A correction to sonic boom theory

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ABSTRACT

Current sonic boom theory is based on linear midfield solutions coupled with acoustic propagation models. Approximate corrections are made within the theory to account for non-linearities, in particular for the coalescence of compression waves and the formation of weak shocks. A very large adjustment is made to account for the increasing acoustic impedance that the waves encounter as they propagate from the low density air at cruise altitude to the high density air at sea level. Typically this correction reduces the calculated over pressure levels by a factor of three. Here the method of characteristics (MOC) is used to prove that the density gradient within a hydrostatic atmosphere has no direct effect on the propagation or intensity of the wave. However gravity and ambient temperature both affect the wave propagation and the combined pressure level attenuation is not dissimilar to that previously attributed to acoustic impedance. Although the flawed acoustic theory has given reasonable predictions of measured sonic booms, the omission of gravity from the equation of motion and the inclusion of a false impedance modification, makes the model unreliable for prediction of future designs, particularly those focused on boom minimisation. As an aid to quiet supersonic aircraft design, Whitham's theory is extended to include gravity and ambient temperature variation and shown to be in good agreement with a MOC solution for the real atmosphere.

NOMENCLATURE

a	speed of sound (ms^{-1})
C_p	specific heat of cool air at constant pressure, 1004 $J.kg^{-1}.K^{-1}$
f	source function (m)
F	Whitham's function ($m^{½}$)
g	local gravity (ms^{-2})
h	geopotential altitude (m)
H	potential function, ($J.kg^{-1}$)
k	function of M, Equation (22)
L	reference length (m)
M	free stream Mach number
P	pressure (Pa)
p	Riemann invariant (ms^{-1})
q	radial function, Equation (32) ($m^{-½}$)
r	radius (m)
R	gas constant of air, (287 $J.kg^{-1}.K^{-1}$)
R_b	radius of body (m)
S	entropy ($J.kg^{-1}.K^{-1}$)
t	time, or running variable, (s or m)
T	temperature (K)
u	x component of velocity (ms^{-1})
$\hat{u}$	perturbation of u (ms^{-1})
v	y component of velocity (ms^{-1})
V	total velocity (ms^{-1})

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w	z component of velocity (ms^{-1})
x	horizontal axis, +ve from nose to tail (m)
y	spanwise axis, +ve to starboard (m)
z	vertical axis, +ve with increasing altitude (m)
β	$\sqrt{M^2 - 1}$
γ	ratio of specific heats for cool air, 7/5
θ	streamline angle about y (rad)
μ	Mach angle (rad)
ρ	density (kg.m^{-3})
χ	x-coordinate of characteristics intercept (m)

Subscripts

1, 2, ..	state at some fixed radius, r_1, r_2,...
c	on the characteristic
g	scale height for q, Equation (39)
s	on the bow shock
ref	reference
∞	unperturbed value, can be a function of altitude

INTRODUCTION

Loud sonic boom is a barrier to supersonic flight over populated areas but a restriction to flight over water is a major impediment to the economic viability of a supersonic transport aircraft. The theory used to predict boom levels is credited to Whitham and Walkden from the University of Manchester and Jones from English Electric Aviation Ltd who used the Whitham-Walkden method to establish a lower bound for the sonic boom. Whitham(1) showed how linear theory could be improved by accounting for the curvature in the characteristics (Mach lines) that results from perturbations in the local speed of sound. He used the improved method to derive an analytical result for the N wave found far from a body of revolution in a uniform atmosphere. Walkden(2) found that the perturbation potential for a wing had the same form in the far field as that of an axisymmetric body, and applying Whitham's theory was able to calculate the strength of an N wave produced by the entire aircraft. Jones(3) sought the optimum shape of an aircraft governed by the Whitham-Walkden theory. He found that an aircraft whose equivalent axisymmetric body had a cross-sectional area that varied with the square root of distance from the nose would produce the lowest possible overpressure. Jones also reduced the ratio of the overpressure to the local pressure, by the factor $\sqrt{(p_2/p_1)}$ in order to account for the effect of pressure variation within the atmosphere. In a later paper(4) he attributes this correction to Randall(5) from the Royal Aircraft Establishment who in the report referenced, introduces it as a 'crude' approximation without further justification.

Subsequently George and Plotkin(6) and Hayes, Haefeli and Kulsrud(7) altered the factor to $\sqrt{(\rho_2 a_2/\rho_1 a_1)}$ and noted, with a reference to Rayleigh(8), that in this way energy transmitted by the wave is conserved. Interestingly in his *Theory of Sound*, Rayleigh upon demonstrating that conservation of energy appears to require the amplitude of a wave (expressed as the perturbation in velocity) to vary inversely with the square root of density when travelling through a density gradient, adds the footnote: "A delicate question arises as to the ultimate fate of sonerous waves propagated upwards. It should be remarked that in rare air the deadening influence of viscosity is much increased.". Here it is demonstrated that one must include potential energy due to gravity in order to determine the true fate of a wave travelling vertically in the atmosphere, and that once this is done the density variations are seen to have no effect on the wave.

References 6 and 7 are particularly relevant to the implications of this correction. The report by Hayes *et al*(7) describes a sonic boom prediction code, ARAP, that for many years has been the standard by which others are judged, and the paper by George and Plotkin(6) describes the non-linear effect of the density variation on the wave propagation. It is this supposed non linear effect that results in delayed development of the N wave and increases the possibility of achieving an overpressure below that calculated by Jones. Once an N wave has formed the George-Seebass-Jones' minimisation theory reduces to that of Jones, and hence the extension of the theory that underpinned the boom minimisation work in the US High Speed Research programme(9), and the DARPA/NASA Shaped Sonic Boom Demonstrator (SSBD)(10) is removed. This is not to say that N waves are inevitable, on the contrary, even a cursory exploration of the problem using the MOC reveals the possibility of: signatures with character; and bow shock overpressures much less than predicted by Jones.

At this stage it is necessary to mention that this is not the first application of MOC to the sonic boom problem. Ferri and colleagues(11) at New York University wrote a MOC code they called MMOC for Sonic Boom prediction. The code, which included an approximate correction for circumferential velocity gradient allowing small departures from rotational symmetry, was extended by Darden at NASA Langley to include shock coalescence. Darden(12) compared MMOC calculated pressure disturbances with near and far field predictions obtained with ARAP and a code called MUAM. MUAM was based on Whitham's uniform atmosphere theory but applied the Randall correction to account for the density variation. Sufficiently good agreement was obtained between the three methods that the underlying physics were not questioned and the acoustic theory went unchallenged.

The MOC code written for this work was based on subroutines we use for intake and nozzle design. Gravitational body force is negligible in these applications and thus had not been included in the compatibility relations or in the thermodynamic potential function. Early in the development of the new code, the atmospheric pressure variation was modelled but the thermodynamic potential had not been corrected for gravity with the result that the streamlines curved away from the ground under the influence of the hydrostatic pressure gradient. Although in retrospect this seems an obvious result of including a static pressure gradient but neglecting the body force from which it originates, this same error is made within current sonic boom acoustic theory.

2.0 A PLANAR ACOUSTIC WAVE IN A HYDROSTATIC ATMOSPHERE

The problem with the acoustic theory is best demonstrated and resolved by a MOC solution of one dimensional unsteady gas dynamics. In this way, the clearly three dimensional problem of an asymmetric aircraft flying in a stratified atmosphere can be left until the factors which allow good approximation with an axisymmetric solution become clear.

Consider a vertical tube aligned with both gravity and the coordinate z, which is positive upwards. The tube contains air which initially is at rest and at uniform temperature. Like the atmosphere the air is in hydrostatic equilibrium such that the pressure at small z is higher than at large z due to the extra weight of air above. The pressure gradient in the tube, prior to disturbance, being just,

$$\frac{\partial p}{\partial z} = -\rho g = -\frac{p g}{R T} \quad \ldots (1)$$

The equation describing the effect of waves propagating in the +z direction (P waves) is after Kantrowitz(13).

$$\frac{\partial P}{\partial t} + (w+a)\frac{\partial P}{\partial z} - T\frac{\partial S}{\partial z} - \frac{a}{R}\left(\frac{\partial S}{\partial t} + w\frac{\partial S}{\partial z}\right) = -g \quad \ldots (2)$$

where $P = 2a/(\gamma\text{-}1) + w$ is the Riemann invariant. Gravitational acceleration has been added to Kantrowitz's model by the author, in order to apply the equation to the vertical tube. The last term on the left hand side represents the generation and convection of entropy and is identically zero in the acoustic problem for which the entropy of the air remains fixed. The only change in entropy at any station is that due the arrival of air of a different state from another part of the column and thus,

$$\frac{\partial S}{\partial t} = -w\frac{\partial S}{\partial z} \qquad \ldots (3)$$

The third term is initially (with the air undisturbed),

$$T\frac{\partial S}{\partial z} = -\frac{RT}{p}\frac{\partial p}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} = g \qquad \ldots (4)$$

after substitution of Equation (1). Since acoustic perturbations are very small, the displacement is minute, and the entropy gradient within the air will not depart significantly from its initial value. Thus the equation describing acoustic propagation in a column of isothermal air is,

$$\frac{\partial P}{\partial t} = -(w+a)\frac{\partial P}{\partial z} \qquad \ldots (5)$$

Although somewhat trivial, this is the key result of this paper. The equation describing the propagation of an acoustic wave in a vertical column of hydrostatic air is the same as that in a horizontal tube in which there are no density gradients. This means that no adjustments need be made to account for the variation in pressure or acoustic impedance as a wave propagates in the stratosphere. If the density/pressure/entropy variations are to be taken into account then so must the body force that produced them.

Rayleigh's question still needs to be answered. Since the intensity of the wave is unaffected by density, the energy transmitted by the wave remains in proportion to density and thus decreases with altitude, or increases when descending. The changes in energy can be equated directly to the change in potential energy of the air column due to its displacement by the wave. In the case of a downward propagating *z*, in which the compression is followed by an expansion and there is no net displacement of the air in its wake, the additional energy (now second order in comparison to the positive and negative deviations in internal energy) is due to the displacement of air within the wave itself. More mass is found within the lower compressed section and less in the higher expansion region.

3.0 MOC COORDINATE SYSTEM

The above picture of a wave gathering energy from the column of air lowered behind it, albeit very slightly, makes the boom look unstoppable. Fortunately the steady axisymmetric MOC provides a clearer picture of the propagation of the disturbance from a supersonic aircraft.

Consider an aircraft cruising in supersonic level flight, with its velocity vector aligned with the horizontal axis *x*. Following Walkden[2], we will assume the aircraft's volume and lift can be represented with an equivalent axisymmetric body, aligned with, and centred on the *x* axis. Gravity acts in the -*z* direction which coincides with the radial coordinate, *r* in the vertical plane on which we will develop the solution.

4.0 CHARACTERISTICS IN A HYDROSTATIC ATMOSPHERE

Along characteristics within an axisymmetrical flow, the conservation equations become[12]:

$$\cot\mu\frac{dV}{V} \mp d\,\theta + \frac{\cot\mu}{V^2}(TdS - dH) - \frac{\sin\theta\sin\mu\,dr}{\sin(\theta\pm\mu)}\left(\frac{1}{r}+\frac{g}{a^2}\right) = 0 \qquad \ldots (6)$$

where the characteristics are the Mach lines

$$\frac{dr}{dx} = \tan(\theta\pm\mu) \qquad \ldots (7)$$

The inclusion of gravity in the momentum balance and Equation (6) implies a cylindrical gravitational field radiating outward from the centreline. Fortunately the gravitational term is small compared to the $1/r$ term until a radius of about $a^2/g/10$ = 0·9km, and at such a radius the circumferential gradients due to the relative inclination of the gravity vector are almost certainly negligible in comparison to other gradients over the typically 0·1km axial length of the disturbance. However the assumption that the axisymmetric solution will remain valid on the plane of symmetry, despite the presence of this term would benefit from a rigorous proof.

Far more significant is the inclusion of gravity in the energy balance, and it appears within the compatability relation as part of the potential term *H*, which combines the streamlines total enthalpy with its undisturbed gravitational potential.

$$H(x,r) = H(h_\infty) = C_p\left(T_\infty - T_{ref}\right) + \frac{1}{2}\left(u_\infty^2 - u_{ref}^2\right) + g_{ref}\left(h_\infty - h_{ref}\right) \qquad \ldots (8)$$

Here for convenience the geopotential height, h_∞, is used rather than the altitude as by its definition it allows the above simple expression for gravitational potential energy while still allowing for variation of gravitational acceleration with altitude. The entropy of the air on each streamline is simply determined by its initial value at height h_∞ with the addition of the entropy increase across each upstream shock.

$$S(x,r) = S(h_\infty) + \sum\Delta S = C_p\log\left(\frac{T_\infty}{T_{ref}}\right) - R\log\left(\frac{p_\infty}{p_{ref}}\right) + \sum\Delta S \qquad \ldots (9)$$

The differential sum within the third term of the compatibility relation is,

$$\begin{aligned} TdS - dH &= C_p dT - RT\frac{dp}{p} + T\sum\Delta S - C_p dT - u_\infty du_\infty - g_{ref}d \\ &= T\sum\Delta S - u_\infty du_\infty \end{aligned} \qquad \ldots (10)$$

The first term is due to shocks upstream of the point on the streamline and the second would normally be zero although it could be used to explore the effect of wind in the direction of flight.

Just like the one dimensional unsteady case, the compatibility relation for two dimensional steady flow is unaffected by the pressure gradients within a hydrostatic atmosphere. Therefore the circumferential variation in ambient pressure will not break the axial symmetry. Gravity and ambient temperature will cause a loss of symmetry but since the gravitational term is not very significant until a radius of about 1km and the waves don't encounter the troposphere until typically 5km from the aircraft, it seems reasonable to assume that treating the horizontal layers as if they were circumferential will not result in a large error on the $y = 0$ plane.

Although this lack of sensitivity to density may seem fortuitous, it is in fact inevitable. Consider the compatibility relation when the 'aircraft' produces no disturbance and hence streamline angles are zero everywhere. In the case of no wind, dV is zero and the third term must also be zero. In the case of wind, the first and third terms are equal and opposite.

5.0 WHITHAM'S FUNCTION

The Whitham function, $F(\chi)$, has a central role in sonic boom prediction, even when near field pressure signatures are obtained from experiment[14] or CFD[15] . The cited references both review extrapolation methods, and indicate that the Whitham function is calculated directly from near field data, excluding all but the far field

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terms in the relationship between axial source distribution and velocity perturbations. The excluded terms do not arise from the additional complications of non-linearity or three dimensional effects, but are part of the general expressions for the perturbed velocities as presented by Whitham:

$$\frac{\hat{u}}{u_\infty} = -\int_0^{\chi} \frac{f'(t)\ dt}{\sqrt{(\chi - t)(\chi - t + 2\beta r)}} \qquad \ldots (11)$$

$$\theta = \frac{\hat{w}}{u_\infty} = \frac{1}{r}\int_0^{\chi} \frac{(\chi - t + \beta r) f'(t)\ dt}{\sqrt{(\chi - t)(\chi - t + 2\beta r)}} \qquad \ldots (12)$$

These results are a modified form of linear theory, in which the coordinate χ replaces x-βr. In linear theory, local perturbation of the velocity and speed of sound are assumed sufficiently small that the characteristics are straight lines with slope β. With Whitham's modification a characteristic is denoted by constant χ where χ is its x coordinate at r = 0, but as an improvement on linear theory, the characteristics respond to first order perturbations in Mach angle and streamline angle and thus are curved in x,r space. In the far field where r>>χ Equations (11) and (12) simplify to,

$$\frac{\hat{u}}{u_\infty} = \frac{-F(\chi)}{\sqrt{2\beta r}} \qquad \ldots (13)$$

$$\theta = \frac{F(\chi)\beta}{\sqrt{2\beta r}} \qquad \ldots (14)$$

where,

$$F(\chi) = \int_0^{\chi} \frac{f'(t)}{\sqrt{\chi - t}} dt \qquad \ldots (15)$$

As is easily demonstrated, sufficiently far from the axis, pressure perturbation is directly proportional to $\hat{u}/u_\infty$ [16],

$$\frac{\Delta p}{p} = -\gamma M^2 \frac{\hat{u}}{u_\infty} = \gamma M^2 \frac{F(\chi)}{\sqrt{2\beta r}} \qquad \ldots (16)$$

The other extreme, very small r, is also important in Whitham's theory, for it is on this limit that the source distribution function $f'(t)$ is normally defined. As r goes to zero, Equation (12) may be written θ = f(χ)/r or f(χ) = rr' = $A'(x)/2\pi$. Hence $f'(t)$ is normally replaced by $A''(x)/2\pi$ which is defined by the area distribution of the body. Note that this relationship assumes $f(0)$=0.

When pressure is defined at an intermediate radius, Equation (11) should be used to infer $f'(t)$. However it is generally assumed that regions where χ-t is small contribute most to the integral, and therefore the second factor in the denominator can be approximated as 2βr even where r is small compared to χ. This far field approximation is widely (perhaps universally) used in extrapolation techniques, but here we will demonstrate how Equation (11) can be inverted to provide a better approximation of source distribution (and hence Whitham's function) from near field pressure perturbations. The difference is significant towards the rear of the body of length L when r is less than about 2L and similar differences have previously been attributed to three dimensional effects[14] .

6.0 NUMERICAL EVALUATION OF WHITHAM'S FUNCTION

Evaluation of F(χ) presents numerical difficulties even for bodies with continuous curvature, in part due to the singularity at t = χ . The same integral appears in the calculation of heat transfer from temperatures measured at the surface of a semi-infinite substrate. Following Cook and Felderman's[17] approach in which the general function $f(t)$ is treated as piecewise linear, the numerical solution may be evaluated as.

$$F(n) = \sum_{i=2}^{n} \int_{\chi_{i-1}}^{\chi_i} \frac{f'(t)}{\sqrt{\chi_n - t}} dt = \sum_{i=2}^{n} \frac{f_i - f_{i-1}}{\chi_i - \chi_{i-1}} \int_{\chi_{i-1}}^{\chi_i} \frac{dt}{\sqrt{\chi_n - t}} = 2\sum_{i=2}^{n} \frac{f_i - f_{i-1}}{\sqrt{\chi_n - \chi_i} + \sqrt{\chi_n - \chi_{i-1}}} \qquad \ldots (17)$$

Application to Whitham's function for an area distribution defined along the axis presents no difficulty since $f_i = A_i'/2\pi$.

The same numerical approach can be used to evaluate the full expression for velocity perturbation and allow comparison with the far field approximation.

$$\frac{\hat{u}_n}{u_\infty} = -\sum_{i=2}^{n} \int_{\chi_{i-1}}^{\chi_i} \frac{f'(t)\ dt}{\sqrt{(\chi_n - t)(\chi_n - t + 2\beta r)}} \qquad \ldots (18)$$

$$= \sum_{i=2}^{n} \frac{f_i - f_{i-1}}{\chi_i - \chi_{i-1}} \ln\left(\frac{\sqrt{(\chi_n - \chi_i)(\chi_n - \chi_i + 2\beta r)} + \chi_n - \chi_i + \beta r}{\sqrt{(\chi_n - \chi_{i-1})(\chi_n - \chi_{i-1} + 2\beta r)} + \chi_n - \chi_{i-1} + \beta r} \right)$$

Isolation of the last term in the summation, allows both equations to be rearranged to give f from a known velocity perturbation or prescribed F. For example,

$$f(n) = f(n-1) + \sqrt{\chi_n - \chi_{n-1}} \left(\frac{F(n)}{2} - \sum_{i=2}^{n-1} \frac{f_i - f_{i-1}}{\sqrt{\chi_n - \chi_i} + \sqrt{\chi_n - \chi_{i-1}}} \right) \qquad \ldots (19)$$

Equation (19) allows the calculation of an equivalent body for a given F(χ), assuming a piecewise linear distribution of f.

7.0 N WAVE

As demonstrated by Whitham, the effect of first order perturbations in local velocity and speed of sound on the local slope of a characteristic is given by,

$$\frac{\partial x_c}{\partial r} = \beta + \frac{(\gamma+1)M^4}{2\beta} \frac{\hat{u}}{u_\infty} - M^2\left(\theta + \beta \frac{\hat{u}}{u_\infty}\right) \qquad \ldots (20)$$

Noting that θ = -β$\hat{u}/u_\infty$ except very close to the axis, Equation (20) is integrated along a characteristic to give,

$$x_c = \beta r + \frac{k\sqrt{\beta/2}}{u_\infty} \int_0^r \hat{u}\, dr + \chi \qquad \ldots (21)$$

where,

$$k = (\gamma+1)M^4/\sqrt{2\beta^3} \qquad \ldots (22)$$

Since weak shocks bisect the intersecting upstream and downstream characteristics, the slope of the shock is,

$$\frac{dx_s}{dr} = \beta + \frac{k\sqrt{\beta/2}}{2} \frac{\hat{u}}{u_\infty} \qquad \ldots (23)$$

Where $\hat{u}/u_\infty$ is determined on the downstream characteristic. By equating the position of the shock and the downstream characteristic, Whitham demonstrated that the characteristic which is coincident with the bow shock at some radius r, was one with F given by,

$$F^2(\chi_s) = \frac{2}{k\sqrt{r}} \int_0^{\chi} F(t)\,dt \qquad \ldots (24)$$

In the far field the characteristic that meets the shock always originates near F(χ_0) = 0. Characteristics originating much upstream of χ_0 advance faster than the shock and catch it midfield, and those further downstream (with negative F values) travel slower than a freestream characteristic and never catch the bow shock. Thus the integral in Equation (24) can be treated as a constant with χ = χ_0, or in other words, the value of the integral can be taken at its maximum with little error in the calculation of bow shock strength.

The slope of the N wave pressure signature (at constant r) is calculated by differentiation of Equations (16) and (21) with respect

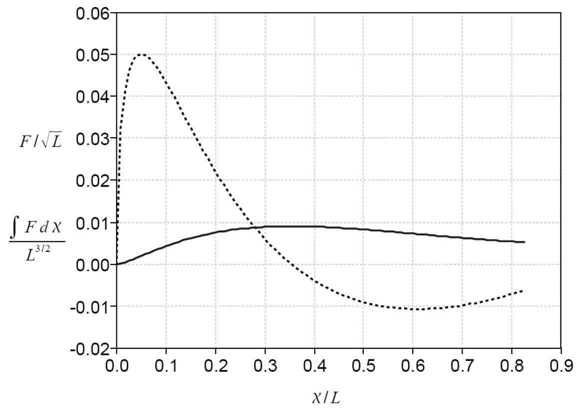

Figure 1. Whitham function and its integral for the body defined by Equation (36).

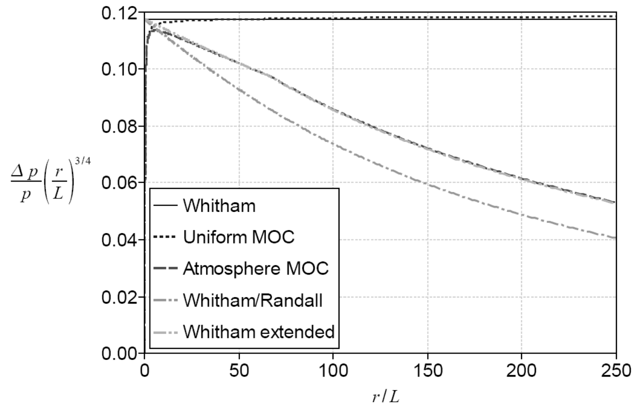

Figure 2. Bow shock overpressure ratio in a uniform atmosphere (without gravity) and in the standard atmosphere.

to x, again noting that $\chi_\infty\chi_0$ =constant, giving,

$$\frac{\partial p}{\partial x}=\frac{-\gamma\beta p}{(\gamma+1)M^2 r} \qquad \ldots (25)$$

Whitham's elegant analysis is the cornerstone of all sonic boom theory and given its central role, a direct comparison with a numerical calculation using MOC is long overdue.

8.0 A MOC TEST OF WHITHAM THEORY

One of Whitham's test cases is a body with a radius R_b defined by,

$$\begin{aligned} R_b &= 0{\cdot}1L\left(1-(1-x/L)^3\right), \quad 0\le x\le L \\ &= 0{\cdot}1\mathrm{L} \qquad\qquad\qquad\qquad L<x \end{aligned} \qquad \ldots (26)$$

For this body the source function is finite in the range $0 \le x \le L$ and given by,

$$f(x)=R_b R_b{}'=0{\cdot}3R_b(1-x/L)^2 \qquad \ldots (27)$$

The characteristic at x on the surface of the body appears to originate on the axis at,

$$X=x-\beta R_b \qquad \ldots (28)$$

and it is with respect to this distorted, Mach number dependent, coordinate that the derivative of the source distribution and F are evaluated. For flight at Mach 2, a plot of F and its integral, F_i, are presented in Fig. 1. Using the maximum of F_i ($= 0{\cdot}00907\,L^{3/2}$) within Equation (24) and substituting this expression for F at the bow shock into Equation (16),

$$\frac{\Delta p}{p}=0{\cdot}117\left(\frac{r}{L}\right)^{-3/4} \qquad \ldots (29)$$

Comparison with a MOC solution for a uniform atmosphere without gravity is made in Fig. 2. The agreement is remarkable and a tribute to Whitham's insight and the power of analytical methods.

A MOC solution for the same body with an L = 60m nose flying at 15km in the US Standard Atmosphere (1976) is also plotted in 2. The steady decrease in bow shock pressure jump reflects the effect of gravity on the flowfield. The kink in the curve near r/L = 66 is at the top of the troposphere within which the increasing temperature adds to the decline of the pressure jump. Randall's 'crude approximation' for the supposed effect of increasing pressure is seen to give a reasonable representation of the gravitational effect and this probably helps explain why the acoustic impedance interpretation of the phenomenon has gone unchallenged.

The line in near perfect agreement with the MOC solution is an extension of Whitham theory, developed in the following section, to account for gravity and the temperature variation.

9.0 WHITHAM THEORY EXTENSION FOR THE REAL ATMOSPHERE

Equation (6) describes the variation of velocity and streamline angle along a characteristic in the real atmosphere. We have seen that the third term is zero within a hydrostatic atmosphere if the shocks are weak allowing entropy rise to be neglected. If we now also assume small perturbations, then sufficiently far from the axis where $\theta = -\beta\hat{u}/u_\infty$ and $\theta \ll \mu$, the compatibility relation simplifies to,

$$2\frac{d\hat{u}}{\hat{u}}+\frac{d\beta}{\beta}+dr\left(\frac{1}{r}+\frac{g}{a^2}\right)=0 \qquad \ldots (30)$$

Noting that $g/a^2 = dp_\infty/(\gamma p_\infty dr)$ from Equation (1) and integrating between r_1 and r_2,

$$\frac{\hat{u}_2}{\hat{u}_1}=\sqrt{\frac{r_1\beta_1}{r_2\beta_2}}\left(\frac{p_1}{p_2}\right)^{\frac{1}{2\gamma}} \qquad \ldots (31)$$

Therefore the perturbation velocity in the real atmosphere exhibits the same geometric radial dependence as that in a uniform stream, with the action of gravity producing an additional radial dependence that can be linked to changes in ambient pressure.

To calculate the pressure jump at the bow shock we need to establish which characteristic intersects the shock as a function of radius. Whitham's physical approach remains valid but the mathematics must be altered to account for the new radial dependence. Taking r_1 close enough to the body that the influence of gravity is small and substituting u_1 from Equation (13) into Equation (31), the dependence of the perturbation velocity on the characteristic and radius can be written as,

$$\hat{u}=-u_\infty F(X)q(r)/\sqrt{2\beta} \qquad \ldots (32)$$

where $q(r) = (p_1/p)^{(1/2\gamma)}/\sqrt{r}$ is a function representing the radial dependence that includes the effect of gravity. Then, substitution into Whitham's equation describing the characteristic path (Equation (20)) gives,

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$$\frac{\partial x_c}{\partial r} = \beta(r) - \frac{1}{2} F(\chi) k(r) q(r) \qquad \ldots (33)$$

where the Mach number dependence of k makes it also radius dependent. Integrating along the characteristic between r_1 and r_2,

$$x_{c2} - \chi = \int_{r_1}^{r_2} \beta\, dr - \frac{F(\chi)}{2} \int_{r_1}^{r_2} k\, q\, dr \qquad \ldots (34)$$

where the x coordinate of the characteristic at r_1 has defined χ. This is a departure from Whitham's definition who defined χ as the characteristics x-coordinate on the axis. To recover Whitham's result r_1 can be taken as zero but then some justification is required for using far field approximations for characteristic curvature in the near field. When extrapolating from CFD or wind-tunnel data, it is more convenient and accurate to start the characteristics at a radius sufficiently large that both: three dimensional effects are negligible; and the velocity perturbations are well described by small perturbation theory.

Along the bow shock,

$$\frac{dx_s}{dr} = \frac{\partial x_c}{\partial r} + \frac{\partial x_c}{\partial \chi}\frac{d\chi}{dr} = \beta - \frac{kFq}{2} - \frac{1}{2}\frac{dF}{dr}\int_{r1}^{r2} k\, q\, dr + \frac{d\chi}{dr} \qquad \ldots (35)$$

Equating to the shock gradient from Equation (23),

$$\beta - \frac{kFq}{4} = \beta - \frac{kFq}{2} - \frac{1}{2}\frac{dF}{dr}\int_{r1}^{r2} k\, q\, dr + \frac{d\chi}{dr} \qquad \ldots (36)$$

therefore,

$$\frac{d}{d\chi}\left(F^2 \int_{r_1}^{r_2} k\, q\, dr\right) = 4F \qquad \ldots (37)$$

and,

$$F^2 = \frac{4\int_0^{\chi} F\, dt}{\int_{r_1}^{r_2} k\, q\, dr} \qquad \ldots (38)$$

For uniform flow without gravity, k is a constant, $q = 1/\sqrt{r}$ and Whitham's result (Equation (24)) is recovered when $r_1=0$. For an isothermal atmosphere, k is a constant, $q = (p_1/p)^{(1/2\gamma)}/\sqrt{r}$ and the integral from 0 to r is,

$$\int_0^r k\, q\, dr = 2k\sqrt{h_g}\int_0^{\sqrt{r/h_g}} e^{-t^2} dt = k\sqrt{\pi h_g}\ \ erf\left(\sqrt{r/h_g}\right) \qquad \ldots (39)$$

where $h_g = 2\gamma RT/g$ is a scale height that reflects the influence of gravity on the flowfield. Note that it is a factor 2γ larger than the atmospheric scale height associated with changes in pressure (Equation (1)).

For the real atmosphere the integration can be performed numerically, and this is how F at the bow shock and hence pressure jump were determined in the application of the extended theory in Fig. 2. One method of avoiding the singularity at $r = 0$ (when F is defined along the axis) is to use Equation (39) within the stratosphere, out to r_2 say, and then recast Equation (38) as;

$$\frac{4}{F^2}\int_0^{\chi} F\, dt = k\sqrt{\pi h_g}\ \ erf\left(\sqrt{r_1/h_g}\right) + \int_{r_2}^{r_3} k\, q\, dr \qquad \ldots (40)$$

for propagation through the troposphere from r_2 to the ground at r_3.

10 SIGNATURE FREEZING

The combined effect of gravity and ambient temperature on wave propagation, described above, is similar to the supposed effect of acoustic impedance, primarily due to the fundamental link between gravity and the pressure profile of the atmosphere in hydrostatic equilibrium. However that is all there is to the relationship; the acoustic model is invalid. Despite this, many of the ideas developed in sonic boom research are still relevant. For example, the recognition that: N waves are not inevitable (which is obvious from measured signatures); and that the real atmosphere helps to prevent, or at least slow, N wave formation. The latter is apparent in the kq integral of Equation (39). Note how this integral becomes essentially independent of r once r exceeds say $1{\cdot}4h_g$ and the error function has reached 95% of the limit value. This process is known as signature freezing[7] and it results in the characteristics running (nearly) parallel to a freestream Mach line, even though $F > 0$. This can be seen directly from Equation (33) which shows the slope of the characteristic approaches β as $q(r)$ goes to zero, and the gravity factor within the q product greatly accelerates its asymptote to zero. Thus the extended Whitham theory still supports the frozen signature concept but shows that the relevant scale height, h_g, is a factor γ longer than that deduced from the acoustic theory. That means that the arguments[18] concerning the techniques applicability to real aircraft are now even harder to make.

Of course, a signature does not have to be frozen for it to have character since its development is interrupted by the ground. One can prescribe any appropriate ground signature and via the use of Equation (34) calculate a Whitham function distribution at an altitude close to the aircraft. As shown by Walkden[2], the Whitham function can be related to most of the essential features of the aircraft, including its lift distribution. This low boom design method is the subject of a following paper.

CONCLUSIONS

The omission of gravity from the acoustic propagation equations and the introduction of an artificial acoustic impedance effect, has allowed sonic boom theory to give a satisfactory match with ground pressure measurements. Proof that gravity is significant to acoustic propagation over a large altitude range and that there is no direct effect of ambient density or pressure has been presented in this paper. This finding is also relevant to other noise prediction codes such as those used to estimate the noise from subsonic overflight.

Whitham's theory for bow shock strength was compared with a MOC calculation in a uniform atmosphere (without gravity) and found to be very accurate. His theory was extended to include the effect of gravity and ambient temperature and the results were in near perfect agreement with the numerical MOC solution for the same case. The extended theory should prove useful for the design of low boom aircraft, as it allows an aircraft's essential features (from a noise perspective) to be derived from a prescribed ground signature.

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