \mathcal{B}(\mathbb{R}_+).$$

Note that M_∞ is the Lévy measure of the distribution $\mathrm{PS}_\alpha(\alpha)$.

Lemma 5. *The following hold:*

$$\lim_{n\to\infty} M_n((s,\infty)) = M_\infty((s,\infty)), \quad s > 0, \qquad \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{[0,\varepsilon)} x M_n(\mathrm{d}x) = 0.$$

Proof. We have, for $s > 0$,

$$\lim_{n\to\infty} M_n((s,\infty)) = \lim_{n\to\infty} V(V^{\leftarrow}(n))\mu\biggl(\biggl(\frac{s}{a_n},\infty\biggr)\biggr)$$
$$= \lim_{n\to\infty} V\biggl(\frac{1}{a_n}\biggr)\frac{1}{V(s/a_n)}$$
$$= s^{-\alpha}$$
$$= M_\infty((s,\infty))$$

and, recalling that $\alpha \in (0, 1)$ yields

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{[0,\varepsilon)} x M_n(\mathrm{d}x) = \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} n a_n \int_{[0,\varepsilon/a_n)} x\mu(\mathrm{d}x)$$
$$= \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} V\biggl(\frac{1}{a_n}\biggr) a_n \int_{[0,\varepsilon/a_n)} x\mu(\mathrm{d}x)$$
$$= \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} \varepsilon^{-\alpha} V\biggl(\frac{\varepsilon}{a_n}\biggr) a_n \int_{[0,\varepsilon/a_n)} x\mu(\mathrm{d}x)$$
$$= \lim_{\varepsilon\downarrow 0}\varepsilon^{1-\alpha}\limsup_{n\to\infty}\frac{\int_{[0,\varepsilon/a_n)} x\mu(\mathrm{d}x)}{(\varepsilon/a_n)\int_{(\varepsilon/a_n,\infty)}\mu(\mathrm{d}x)}$$
$$= \lim_{\varepsilon\downarrow 0}\varepsilon^{1-\alpha}\frac{\alpha}{1-\alpha}$$
$$= 0,$$

where the first convergence follows by Theorem 2 of [11, p. 283]. □

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Proof of Lemma 1. Since $a_n \to 0$, Slutsky's theorem implies that $a_n X_1 \xrightarrow{\mathbb{P}} 0$. From here, the result follows by combining Lemma 5 with Proposition 3. □

Lemma 6. *Let $h, h_1, h_2, \dots$ be a sequence of Borel functions and let D be a Borel set with Lebesgue measure 0 such that, for any $x \in D^c$ and any sequence of real numbers $x_1, x_2, \dots$ with $x_n \to x$, we have $h_n(x_n) \to h(x)$. Then, for any $s > 0$,*

$$\lim_{n\to\infty} \int_{(s,\infty)} h_n(x) M_n(\mathrm{d}x) = \int_{(s,\infty)} h(x) M_\infty(\mathrm{d}x).$$

Proof. Fix $s > 0$. Let $m_n = M_n((s,\infty))$, $m_\infty = M_\infty((s,\infty))$, and define the probability measures $M_n^{(s)}(\mathrm{d}x) = m_n^{-1} \mathbf{1}_{\{x>s\}} M_n(\mathrm{d}x)$ and $M_\infty^{(s)}(\mathrm{d}x) = m_\infty^{-1} \mathbf{1}_{\{x>s\}} M_\infty(\mathrm{d}x)$. From Lemma 5 and the Portmanteau theorem, it follows that $M_n^{(s)} \xrightarrow{\mathrm{w}} M_\infty^{(s)}$. Further, since $M_\infty^{(s)}$ is absolutely continuous with respect to the Lebesgue measure, it follows that $M_\infty^{(s)}(D) = 0$. From here, a standard result about weak convergence, see, e.g. Example 32 of [27, p. 58], implies that

$$\lim_{n\to\infty} \int_{(s,\infty)} h_n(x) M_n^{(s)}(\mathrm{d}x) = \int_{(s,\infty)} h(x) M_\infty^{(s)}(\mathrm{d}x).$$

The result follows by combining this with the fact that $m_n \to m_\infty$. □

Proof of Theorem 1. The proof is based on verifying that the assumptions of Proposition 3 hold. Toward this end, note that

$$\begin{aligned} n\mathbb{P}(a_n X_{1n} > s) &= n c_{\ell_n} \int_{(s/a_n,\infty)} q_{\ell_n}(x)\mu(\mathrm{d}x) \\ &\sim n \int_{(s/a_n,\infty)} q\left(\frac{x}{\ell_n}\right)\mu(\mathrm{d}x) \\ &= \int_{(s,\infty)} q\left(\frac{x}{a_n \ell_n}\right) M_n(\mathrm{d}x). \end{aligned}$$

By Lemma 6, this converges to

$$\int_{(s,\infty)} q\left(\frac{x}{c}\right) M_\infty(\mathrm{d}x) = \alpha \int_{(s,\infty)} q_c(x) x^{-1-\alpha}\,\mathrm{d}x,$$

where we interpret $q(x/c) = 1$ if $c = \infty$ and $q(x/c) = \zeta$ if $c = 0$. Further,

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} n\mathbb{E}[a_n X_1 \mathbf{1}_{\{a_n X_1 < \varepsilon\}}] &= \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} n a_n c_{\ell_n} \int_{[0,\varepsilon/a_n)} x q\left(\frac{x}{\ell_n}\right)\mu(\mathrm{d}x) \\ &= \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} n a_n \int_{[0,\varepsilon/a_n)} x q\left(\frac{x}{\ell_n}\right)\mu(\mathrm{d}x) \\ &\le K \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \int_{[0,\varepsilon)} x M_n(\mathrm{d}x) \\ &= 0, \end{aligned}$$

where the penultimate line follows by Lemma 5 and K is any upper bound on q. □

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Proof of Lemma 4. First note that, by Slutsky's theorem, for any $t > 0$,

$$-X_n \log(\hat{\mu}_{\theta_n}(z)) = -\theta_n X_n \frac{\log(\hat{\mu}_{\theta_n}(z))}{\theta_n} \stackrel{\mathrm{D}}{\to} -X \log(\hat{\mu}(z)).$$

Let L_n be the Laplace transform of the distribution of X_n. The Laplace transform of the distribution of $\theta_n \circ_{\mathcal{P}} X_n$ is then $L_n(-\log(\hat{\mu}_{\theta_n}(z))$. Since convergence in distribution implies convergence of Laplace transforms,

$$\lim_{n\to\infty} L_n(-\log(\hat{\mu}_{\theta_n}(z)) = \lim_{n\to\infty} \mathbb{E}[\exp(X_n \log(\hat{\mu}_{\theta_n}(z)))] = \mathbb{E}[\exp(X \log(\hat{\mu}(z)))].$$

Observing that

$$\mathbb{E}[\mathrm{e}^{-zZ_X}] = \mathbb{E}[\mathbb{E}[\mathrm{e}^{-zZ_X} \mid X]] = \mathbb{E}[(\hat{\mu}(z))^X] = \mathbb{E}[\exp(X \log(\hat{\mu}(z)))]$$

yields the result. □

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