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Change of stability for symmetric bifurcating solutions in the Ginzburg–Landau equations

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We consider the bifurcating solutions for the Ginzburg–Landau equations when the superconductor is a film of thickness $2d$ submitted to an external magnetic field. We refine some results obtained earlier [1] on the stability of bifurcating solutions starting from normal solutions. We prove, in particular, the existence of curves $d \mapsto \kappa_0(d)$, defined for large d and tending to $2^{-1/2}$ when $d \mapsto +\infty$ and $\kappa \mapsto d_1(\kappa)$, defined for small κ and tending to $\sqrt{5}/2$ when $\kappa \mapsto 0$, which separate the sets of pairs (κ, d) corresponding to different behaviour of the symmetric bifurcating solutions. In this way, we give in particular a complete answer to the question of stability of symmetric bifurcating solutions in the asymptotics 'κ fixed-d large' or 'd fixed-κ small'.

1 Introduction

Continuing from our previous paper [1], in this paper we consider the Ginzburg–Landau equations for a film subjected to an exterior magnetic field. Let $2d$ be the thickness of the film, κ the Ginzburg–Landau parameter and $h \in]0, +\infty[$ a constant proportional to the exterior magnetic field; these equations can then be written

$$(GL)_d \quad \begin{cases} \text{(a)} \ -\kappa^{-2} f'' - f + f^3 + A^2 f = 0 \quad \text{in }]-d, d[, \\ \text{(b)} \ f'(\pm d) = 0, \\ \text{(c)} \ -A'' + f^2 A = 0 \quad \text{in }]-d, d[, \\ \text{(d)} \ A'(\pm d) = h, \end{cases} \tag{1.1}$$

where f and A are in $H^2(]-d, d[)$. The function f is a real wave function, and in this model A represents the only non-zero component of the interior magnetic potential. They characterize the state of the material when it is a film (see Ginzburg & Landau [2] and Ginzburg [3]).

A solution of $(GL)_d$ is a triple $(f, A; h)$ in $H^2(]-d, d[) \times H^2(]-d, d[) \times]0, +\infty[$ satisfying $(GL)_d$. For given $\kappa > 0$, $h \geqslant 0$ and $e \in \mathbb{R}$, the triple $(f, A; h) = (0, h(x+e); h)$ is clearly a solution of $(GL)_d$. These solutions are called *normal solutions*. A *superconducting solution* is a solution $(f, A; h)$ of $(GL)_d$, such that f is not identically 0.

In this paper we are interested in bifurcating superconducting solutions starting from normal solutions. A continuous curve $\varepsilon \mapsto (f(\cdot, \varepsilon), A(\cdot, \varepsilon); h(\varepsilon))$ will be called a *bifurcating curve of solutions starting from a normal solution* $(0, h_0(x+e); h_0)$, with $h_0 > 0$ and $e \in \mathbb{R}$, if there exists $\varepsilon_0 > 0$ such that:

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(i) $(f(\cdot, \varepsilon), A(\cdot,\varepsilon); h(\varepsilon))$ is a solution for $|\varepsilon| \leqslant \varepsilon_0$,

(ii) this solution is superconducting for $\varepsilon \neq 0$,

(iii) the map $\varepsilon \mapsto (f(\cdot,\varepsilon), A(\cdot,\varepsilon); h(\varepsilon))$ is continuous from $[-\varepsilon_0, \varepsilon_0]$ to $H^2(]-d,d[) \times H^2(]-d,d[) \times I\!R$, and satisfies

$$(f(\cdot,0),\ A(\cdot,0);\ h(0)) = (0,\ h_0(x+e);\ h_0).$$

In that case, the normal solution $(0, h_0(x+e); h_0)$ is called a bifurcation point for the problem $(GL)_d$.

We say that a bifurcating solution $(f(\cdot,\varepsilon), A(\cdot,\varepsilon); h(\varepsilon))$, with $\varepsilon \in]-\varepsilon_0, \varepsilon_0[$, is *symmetric* if $f(\cdot,\varepsilon)$ is even and $A(\cdot,\varepsilon)$ is odd on $[-d,d]$, and that it is *asymmetric* otherwise.

The existence of symmetric or asymmetric bifurcating solutions starting from normal solutions has already been studied [5–6]. Necessary conditions for the existence can also be found elsewhere [7–8]. These studies have been recently carried on by Dancer & Hasting [9].

Earlier [1], we studied the structure and stability of the bifurcating solutions starting from normal solutions. Our study gave precise results in several asymptotic regimes, like κd small or κd large for the symmetric solutions, or in the regime κd large for the asymmetric solutions, excluding in each regime a small set where the study was not complete. Our purpose in this paper is to complete our previous work in the case of the symmetric bifurcating solutions by describing the transition between the various regimes. An analogous study in the case of the asymmetric solutions is probably possible.

New numerical computations by Aftalion & Troy [10] are in good agreement with our results. They show the existence of three curves in the (κ, d) plane, separating the pairs (κ, d) with different behaviour. These transitions are described, by using formal expansions, in the recent paper by Aftalion & Chapman [11]. Our paper proves mathematically, and independently, the existence of two of these curves, denoted $\kappa_1(d)$ and $\kappa_2(d)$ in Aftalion & Troy [10] and Aftalion & Chapman [11], in the asymptotic limits $(\kappa \mapsto 2^{-1/2}, d \mapsto +\infty)$ and $(\kappa \mapsto 0, d \mapsto \sqrt{5}/2)$.

The plan of this study is as follows. In § 2, after recalling some of the results on bifurcating solutions obtained mainly in our earlier work [1], we present the main results of this paper. § 3 describes the scalings used in subsequent sections, and gives some useful formulae. § 4 is devoted to the study of the symmetric bifurcating solutions when κd is large. § 5 presents an analogous study when κd is small. In § 6, we add some remarks on the stability of the symmetric bifurcating solutions when κd is large or small.

2 Main results on bifurcating solutions

Before giving the principal results of this paper, we recall some properties of the symmetric bifurcating solutions.

2.1 Previous results on bifurcating solutions

The occurrence of bifurcation points for $(GL)_d$ is analysed by considering the spectral properties of a system obtained by linearization of the Ginzburg–Landau equations at a normal solution $(0,\, h_0(x+e), h_0)$, that is the problem

$$\begin{cases} -\kappa^{-2}\phi'' + h_0^2\,(x+e)^2\,\phi = \tau\,\phi & \text{in }]-d,d[, \\ \phi'(\pm d) = 0, \\ \phi = \phi(\cdot;\kappa,d,h_0) \in H^2(]-d,d[), \end{cases} \tag{2.1}$$

(see Bolley & Helffer [1] for details). We consider, as in our previous paper, pairs (ϕ,τ) which are solutions of (2.1) depending upon κ, d, h_0 and e, such that ϕ is strictly positive and satisfies the normalization condition

$$||\phi||_{L^2(]-d,d[)} = 1. \tag{2.2}$$

$\tau = \tau(\kappa,d,h_0,e)$ is consequently the lowest eigenvalue of the operator. We have proved (see Bolley & Helffer [12, Proposition 0.1]) that, for all $\kappa > 0$, $d > 0$ and $e \in I\!R$, there exists a unique $h_0 = \bar{h}(\kappa,d,e)$ such that

$$\tau(\kappa,d,e,\bar{h}(\kappa,d,e)) = 1. \tag{2.3}$$

Theorem 2.1 of Bolley & Helffer [6] gives sufficient conditions on h_0 and e for the existence of solutions bifurcating from $(0,h_0(x+e);\,h_0)$. However, because in this paper we only improve the preceding results in the case of symmetric bifurcating solutions, in the following we restrict our review to these particular solutions. We first note that the corresponding bifurcation points are also symmetric. This is equivalent to the condition $e = 0$, so that a bifurcation point will be in the form $(0,h_0\,x;\,h_0)$ and we shall, in general, skip the e-dependence of h_0, τ, ϕ and write $\bar{h}(\kappa,d)$, $\tau(\kappa,d,\bar{h}(\kappa,d))$, $\phi(\cdot;\kappa,d)$, etc. Moreover, because of the symmetry, the bifurcation problem is here associated with a one-dimensional null space, and the existence of a bifurcation is then a standard result (see, for example, Chow & Hale [13]). We recall, however, a previous result which also gives the uniqueness of the bifurcating solutions.

We have then proved:

Theorem 2.1 (Theorem 2.1 in Bolley & Helffer [6] with $e = 0$)
Let $(\kappa,d) \in]0,+\infty[^2$ *and let* $h_0 \in I\!R \times]0,+\infty[$ *satisfying*

$$\begin{aligned} &\text{(a)} \quad h_0 = \bar{h}(\kappa,d), \\ &\text{(b)} \quad \frac{\partial^2\tau}{\partial e^2}(\kappa,d,h_0) \neq 0. \end{aligned} \tag{2.4}$$

Then, there exists a constant $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\kappa,d) > 0$ *and a* C^∞ *curve,* $]-\tilde{\varepsilon}_0,\tilde{\varepsilon}_0[\ni \varepsilon \mapsto (f(\cdot,\varepsilon),A(\cdot,\varepsilon);h(\varepsilon))$ *of superconducting bifurcating solutions such that*

$$\begin{aligned} f(x,\varepsilon) &= \varepsilon\, f_0(x) + \varepsilon^3\, f_1(x) + o(\varepsilon^3) \quad \text{in } H^2(]-d,d[), \\ A(x,\varepsilon) &= A_0(x) + \varepsilon^2\, A_1(x) + o(\varepsilon^2) \quad \text{in } H^2(]-d,d[), \\ h(\varepsilon) &= h_0 + \varepsilon^2\, h_1 + o(\varepsilon^2), \end{aligned} \tag{2.5}$$

where f_0 *is the principal normalized positive eigenfunction* ϕ *defined by* (2.1)*, with* $h_0 = \bar{h}(\kappa,d)$*, so that* $\tau = 1$*, and* $A_0 = h_0\,x$*. Moreover, there exist constants* $\varepsilon_0 = \varepsilon_0(\kappa,d) > 0$ *with* $\varepsilon_0 \leqslant \tilde{\varepsilon}_0$*, and* $\gamma_0 = \gamma_0(\kappa,d) > 0$ *such that, for* $0 < |\varepsilon| \leqslant \varepsilon_0$*, the solution* $(f(\cdot,\varepsilon),A(\cdot,\varepsilon);h(\varepsilon))$ *is*

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the unique solution of $(GL)_d$ *such that*

$$\text{(i)}\ \ ||f(\cdot,\varepsilon)||_{H^2(]-d,d[)} \leqslant \gamma_0, \qquad \text{(ii)}\ \ ||A(\cdot,\varepsilon)-A_0||_{H^2(]-d,d[)} \leqslant \gamma_0,$$
$$\text{(iii)}\ \ |h(\varepsilon)-h_0| \leqslant \gamma_0, \qquad \text{(iv)}\ \ (f(\cdot,\varepsilon),\ f_0)_{L^2(]-d,d[)} = \varepsilon.$$

We notice that the functions f_i, A_i and the coefficients h_i (for $i = 0, 1$) depend, as does the function ϕ, upon κ and d. We have in particular:

$$f_0(\cdot) = f_0(\cdot\,;\kappa,d), \qquad A_0(\cdot) = A_0(\cdot\,;\kappa,d), \qquad h_1 = h_1(\kappa,d). \tag{2.6}$$

2.2 Main results on the structure of the bifurcating solutions

The first purpose of this paper is to study the sign of h_1, which is the second term in the expansion of $h(\varepsilon)$ defined in (2.5), and to prove the existence, for d large enough, of a curve separating the set of pairs (κ,d) such that $h_1 > 0$ and the set of pairs such that $h_1 < 0$.

We prove, in particular, the following theorem:

Theorem 2.2 [d large, κd large]
There exists a constant $d_0 > 0$ *and a* C^1 *function* $\kappa_0 : d \mapsto \kappa_0(d)$ *defined for* $d \geqslant d_0$*, satisfying*

$$\lim_{d\to+\infty} \kappa_0(d) = 2^{-1/2},$$

and such that, for $\kappa = \kappa_0(d)$ *and* $d \geqslant d_0$*, we have:*

$$h_1(\kappa_0(d), d) = 0,$$

where $h_1(\kappa,d)$ *is defined by* (2.5)*. Moreover, there exists a constant* $a_1 > 0$ *such that for all pairs* (κ,d) *with* $\kappa d \geqslant a_1$*,* $d \geqslant d_0$ *and* $\kappa \neq \kappa_0(d)$*, we have*

$$(\kappa_0(d) - \kappa)\cdot h_1(\kappa,d) > 0.$$

As a corollary, we get for the bifurcating solutions:

Corollary 2.3 *There exists* $a_1 > 0$ *such that, for all pairs* (κ,d) *with* $\kappa d \geqslant a_1$*,* $d \geqslant d_0$ *and* $\kappa \neq \kappa_0(d)$*, and for* $h_0 = \bar{h}(\kappa,d)$*, there exists* $\varepsilon_1 = \varepsilon_1(\kappa,d) > 0$ *such that the curve of bifurcating solutions starting from the normal solution* $(0, h_0 x; h_0)$ *satisfies, for* $0 < |\varepsilon| \leqslant \varepsilon_1$*,*

$$(\kappa_0(d) - \kappa)\cdot(h(\varepsilon) - h_0) > 0.$$

We studied the sign of h_1 earlier [1], but our study left the sign of h_1 undetermined in a domain defined, for any $\eta > 0$, by $\Omega^1_\eta \equiv \{(\kappa,d)\ :\ |\kappa - 2^{-1/2}| \leqslant \eta, \kappa d \geqslant a_1\}$, where a_1 is some large constant depending upon $\eta > 0$. Theorem 2.2 completes this preceding study in Ω^1_η.

We also get (see §4) an analogous result when κd is small, which gives the existence, for κ small, of a curve $\kappa \mapsto d_1(\kappa)$ separating the values of the parameters κ and d, where $h_1(\kappa,d) > 0$ of those where $h_1(\kappa,d) < 0$. We prove the following theorem which improves Theorem 5.8 in Bolley & Helffer [1]:

Theorem 2.4 [κ small, κd small]
There exists a constant $\kappa_1 > 0$ *and a* C^1 *function* $\kappa \mapsto d_1(\kappa)$ *defined for* $0 < \kappa \leqslant \kappa_1$, *satisfying*

$$\lim_{\kappa \to 0} d_1(\kappa) = \frac{\sqrt{5}}{2},$$

and such that, for the pairs (κ, d) *with* $0 < \kappa \leqslant \kappa_1$ *and* $d = d_1(\kappa)$, *we have*

$$h_1(\kappa, d_1(\kappa)) = 0,$$

where $h_1(\kappa, d)$ *is defined in* (2.5). *Moreover, there exists a constant* $a_2 > 0$ *such that for* (κ, d) *satisfying* $\kappa d \leqslant a_2$, $0 < \kappa \leqslant \kappa_1$ *and* $d \neq d_1(\kappa)$,

$$(d_1(\kappa) - d) \cdot h_1 > 0.$$

As in the case when κd is large, we deduce a corresponding result for the bifurcating solutions:

Corollary 2.5 *There exists* $a_2 > 0$ *such that, for* (κ, d) *satisfying* $\kappa d \leqslant a_2$, $0 < \kappa \leqslant \kappa_1$, $d \neq d_1(\kappa)$ *and for* $h_0 = \bar{h}(\kappa, d)$, *there exists* $\varepsilon_1 = \varepsilon_1(\kappa, d)$ *such that the unique curve of bifurcating solutions starting from the normal solution* $(0, h_0\, x;\, h_0)$ *satisfies, for* $0 < |\varepsilon| \leqslant \varepsilon_1$,

$$(d_1(\kappa) - d) \cdot (h(\varepsilon) - h_0) > 0. \tag{2.7}$$

The sign of h_1 was also left undetermined by Bolley & Helffer [1, Theorem 5.8], in a domain $\Omega^2_\eta \equiv \{(\kappa, d)\ ;\ |d - \frac{\sqrt{5}}{2}| \leqslant \eta,\ \kappa d \leqslant a_2\}$, for some $a_2 > 0$ depending upon η. Theorem 2.4 allows us to complete this result in Ω^2_η.

3 Study of $h_1(\kappa, d)$ in other scalings

To prove Theorem 2.2, we rewrite the problem in other units, as previously [1].

3.1 Scalings

We have used, in preceding papers, and in particular in Bolley & Helffer [1], two different scalings for studying (2.1). By using these units once again, we derive the main results of this paper. Let us recall these scalings, and make the inverse transformation explicit; it was only implicit in Bolley & Helffer [1].

(i) The first scaling is given by

$$y = \sqrt{\kappa\, h_0}\, x, \quad a = d\, \sqrt{\kappa\, h_0}, \quad \mu = \frac{\kappa\, \tau}{h_0}. \tag{3.1}$$

It has the advantage of giving an operator which is independent of the parameters. If P is the harmonic oscillator

$$P \equiv -\frac{d^2}{dy^2} + y^2, \tag{3.2}$$

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the spectral problem (2.1)–(2.2) becomes, for $a > 0$,

$$\begin{cases} P\,\bar{f} = \mu\,\bar{f} & \text{in }]-a,a[, \\ (\partial_y \bar{f})(\pm a;a) = 0, \\ \bar{f}(\cdot\,;a) \in H^2(]-a,a[), \end{cases} \tag{3.3}$$

where $\bar{f}$ is considered as a function of y and a, and μ as a function of a, and where we denote by $\partial_y \bar{f}$ the partial derivative of $\bar{f}$ with respect to y.

The eigenfunction $\bar{f}$ is assumed to be normalized so that

$$||\bar{f}(\cdot\,;a)||_{L^2(]-a,a[)} = 1. \tag{3.4}$$

We have then $\mu = \mu(a)$ and $\bar{f} = \bar{f}(y;a)$ and get the relation

$$\bar{f}(y;a) = (\kappa\, h_0)^{-1/4}\, \phi(x;\kappa,d,h_0), \tag{3.5}$$

with the parameters related by (3.1).

(ii) The second scaling gives an interval independent of the parameters. It is defined by

$$u = \frac{x}{2d}, \quad a = d\sqrt{\kappa h_0}, \quad \bar{\lambda} = 4\,\kappa^2\, d^2\, \tau. \tag{3.6}$$

It will be used when κd is small or for differentiating $\bar{f}$ with respect to a.

The spectral problem becomes

$$\begin{cases} -\dfrac{\partial^2\,\bar{\phi}}{\partial u^2} + 16\,a^4\,u^2\,\bar{\phi} = \bar{\lambda}\,\bar{\phi} & \text{in }]-1/2,1/2[, \\ (\partial_u \bar{\phi})(\pm 1/2;a) = 0, \\ \bar{\phi}(\cdot\,;a) \in H^2(]-1/2,1/2]), \end{cases} \tag{3.7}$$

where $\bar{\phi} = \bar{\phi}(u;a)$ and $\bar{\lambda} = \bar{\lambda}(a)$. The eigenfunction $\bar{\phi}$ is normalized so that

$$||\,\bar{\phi}(\cdot\,;a)\,||_{L^2(]-1/2,1/2[)} = 1, \tag{3.8}$$

and we then get the relation

$$\bar{\phi}(u;a) = (2d)^{1/2}\,\phi(x;\kappa,d,h).$$

(iii) The scalings (3.1) and (3.6) are connected by the change of variables and parameters

$$y = 2\,a\,u, \quad \mu(a) = \frac{\bar{\lambda}(a)}{4a^2} = \frac{\kappa^2\,d^2\,\tau}{a^2}, \tag{3.9}$$

and we have

$$\bar{\phi}(u;a) = (2a)^{1/2}\,\bar{f}(y;a). \tag{3.10}$$

Remark 3.1 *These two scalings will be used when $h_0 = \bar{h}(\kappa,d)$. In that case, the normalized eigenfunction ϕ of* (2.1)–(2.2) *depends only upon x, κ and d, and we have*

$$\mu(a) = \kappa\,\left(\bar{h}(\kappa,d)\right)^{-1} \quad \text{and} \quad \bar{\lambda}(a) = 4\,\kappa^2\,d^2. \tag{3.11}$$

We shall give the inverse transformation in the following section.

3.2 Variations of the parameters – an inverse scaling

The results stated in § 2.2 concern asymptotics when the product κd is large or small. The following lemma gives an equivalent by using the parameter a.

Lemma 3.2 (cf. Bolley & Helffer [1, § 2.2.2])
Let $\bar{\lambda}(a)$ be the principal eigenvalue of (3.7)*, then $a \mapsto \bar{\lambda}(a)$ is a strictly increasing function of a on* $]0, +\infty[$ *which satisfies*

$$\bar{\lambda}(0) = 0 \quad \textit{and} \quad \bar{\lambda}(+\infty) = +\infty.$$

More precisely, when $a \mapsto 0$,

$$\bar{\lambda}(a) = \tfrac{4}{3}\, a^4 + \mathcal{O}(a^8), \tag{3.12}$$

and when $a \mapsto +\infty$,

$$\bar{\lambda}(a) = 4\, a^2 + \mathcal{O}\left(\exp\left(-\tfrac{5}{8}\, a^2\right)\right). \tag{3.13}$$

Note that (3.12) results from a simple argument of perturbation theory.

From Lemma 3.2, the function $a \mapsto \bar{\lambda}(a)$ is one-to-one from $[0, +\infty[$ onto $[0, +\infty[$ and we can define a function α on $[0, +\infty[$ by

$$\alpha \;:\; s \in]0, +\infty[\mapsto \; \alpha(s) = \bar{\lambda}^{-1}(s^2). \tag{3.14}$$

The function α is also one-to-one from $[0, +\infty[$ onto $[0, +\infty[$, and is strictly increasing from 0 to $+\infty$.

Now, when $h_0 = \bar{h}(\kappa, d)$, we have, from (3.11), $\bar{\lambda}(a) = 4\, \kappa^2\, d^2$. We then get

Lemma 3.3 *Let $a(\kappa, d)$ be defined in* (3.1) *with $h_0 = \bar{h}(\kappa, d)$. Then a depends only upon the product κd by the formula*

$$a(\kappa, d) = \alpha(\kappa\, d), \tag{3.15}$$

where α is defined in (3.14)*. Moreover, the function $s \mapsto \alpha(s)$ is a C^∞ function on* $]0, +\infty[$ *satisfying:*

(i)

$$\alpha(s) = \left(\tfrac{3}{4}\right)^{\frac{1}{4}} \, (s)^{\frac{1}{2}} \, (1 + \mathcal{O}(s^2)) \quad \text{when } s \mapsto 0,$$

and

(ii)

$$\alpha(s) = \frac{s}{2} \left(1 + \mathcal{O}\left(\frac{1}{s^2} \exp\left(-\frac{5}{32} s^2\right)\right)\right) \quad \text{when } s \mapsto +\infty. \tag{3.16}$$

The lemma results from Lemma 3.2 and from the relation $2\, \kappa\, d = (\bar{\lambda}(a))^{1/2}$ recalled in (3.11). The regularity of α results from the regularity of the principal eigenvalue with respect to the coefficients in equation (3.7).

This inverse transformation was not made explicit in Bolley & Helffer [1], so this result completes § 3 of that work.

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3.3 Formulae for h_1

We recall that h_1 is defined in (2.6), and that it is a function of κ and d. It is given by the following formula (see Bolley & Helffer [6, § 2], or Millman & Keller [14]):

$$\begin{aligned} &2\,\frac{h_1(\kappa,d)}{\bar h(\kappa,d)}\int_{-d}^{d}(A_0(x;\kappa,d))^2\,(f_0(x;\kappa,d))^2\,dx \\ &\qquad= -\int_{-d}^{d}(f_0(x;\kappa,d))^4 dx + 2\int_{-d}^{d}\left(\frac{\partial A_{1,0}}{\partial x}(x;\kappa,d)\right)^2 dx, \end{aligned} \tag{3.17}$$

where, with the notation of Bolley & Helffer [6], $A_1(\cdot\,;\kappa,d)$ defined in (2.5) is split as

$$A_1(x;\kappa,d) = A_{1,0}(x;\kappa,d) + h_1(\kappa,d)\,x,$$

and where $A_{1,0}(\cdot\,;\kappa,d)$ is the unique solution in $H^2(]-d,d[)$ of the following problem:

$$\begin{cases} -\dfrac{\partial^2 A_{1,0}}{\partial x^2} + f_0^2\,A_0 = 0 \quad \text{in }]-d,d[, \\ \dfrac{\partial A_{1,0}}{\partial x}(\pm d;\kappa,d) = 0\,, \\ A_{1,0}(0;\kappa,d) = 0. \end{cases} \tag{3.18}$$

As in Bolley & Helffer [1], we would like to consider h_1 as a function of the parameters κ and a by using the scaling (3.1). So we get

$$\widetilde{h_1}(\kappa,a) = h_1(\kappa,(2\kappa)^{-1}\bar\lambda^{1/2}(a)) \quad \text{for } (\kappa,a)\in(I\!R^{+*})^2\,,$$

where we have used (3.14) and (3.15). In this scaling, $\widetilde{h_1}$ is given by

$$2\,\frac{\widetilde{h_1}(\kappa,a)}{\kappa}\int_{-a}^{a} y^2\,\bar f(y;a)^2\,dy = \frac{\kappa}{\sqrt{\mu(a)}}\cdot\widetilde{A}(\kappa,a), \tag{3.19}$$

where

$$\widetilde{A}(\kappa,a) \equiv -\int_{-a}^{a}\bar f(y;a)^4\,dy\; + 2\,\kappa^{-2}\int_{-a}^{a}\left(\frac{\partial\bar B}{\partial y}(y;a)\right)^2 dy, \tag{3.20}$$

and where $\bar B \in H^2(]-a,a[)$ is defined by

$$\bar B(y;a) = \kappa\,A_{1,0}(x;\kappa,d) \quad \text{for } y\in]-a,a[.$$

We verify that the function $\bar B$ depends only upon y and a. It is indeed the solution of the following equation which results from (3.18):

$$\begin{cases} \dfrac{\partial^2\bar B}{\partial y^2}(y;a) = y\,\bar f(y;a)^2 \quad \text{for } y\in]-a,a[, \\ \dfrac{\partial\bar B}{\partial y}(\pm a;a) = 0, \\ \bar B(0;a) = 0\,, \\ \bar B(\cdot\,;a)\in H^2(]-a,a[). \end{cases} \tag{3.21}$$

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3.4 Approximations of eigenfunctions for κd large

Accurate approximations for the eigen-elements of the spectral problem (3.3) will also be needed. They have been calculated by Bolley [4], when a tends to $+\infty$, by using a semi-classical analysis. Let us now describe these results.

In the spirit of Helffer & Sjöstrand [15], we have constructed an approximation of $\bar{f}$ and of μ valid when a tends to $+\infty$. For this purpose, we have considered, instead of (3.3), the problem on $\mathbb{R}$ with $\mu = 1$, i.e.

$$Pf = f \quad \text{in } \mathbb{R},$$

and a basis $\{\phi_1, \phi_2\}$, of solutions of this equation chosen such that (see Sibuya [21])

$$\phi_1(y) = \exp\left(-\frac{y^2}{2}\right) \quad \text{for } y \in \mathbb{R}, \tag{3.22}$$

$$\phi_2(y) = \exp\left(\frac{y^2}{2}\right) \cdot \frac{1}{y} \cdot \left(1 + \mathcal{O}\left(\frac{1}{y^2}\right)\right) \quad \text{as } y \to +\infty; \tag{3.23}$$

also, in this case,

$$\phi_2'(y) = \exp\left(\frac{y^2}{2}\right) \cdot \left(1 + \mathcal{O}\left(\frac{1}{y}\right)\right) \quad \text{as } y \to +\infty. \tag{3.24}$$

We have proved, using techniques from Helffer & Sjöstrand [15], the following result:

Proposition 3.4 (see Proposition 2.9 in Bolley [5])
There exist positive constants C_1, C_2 and a_0 such that for $a \geqslant a_0$,

(i)

$$|\mu(a) - 1| \leqslant C_1 \exp\left(-\tfrac{11}{16}\, a^2\right). \tag{3.25}$$

(ii)

$$\sup_{y \in [-a,a]} |\bar{f}(y) - \bar{f}^{app}(y)| \leqslant C_2 \exp\left(-\tfrac{5}{8}\, a^2\right), \tag{3.26}$$

where $\bar{f}^{app}$ is an even approximation of $\bar{f}$ in the form

$$\bar{f}^{app}(y) = \beta(a) \exp\left(-\frac{y^2}{2}\right) + \rho(a) \cdot \phi_2(|y|) \cdot \Xi\left(\frac{y}{a}\right), \tag{3.27}$$

with

$$\beta(a) = \pi^{-1/4} + \mathcal{O}(a \exp(-a^2)) \quad \textit{as } a \to +\infty, \tag{3.28}$$

$$\rho(a) = \pi^{-1/4}\, a \exp(-a^2) \cdot \left(1 + \mathcal{O}\left(\frac{1}{a^2}\right)\right), \tag{3.29}$$

and where Ξ is an even $C^\infty(\mathbb{R})$ cutoff function such that

$$\Xi(x) = 0 \textit{ if } |x| \leqslant 1/2\,; \quad \Xi(x) = 1 \textit{ if } |x| \geqslant 3/4.$$

Remark 3.5 *The coefficients $\beta(a)$ and $\rho(a)$ are defined in such a way that*

$$\left(\frac{\partial \bar{f}^{app}}{\partial y}\right)(\pm a; a) = 0, \tag{3.30}$$

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and

$$\|\bar{f}^{app}(\;\cdot\;;a)\|_{L^2(]-a,a[)} = 1. \tag{3.31}$$

Remark 3.6 *Proposition 3.4 immediately gives the following estimate for $\bar{f}(y)$, when $a \geqslant a_0$:*

$$\forall |y| \in [a/2, a]\,, \quad |\bar{f}(y)| \leqslant 2\pi^{-1/4} \exp\left(-\frac{y^2}{2}\right)\left(1 + \mathcal{O}\left(\frac{1}{y^2}\right)\right).$$

4 Localization of the zeros of $\widetilde{h_1}(\kappa, a)$ when a is large

4.1 Reduction to an implicit theorem

Let us consider the bifurcating solutions starting from symmetric normal solutions like $(0, h_0\,x;\, h_0)$ when $h_0 = \bar{h}(\kappa, d)$ for some pair (κ, d) of parameters. As proved in Theorem 2.1, when

$$\frac{\partial^2 \tau}{\partial e^2}(\kappa, d, h_0) \neq 0$$

there exists a unique curve of bifurcating solutions starting from $(0, h_0\,x;\, h_0)$. This is the case when κd is large enough (or small enough) (see Theorem 3.4 in Bolley & Helffer BoHe1997). Therefore, using Lemma 3.3, this is the case when a is large (or small) enough. Let us consider, in this section, the case when a is large.

To prove Theorem 2.2, in the next section we show the existence of a curve $d \to \kappa_0(d)$, defined for d large enough, and such that

$$\begin{cases} h_1(\kappa_0(d), d) = 0, \\ \lim\limits_{d \to +\infty} \kappa_0(d) = 2^{-1/2}, \end{cases} \tag{4.1}$$

which is the boundary line between $h_1^{-1}(]0, +\infty[)$ and $h_1^{-1}(]-\infty, 0[)$. Following the ideas of Bolley & Helffer [1], we first study h_1 in the scaling (3.1) by using (3.19); this will be a first step of the proof of Theorem 2.2.

According to (3.19) and the study of § 3.3, we note that

$$\widetilde{h_1}(\kappa, a) \;=\; 0 \iff \widetilde{\Lambda}(\kappa, a) \;=\; 0.$$

So, our purpose is to study the zero set of the map $(\kappa, a) \mapsto \widetilde{\Lambda}(\kappa, a)$.

We prove the following proposition.

Proposition 4.1 *There exists a constant $a_0 > 0$ and a C^1 function $r_0 : a \mapsto r_0(a)$ defined for $a \geqslant a_0$, satisfying*

$$\lim_{a \to +\infty} r_0(a) = 2^{-1/2}, \tag{4.2}$$

and such that, for $a \geqslant a_0$,

(i) $\widetilde{\Lambda}(r_0(a), a) = 0,$

(ii) for, $\kappa \neq r_0(a) : (r_0(a) - \kappa) \cdot \widetilde{\Lambda}(\kappa, a) > 0.$

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This function r_0 *satisfies, for some constants* $C > 0$,

$$\begin{cases} \text{(a)} \quad |r_0(a) - 2^{-1/2}| \leqslant C\, a^4 \;\exp\left(-\tfrac{1}{2}a^2\right), \\ \text{(b)} \quad |r_0'(a)| \leqslant C\, a^3 \;\exp\left(-\tfrac{1}{2}a^2\right) \quad \text{when } a \to +\infty\,. \end{cases} \tag{4.3}$$

This result improves Lemma 5.3 in Bolley & Helffer [1] and then, using (3.19), it improves Proposition 5.4 of the same paper.

4.2 Approximations of some partial derivatives with respect to a or y

The proof of Proposition 4.1 will use an Implicit Function Theorem and then estimates for the first partial derivative of $\bar f(\cdot\,;a)$ with respect to a.

Proposition 4.2 . *There exist constants* $C > 0$ *and* $a_0 > 0$ *such that for* $a \geqslant a_0$,

$$\left|\left| \frac{\partial \bar f}{\partial a}(\cdot\,;a) \right|\right|_{C^1([-a,a])} \leqslant C\, a^5 \exp\left(-\frac{a^2}{2}\right). \tag{4.4}$$

Proof Let us write an equation satisfied by

$$y \mapsto \frac{\partial \bar f}{\partial a}(y\,;a).$$

By differentiating the equation (3.3) with respect to a in any fixed interval of $]-a,a[$, we get

$$\begin{cases} -\dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial \bar f}{\partial a}\right)(y\,;a) + y^2\,\dfrac{\partial \bar f}{\partial a}(y\,;a) - \mu(a)\,\dfrac{\partial \bar f}{\partial a}(y\,;a) \\ \qquad\qquad\qquad = \mu'(a)\,\bar f(y\,;a) \quad \text{in }]-a,a[, \\ \dfrac{\partial \bar f}{\partial a}(\cdot\,;a) \in H^2(]-a,a[). \end{cases} \tag{4.5}$$

Now, using the normalization condition $||\,\bar f(\cdot\,;a)\,||_{L^2(]-a,a[)} = 1$, we get

$$2\int_a^a \bar f(y\,;a)\,\frac{\partial \bar f}{\partial a}(y\,;a)\,dy \;+\; [\bar f(a\,;a)^2 - \bar f(-a\,;a)^2] = 0,$$

and because $\bar f(\cdot\,;a)$ is even,

$$\left(\frac{\partial \bar f}{\partial a}(\cdot\,;a),\, \bar f(\cdot\,;a)\right)_{L^2(]-a,a[)} = 0. \tag{4.6}$$

It remains to compute the first partial derivative of $\bar f(\cdot\,;a)$ with respect to a at $y = \pm a$ By using that, for any $a > 0$

$$\frac{\partial \bar f}{\partial y}(\pm a\,;a) = 0,$$

we get, for example at $y = a$ (and an analogous result at $y = -a$);

$$\frac{\partial^2 \bar f}{\partial y\,\partial a}(a\,;a) = -\frac{\partial^2 \bar f}{\partial y^2}(a\,;a) = (\mu(a) - a^2)\bar f(a\,;a). \tag{4.7}$$

This computation can be justified by using an extension of $\bar f$ into a function $(y,a) \mapsto \bar g(y\,;a)$, which is a C^2 function on $\mathbb{R}\times]0,+\infty[$, and then by differentiating $\bar g$ with respect to a. Now, from Proposition 3.4, we get an estimate of $\bar{f}(a;a)$ when a tends to $+\infty$, and then from (4.7),

$$\frac{\partial^2 \bar{f}}{\partial y\, \partial a}(\pm a;a) = 2\,\pi^{-1/4}\ \left(-a^2+1+\mathcal{O}\left(a^{-2}\right)\right)\,\exp\left(-\frac{1}{2}a^2\right). \tag{4.8}$$

This means, in particular, that the function

$$y \mapsto \frac{\partial \bar{f}}{\partial a}(y;a)$$

does not satisfy the Neumann conditions at $\pm a$. For getting estimates on this function, we add, for large y, some even function in such a way that the sum will satisfy these Neumann conditions.

Let us consider the function $\bar{w}(\cdot;a)$ defined by

$$\bar{w}(y;a) = \frac{\partial \bar{f}}{\partial a}(y;a) + \gamma(a)\ \phi_2(|y|)\,\Xi\left(\frac{y}{a}\right) \quad \text{for } y \in [-a,a]\ ,$$

where ϕ_2 and Ξ are defined in Proposition 3.4, and where

$$\gamma(a) = -\left(\frac{\partial^2 \bar{f}}{\partial y\,\partial a}(a;a)\right)\cdot\left(\phi_2'(a)\right)^{-1} = \frac{(a^2-\mu(a))\,\bar{f}(a;a)}{\phi_2'(a)}.$$

This choice of $\gamma(a)$ gives

$$\frac{\partial \bar{w}}{\partial y}(\pm a;a) = 0.$$

Moreover, by once again using the approximation $\bar{f}^{app}(\cdot;a)$ of $\bar{f}(\cdot;a)$ given in Proposition 3.4, we get

$$\gamma(a) = 2\pi^{-1/4}\ a^2\ \exp(-a^2)\ \left(1+\mathcal{O}(a^{-1})\right) \quad \text{when } a \to +\infty. \tag{4.9}$$

The new function $\bar{w}(\cdot;a)$ satisfies the Neumann conditions at $\pm a$, but not the orthogonality condition necessary for our estimates, so we add to $\bar{w}(\cdot;a)$ another term belonging to the kernel of $P-I$. Let us define

$$\bar{Z}(y;a) = \bar{w}(y;a) + v(a)\,\bar{f}(y;a),$$

with

$$v(a) = -\,\gamma(a)\int_{-a}^{a}\phi_2(|y|)\,\Xi\left(\frac{y}{a}\right)\ \bar{f}(y;a)\,dy,$$

then

$$\left(\bar{Z}(\,\cdot\,;a),\bar{f}(\,\cdot\,;a)\right)_{L^2(]-a,a[)} = 0.$$

We get, using (3.23) and Proposition 3.4,

$$\begin{aligned} v(a) = &-2\,\gamma(a)\int_{a/2}^{a}\Xi\left(\frac{y}{a}\right)\ \left[\beta(a)\,\phi_1(y)\,\phi_2(y)+\rho(a)\,\phi_2(y)\,\Xi\left(\frac{y}{a}\right)\right]\ dy \\ &+\mathcal{O}\left(a\,\gamma(a)\exp\left(-\tfrac{5}{8}a^2\right)\,\phi_2(a)\right) \quad \text{when } a \to +\infty, \end{aligned} \tag{4.10}$$

and then

$$v(a) = \mathcal{O}(\gamma(a)) \quad \text{when } a \to +\infty. \tag{4.11}$$

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We have proved that the function $\bar{Z}(\cdot;a) \in H^2(]-a,a[)$ satisfies the following equations:

$$\begin{cases} -\dfrac{\partial^2 \bar{Z}}{\partial y^2} + y^2\,\bar{Z} - \mu(a)\,\bar{Z} \;=\; \mu'(a)\,\bar{f} - \gamma(a)\,(y^2-1)\,\phi_2(|y|)\,\Xi\left(\dfrac{y}{a}\right) \\ \qquad -\dfrac{\gamma(a)}{a}\,\phi_2'(|y|)\,\Xi'\left(\dfrac{y}{a}\right) - 2\,sign(a)\dfrac{\gamma(a)}{a^2}\,\phi_2(|y|)\,\Xi''\left(\dfrac{y}{a}\right) \quad \text{in }]-a,a[\,, \\ \dfrac{\partial \bar{Z}}{\partial y}(\pm a;a) = 0\,, \\ \left(\bar{Z}(\cdot;a), \bar{f}(\cdot;a)\right)_{L^2(]-a,a[)} = 0. \end{cases} \tag{4.12}$$

□

To obtain an estimate of $\bar{Z}(\cdot;a)$, we need an estimate for $\mu'(a)$.

Lemma 4.3 Approximation of $\mu'(a)$ when $a \to +\infty$
When a tends to $+\infty$, the principal eigenvalue $\mu(a)$ of (3.7) *verifies*

$$\mu'(a) = a\pi^{-1/2}\;\exp(-a^2)\cdot\left(1+\mathcal{O}(a^{-2})\right). \tag{4.13}$$

Proof Let us consider the equation (4.12). A compatibility relation gives, by the Heilman–Feynman formula

$$\mu'(a) + 2 = 4\,a\int_{-a}^{a} y\,\bar{f}(y;a)\,\frac{\partial \bar{f}}{\partial y}(y;a)\,dy, \tag{4.14}$$

because $\bar{f}(\cdot;a)$ is normalized in $L^2(]-a,\,a[)$. Using Proposition 3.4, we then get, when a tends to $+\infty$,

$$\mu'(a)) = 4\,a\,\bar{f}^2(a;a), \tag{4.15}$$

and then the lemma. □

Let us now define the spaces

$$G^a \equiv \{\phi \in L^2(]-a,a[)\;;\;(\phi,\bar{f}(.;a))_{L^2(]-a,a[)}\,\}$$

and

$$F^a \equiv \{\phi \in H^2(]-a,a[)\bigcap G^a;\;\phi'(\pm a) = 0\,\}.$$

We see from (4.5)–(4.7)

$$\begin{aligned} ||\,\bar{Z}(\,\cdot\,;a)\,||_{H^2(]-a,a[)} &\leqslant ||\,(P-\mu(a))^{-1}\,||_{\mathscr{L}(G^a,F^a)} \\ &\cdot\left[|\mu'(a)|\cdot||\,\bar{f}\,||_{L^2(]-a,a[)} + 2\,a^3\,\gamma(a)\phi_2(a) + 2\,\gamma(a)\,(\phi_2'(a)+\phi_2(a))\right], \end{aligned}$$

using that, for some positive constant C and for $a > 0$,

$$||\,(P-\mu(a)\,I)^{-1}\,||_{\mathscr{L}(G^a,F^a)} \leqslant C\;a^2.$$

Using also the estimates of Proposition 3.4 and Lemma 4.3 on $\mu'(a)$, we deduce that, when $a \to +\infty$

$$||\,\bar{Z}(\cdot;a)\,||_{H^2(]-a,a[)} \leqslant C\,a^5\exp\left(-\tfrac{1}{2}a^2\right). \tag{4.16}$$

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Using now that $\bar{Z}(\cdot;a)$ and $\frac{\partial \bar{f}}{\partial a}(\cdot\,;a)$ are related by the following relation:

$$\bar{Z}(y;a) = \frac{\partial \bar{f}}{\partial a}(y;a) + \gamma(a)\,\phi_2(|y|)\,\Xi\left(\frac{y}{a}\right) + v(a)\,\bar{f}(y;a) \quad \text{for } y \in [-a,a],$$

we get the proposition.

We shall also need later $\frac{\partial a}{\partial \kappa}(\kappa,d)$ and $\frac{\partial a}{\partial d}(\kappa,d)$ for κ and $d>0$.

Lemma 4.4 Approximation of $\frac{\partial a}{\partial \kappa}(\kappa,d)$ and of $\frac{\partial a}{\partial d}(\kappa,d)$
When a (or $\kappa\, d$) tends to $+\infty$:

$$\begin{aligned}\frac{\partial a}{\partial \kappa}(\kappa,d) &= \frac{a(\kappa,d)}{\kappa}\cdot\left(1+\mathcal{O}\left(a^2\exp\left(-\frac{5}{8}a^2\right)\right)\right),\\ \frac{\partial a}{\partial d}(\kappa,d) &= \frac{\kappa}{2}\cdot\left(1+\mathcal{O}\left(a^2\exp\left(-\frac{5}{8}a^2\right)\right)\right).\end{aligned}$$

Proof The two partial derivatives of a are given by differentiating the relation $\bar{\lambda}(a) = 4\,\kappa^2 d^2$. We get, by differentiation with respect to κ,

$$\frac{\partial a}{\partial \kappa}(\kappa,d) = 8\,\kappa d^2\,(\bar{\lambda}'(a))^{-1} = \frac{2\,\bar{\lambda}(a)}{\kappa}\,\left(\bar{\lambda}'(a)\right)^{-1}.$$

Now, using (3.13) and Lemma 3.3, we get the first part of the lemma.

Let us now differentiate $\bar{\lambda}(a(\kappa,d))$ with respect to d; we get

$$\frac{\partial a}{\partial d}(\kappa,d) = 8\,\kappa^2 d\,\left(\bar{\lambda}'(a)\right)^{-1} = 4\,\kappa\,(\bar{\lambda}(a))^{1/2}\,\left(\bar{\lambda}'(a)\right)^{-1},$$

and then, using (3.13) and Lemma 3.3, we get the second part of the lemma. □

4.3 Proof of Proposition 4.1

Let us now prove Proposition 4.1. It will be done in several steps.

Step 1: Extension of the function $\widetilde{\Lambda}$
By studying (3.7) and (3.5), then (3.21) and (3.20), we note that we can extend $\bar{f}(\cdot;a)$ as an even function of a to negative a, and $\bar{B}(\cdot;a)$ and $\widetilde{\Lambda}(\kappa,a)$ as odd functions of a. Moreover, from computations in Bolley & Helffer [1, Formula (5.13)]), $\widetilde{\Lambda}(\kappa,a)$ satisfies, for a large enough,

$$\widetilde{\Lambda}(\kappa,a) = 2^{-3/2}\,\pi^{-1/2}\,\kappa^{-2}\,\left[1-2\,\kappa^2+(1+\kappa^2)\,\mathcal{O}\left(a^2\exp\left(-\frac{5\,a^2}{8}\right)\right)\right]. \tag{4.17}$$

We then define a parameter $\bar{a}=a^{-1}$ and a function $\widetilde{\Lambda}_2$ of κ and $\bar{a}$, for $(\kappa,\bar{a})\in]0,+\infty[\times I\!R$ by

$$\begin{cases} \text{(a)} \quad \widetilde{\Lambda}_2(\kappa,\bar{a}) \equiv \widetilde{\Lambda}(\kappa,a) \quad \text{when } (\kappa,\bar{a})\in]0,+\infty[\times I\!R\setminus\{0\},\\ \text{(b)} \quad \widetilde{\Lambda}_2(\kappa,0) \equiv 2^{-3/2}\,\pi^{-1/2}\,\kappa^{-2}\,(1-2\,\kappa^2) \quad \text{when } \kappa>0.\end{cases} \tag{4.18}$$

From (4.18)(b),

$$\widetilde{\Lambda}_2(2^{-1/2},0)=0, \tag{4.19}$$

and we seek to apply the Implicit Function Theorem to the function $\widetilde{\Lambda}_2(\kappa,\bar{a})$ at the point

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$(\kappa, \bar{a}) = (2^{-1/2}, 0)$ to get the existence, for $\bar{a}$ small, of a unique C^1 function $\bar{a} \mapsto \bar{r}_0(\bar{a})$ such that

$$\begin{cases} \widetilde{\Lambda}_2(\bar{r}_0(\bar{a}), \bar{a}) = 0, \\ \bar{r}_0(0) = 2^{-1/2}. \end{cases} \tag{4.20}$$

The function r_0, defined for a positive large enough, by

$$r_0(a) = \bar{r}_0(\bar{a}),$$

will be proved to satisfy Proposition 4.1.

Step 2: The partial derivatives of $\widetilde{\Lambda}_2$

To apply the Implicit Function Theorem, we need to prove that $\widetilde{\Lambda}_2$ is continuously differentiable in a neighbourhood of the point $(\kappa, \bar{d}) = (2^{-1/2}, 0)$, and that $\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(2^{-1/2}, 0)$ is different from 0. The last point is immediate from (4.18)(b). We have, indeed :

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(2^{-1/2}, 0) = -\pi^{-1/2}. \tag{4.21}$$

(a) Regularity of $\widetilde{\Lambda}_2(\kappa, \bar{a})$ on $]0, +\infty[\times I\!R^*$

As functions $\bar{f}(\cdot\,; a)$ and $\partial_y \bar{B}(\cdot\,; a)$ are independent of κ and admit continuous partial derivatives with respect to the parameter a, we deduce that $\widetilde{\Lambda}_2(\kappa, \bar{a})$ is a C^1 function with respect to $(\kappa, \bar{a})$ in $]0, +\infty[\times]0, +\infty[$ and then, by parity, in $]0, +\infty[\times I\!R^*$. It remains to study the regularity of $\widetilde{\Lambda}_2$ on the half straight-line $D_0 \equiv \{(\kappa, 0)\ ;\ \kappa > 0\}$.

(b) **Continuity of $\widetilde{\Lambda}_2(\kappa, \bar{a})$ on D_0**

The continuity of $\widetilde{\Lambda}_2$ on $\{(\kappa, 0)\ ;\ \kappa > 0\}$ results from (4.17) and (4.18)(b).

Continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, 0)$

Let us prove the continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}$ at $(\kappa, 0)$ for $\kappa > 0$. We have, from $(4.18)_{(a)}$ and (3.20), and with $a = \bar{a}^{-1} \neq 0$,

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, \bar{a}) = -4\,\kappa^{-3} \int_{-a}^{a} (\frac{\partial \bar{B}}{\partial y}(y\,; a))^2\, dy. \tag{4.22}$$

Now, using the computations of Bolley & Helffer [1] (proof of Lemma 5.3), which gives the last integral, we get for positive $\bar{a}$ small enough and $\kappa > 0$,

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, \bar{a}) = -\,\kappa^{-3}\, 2^{-1/2}\, \pi^{-1/2} \cdot \left[1 + \mathcal{O}\left(\bar{a}^{-2} \exp\left(-\frac{5}{8}\bar{a}^{-2}\right)\right)\right], \tag{4.23}$$

with a $\mathcal{O}$ uniform in κ. The result is extended to negative $\bar{a}$ with $|\bar{a}|$ small enough, by parity. Consequently, for $\bar{a} \neq 0$:

$$\lim_{(\tilde{\kappa}, \bar{a}) \mapsto (\kappa, 0)} \frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\tilde{\kappa}, \bar{a}) = -\,\kappa^{-3}\, 2^{-1/2}\, \pi^{-1/2}.$$

Thanks to (4.18)(b), the continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}$ at $(\kappa, 0)$ follows.

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The partial derivative $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, 0)$

From the relations (4.17)–(4.18), the partial derivative of $\widetilde{\Lambda}_2$ with respect to $\bar{a}$ calculated at $(\kappa, \bar{a}) = (\kappa, 0)$, is given by the limit as $\eta \to 0$ of

$$\frac{\widetilde{\Lambda}_2(\kappa,\eta) - \widetilde{\Lambda}_2(\kappa,0)}{\eta} = 2^{-3/2}\,\pi^{-1/2}\,(\kappa^{-2}+1)\,\mathcal{O}\left(\bar{\eta}^{-3}\,\exp\left(-\frac{5}{8}\,\bar{\eta}^{-2}\right)\right).$$

Therefore, $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, \bar{a})$ exists and

$$\forall \kappa > 0, \quad \frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, 0) = 0. \tag{4.24}$$

Expression of $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a})$ for $\tilde{\kappa} > 0$ and $\bar{a} \in I\!R^*$

For the continuity of the partial derivative $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}$ at $(\kappa, 0)$ for $\kappa > 0$, we need more computations. Using (4.18), (3.20) and the parity of the different functions, we have

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a}) \;=\; -\,a^2\,\frac{\partial \widetilde{\Lambda}}{\partial a}(\tilde{\kappa}, a), \tag{4.25}$$

with

$$\begin{aligned} \frac{\partial \widetilde{\Lambda}}{\partial a}(\tilde{\kappa}, a) &= -2\bar{f}(a;a)^4 + 2\,\tilde{\kappa}^{-2}\,\left(\frac{\partial \bar{B}}{\partial y}(a;a)\right)^2 \\ &\quad -4\int_{-a}^{a} \bar{f}(y;a)^3\,\frac{\partial \bar{f}}{\partial a}(y;a)\,dy + 4\,\tilde{\kappa}^{-2}\int_{-a}^{a}\frac{\partial \bar{B}}{\partial y}(y;a)\,\frac{\partial^2 \bar{B}}{\partial a\,\partial y}(y;a)\,dy. \end{aligned} \tag{4.26}$$

Equations (4.25) and (4.26) allow us to use preceding computations performed when $a \to +\infty$.

Estimates for the first terms in (4.26)

In (4.26), the parameter $\tilde{\kappa}$ appears only in factor before the second and the third terms. So, we study only terms depending upon a. The analysis performed in Bolley [5] and partially recalled in Proposition 3.4, gave estimates for the three first terms. These terms satisfy, when a tends to $+\infty$,

$$\bar{f}(a;a)^4 \;=\; \mathcal{O}(\exp(-2a^2)), \tag{4.27}$$

$$\frac{\partial \bar{B}}{\partial y}(a;a)^2 \;=\; \mathcal{O}\left(a^2 \exp\left(-\frac{5}{4}a^2\right)\right). \tag{4.28}$$

and as consequence of Lemma 4.2, using also that $\bar{f}$ is bounded,

$$\int_{-a}^{a} \bar{f}^3(y;a)\,\frac{\partial \bar{f}}{\partial a}(y;a)\,dy = \mathcal{O}\left(a^2\,\exp\left(-\frac{1}{2}a^2\right)\right) \quad \text{when } a \to +\infty. \tag{4.29}$$

The partial derivative $\frac{\partial^2 \bar{B}}{\partial a \partial y}$

Let us now consider the partial derivative $\frac{\partial \bar{B}}{\partial a}(\cdot\,; a)$. Integrating (3.21) gives

$$\frac{\partial^2 \bar{B}}{\partial a \, \partial y}(y\,; a) = a \bar{f}(a\,; a)^2 + \int_{-a}^{y} 2t \, \bar{f}(t, a) \, \frac{\partial \bar{f}}{\partial a}(t\,; a) \, dt.$$

Therefore, using this relation and the usual approximation of $\bar{f}$ (Proposition 3.4), we get, as a tends to $+\infty$, uniformly on $[-a, a]$,

$$\frac{\partial^2 \bar{B}}{\partial a \, \partial y}(y\,; a) = \mathcal{O}\left(a^2 \exp\left(-\frac{1}{2}a^2\right)\right). \tag{4.30}$$

Moreover, computations in Bolley & Helffer [1] (proof of Lemma 5.3) give

$$\frac{\partial \bar{B}}{\partial y}(y\,; a) = -2^{-1}\pi^{-1/2} \exp(-y^2) + \mathcal{O}\left(a \exp\left(-\frac{5}{8}a^2\right)\right). \tag{4.31}$$

Consequently, when $a \to +\infty$,

$$\int_{-a}^{a} \frac{\partial \bar{B}}{\partial y} \, \frac{\partial^2 B}{\partial a \, \partial y} \, dy = +\mathcal{O}\left(a^3 \exp\left(-\frac{1}{2}a^2\right)\right). \tag{4.32}$$

In this step, the terms, and then the $\mathcal{O}$, do not depend upon κ.

Coming back to (4.25) and (4.26), we get, from the preceding steps (4.27)–(4.29) and (4.32), an estimate of $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a})$. We have proved

Lemma 4.5 *For $(\tilde{\kappa}, \bar{a}) \in]0, +\infty[\times I\!R^*$ and $\bar{a}$ small enough,*

$$\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a}) \;=\; \mathcal{O}\left(\bar{a}^{-5} \exp\left(-\frac{1}{2\bar{a}^2}\right)\right),$$

where the $\mathcal{O}$ is uniform for $\tilde{\kappa}$ in a compact set of $]0, +\infty[$.

Continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}$ at $(\kappa, 0)$

It results from Lemma 4.5 that, for all $\kappa > 0$, $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\tilde{\kappa}, \bar{a})$ tends to 0 as $(\tilde{\kappa}, \bar{a})$ tends to $(\kappa, 0)$. Using (4.24), we get the continuity of $\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}$ at any point $(\kappa, 0)$.

(c) **Conclusion**

The preceding steps imply that the Implicit Function Theorem can be applied to $\widetilde{\Lambda}_2$ at $(2^{-1/2}, 0)$. The following lemma results:

Lemma 4.6 *There exists a constant $\bar{a}_0 > 0$ and a C^1 function $\bar{a} \mapsto \bar{r}_0(\bar{a})$ defined, for $|\bar{a}| \leqslant \bar{a}_0$ such that* (4.20) *is satisfied, that is such that*

$$\begin{cases} \widetilde{\Lambda}_2(\bar{r}_0(\bar{a}), \bar{a}) = 0, \\ \bar{r}_0(0) = 2^{-1/2}. \end{cases} \tag{4.33}$$

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As a consequence, we get the existence of a C^1 function $a \to r_0(a)$ defined for $a \geqslant \bar{a}_0^{-1}$, by

$$r_0(a) = \bar{r}_0\left(\frac{1}{a}\right),$$

such that (4.2) and Part (i) of Proposition 4.1 are satisfied.

To prove (ii), we note, using (4.18)(b) with the parameter $\bar{a} = a^{-1}$, that for $\kappa > 0$,

$$\widetilde{\Lambda}_2(\kappa, 0) \cdot (\bar{r}_0(0) - \kappa) > 0,$$

with $\bar{r}_0(0) = 2^{-1/2}$. By continuity of $\widetilde{\Lambda}_2$, it results that, for $\kappa > 0$ and $\bar{a}$ small enough,

$$\widetilde{\Lambda}_2(\kappa, \bar{a}) \cdot (\bar{r}_0(\bar{a}) - \kappa) > 0.$$

Part (ii) in Proposition 4.1 follows.

For getting the asymptotic behaviour of r_0', we note that

$$r_0'(a) = -\frac{1}{a^2}\bar{r}_0'(a^{-1}) = \frac{1}{a^2}\left(\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, \bar{a})\right) \cdot \left(\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, \bar{a})\right)^{-1},$$

so that, using (4.23) and Lemma 4.5, we get (4.3)(b), and then (4.3)(a).

4.4 Proof of Theorem 2.2

Let us now prove Theorem 2.2, which is written in the physical parameters κ and d. The proof is a consequence of Proposition 4.1, where the parameters κ and a were used.

The first problem is to prove the existence of a curve $d \mapsto \kappa_0(d)$ defined for large d and $\kappa_0(d)$ near $2^{-1/2}$, such that, for $(\kappa, d) = (\kappa_0(d), d)$, the parameter $h_1(\kappa, d)$ cancels. We have studied this problem in the previous section in the parameters κ and a for large a, where we have used that $h_1 = \widetilde{h}_1(\kappa, a)$ is equal to 0 if and only if $\widetilde{\Lambda}(\kappa, a) = 0$. But, from Lemma 3.3, we have

$$a(\kappa, d) = \alpha(\kappa d),$$

so that $\widetilde{\Lambda}$ can be written as a function of κ and d: we then define

$$\Lambda(\kappa, d) = \widetilde{\Lambda}(\kappa, \alpha(\kappa d)) \quad \text{for } (\kappa, d) \in (\mathbb{R}^{+*})^2.$$

The existence of a curve $r_0 : a \mapsto r_0(a)$ (see Lemma 4.6) defined for $a \geqslant a_0$ such that $\widetilde{\Lambda}(r_0(a), a) = 0$ for $a \geqslant a_0$ will give us the existence of κ_0 for large d such that (4.1) is satisfied.

The zeros of $\widetilde{\Lambda}$ are given by the relation

$$\kappa = r_0(a) \quad \text{with } a = \alpha(\kappa d),$$

so that we get a relation between κ and d giving $\Lambda = 0$.

We then let

$$F(\kappa, d) \equiv \kappa - (r_0 \circ \alpha)(\kappa d), \tag{4.34}$$

and we note that

$$\lim_{d \mapsto +\infty} F(2^{-1/2}, d) = 0.$$

It is then easy to apply an Implicit Function Theorem to the function F at the point $(2^{-1/2}, +\infty)$ (or, if we prefer, at the point $(2^{-1/2}, 0)$ to a function $(\kappa, \bar{d}) \mapsto \bar{F}(\kappa, \bar{d})$ defined by $\bar{F}(\kappa, \bar{d}) = F(\kappa, d^{-1})$ with $\bar{d} = d^{-1}$ and extended to null and negative $\bar{d}$), for getting the existence of a C^1-curve $\kappa_0 \; : \; d \mapsto \kappa_0(d)$ defined for large d, and such that $h_1(\kappa_0(d), d) = 0$ for large d.

The continuity and differentiability of F in a neighbourhood of $(2^{-1/2}, \infty)$ comes, indeed, from the composition of the two functions $a \mapsto r_0(a)$ and $s \mapsto \alpha(s)$ by using that, from Lemma 4.6, r_0 is continuously differentiable in a neighbourhood of ∞. The continuity and the differentiability of α for large s ($\kappa > 0$ is close to $2^{-1/2}$ and d is large) comes from Lemma 3.3. We also have to verify that $\frac{\partial F}{\partial \kappa}(2^{-1/2}, d)$ is different from zero when d tends to 0.

Let us show that $\frac{\partial F}{\partial \kappa}(2^{-1/2}, d) = 1 + \mathcal{O}\left(d^3 \exp(-d^2)\right)$ as $d \to +\infty$. We have, by differentiating (4.34) and with $\sigma = 2^{1/2} d$:

$$\frac{\partial F}{\partial \kappa}(2^{-1/2}, d) = 1 - d\, r_0'(\alpha(\sigma))\, \alpha'(\sigma),$$

with, from Lemma 3.3, $\alpha(\sigma) = 2^{-1/2} d (1 + \mathcal{O}\left(d^{-2} \exp\left(-\frac{5}{16} d^2\right)\right)$ when $d \to \infty$. However,

$$r_0'(a) = -\frac{1}{a^2}\, \bar{r}_0'(a^{-1}),$$

and

$$\bar{r}_0'(\bar{a}) = -\left(\frac{\partial \widetilde{\Lambda}_2}{\partial \bar{a}}(\kappa, \bar{a})\right) \cdot \left(\frac{\partial \widetilde{\Lambda}_2}{\partial \kappa}(\kappa, \bar{a})\right)^{-1},$$

so that, using (4.23) and Lemma 4.5, we get

$$r_0'(\alpha(\sigma)) \; = \mathcal{O}\left(\alpha(\sigma)^3 \exp\left(-\tfrac{1}{2}\alpha(\sigma)^2\right)\right).$$

Now, from Lemma 3.3, we have, when $d \to +\infty$,

$$r_0'(\alpha(\sigma)) = \mathcal{O}\left(d^3 \, \exp\left(-\, d^2\right)\right) \; ,$$

and

$$\alpha'(s) = 2s\, \left(\bar{\lambda}'\right)^{-1} = \left(1 + \mathcal{O}\left(d^{-2} \exp\left(-\tfrac{5}{16} d^2\right)\right)\right) \; .$$

Therefore,

$$\frac{\partial \bar{F}}{\partial \kappa}(2^{-1/2}, d) \; = \; 1 + \mathcal{O}\left(d^3 \exp(-d^2)\right) \quad \text{as } d \to +\infty\,. \tag{4.35}$$

4.5 Conclusion

The Implicit Function Theorem applied to F at the point $(2^{-1/2}, 0)$, gives the existence of a constant $d_0 > 0$ and of a C^1 curve $\kappa_0 \; : \; d \mapsto \kappa_0(d)$ defined for $d \geqslant d_0$, such that

$$F(\kappa_0(d), d) = 0 \quad \text{with} \lim_{d \to +\infty} \kappa_0(d) \; = \; 2^{-1/2}.$$

We then get Theorem 2.2.

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5 The sign of h_1 when κd is small

Let us now consider the case when κd is small. Theorem 5.9 in Bolley & Helffer [1] establishes, in that case, the existence of a family of bifurcating solutions, starting from the normal solution $(0, h_0 x; h_0)$, where $h_0 = \bar{h}(\kappa, d)$, and parametrized, in a neighbourhood of the normal solution, by

$$\sigma = 2\kappa d\ ,\ d\ ,\ \bar{\varepsilon} = (2d)^{-1/2}\,\varepsilon \text{ and } u = \frac{x}{2d} \in [-1/2, 1/2],$$

for $\bar{\varepsilon}$ small enough. These solutions are described by $(g(\cdot\,;\sigma, d, \bar{\varepsilon}), V(\cdot\,;\sigma, d, \bar{\varepsilon});$ $\eta(\sigma, d, \bar{\varepsilon}))$ where

$$\begin{aligned} &g(u;\,\sigma, d, \bar{\varepsilon}) = f(x;\kappa, d, \varepsilon);\\ &V(u;\,\sigma, d, \bar{\varepsilon}) = A(x;\,\kappa, d, \varepsilon) \qquad \text{with } u \in [-1/2, 1/2]\\ &\eta(\sigma, d, \bar{\varepsilon}) = 2d\, h(\kappa, d, \varepsilon). \end{aligned} \tag{5.1}$$

From Bolley & Helffer [1], this family of bifurcating solutions admits an expansion in power of $\bar{\varepsilon}$ such that, for $\bar{\varepsilon}$ small,

$$\begin{aligned} &g(\cdot\,;\sigma, d, \bar{\varepsilon}) = \bar{\varepsilon}\, g_0(\cdot\,;\sigma, d) + \bar{\varepsilon}^3\, \tilde{g}(\cdot\,;\sigma, d, \bar{\varepsilon}),\\ &V(\cdot\,;\,\sigma, d, \bar{\varepsilon}) = V_0(\cdot\,;\,\sigma, d) + \bar{\varepsilon}^2\, V_1(\cdot\,;\,\sigma, d) + o(\bar{\varepsilon}^2) \quad \text{in } H^2(]-1/2, 1/2[),\\ &\partial_u V(\pm 1/2;\,\sigma, d, \bar{\varepsilon}) = \eta(\sigma, d, \bar{\varepsilon}) = \eta_0(\sigma, d) + \bar{\varepsilon}^2\, 2d\, \bar{\bar{h}}_1(\sigma, d) + o(\bar{\varepsilon}^2), \end{aligned}$$

where $\tilde{g}(\cdot\,;\sigma, d, \bar{\varepsilon}) \in H^2(]-1/2, 1/2[)$ satisfies

$$(g_0(\cdot\,;\sigma, d), \tilde{g}(\cdot\,;\sigma, d))_{L^2(]-1/2,1/2[)} = 0,$$

and where $g_0(\cdot\,;\sigma, d)$ is the normalized principal eigenfunction of (3.7) in the new parameters.

Let us define

$$\bar{\bar{h}}_0(\sigma, d) = \bar{h}\left((2d)^{-1}\sigma, d\right) \quad \text{and} \quad \bar{\bar{h}}_1(\sigma, d) = h_1\left((2d)^{-1}\sigma, d\right);$$

we then get

$$\eta_0(\sigma, d) = 2d\, \bar{\bar{h}}_0(\sigma, d).$$

Moreover, in Bolley & Helffer [1, Theorem 5.8], we have obtained results on the sign of $\bar{\bar{h}}_1(\sigma, d)$, outside the domain Ω^2_η defined in § 2. Our purpose in this subsection is first to study the sign of $\bar{\bar{h}}_1(\sigma, d)$ in this domain, and then to come back to the initial units (κ and d) for getting Theorem 2.4.

We get from the expansion of the bifurcating solutions a formula analogous to (3.19) in our new scaling,

$$\begin{aligned} &2\,\frac{\bar{\bar{h}}_1(\sigma, d)}{\bar{\bar{h}}_0(\sigma, d)} \int_{-1/2}^{1/2} V_0(u;\sigma, d)^2\, g_0(u;\sigma, d)^2\, du\\ &\quad = -\int_{-1/2}^{1/2} g_0(u;\sigma, d)^4\, du + 2^{-1}\, d^{-2} \int_{-1/2}^{1/2} \left(\frac{\partial V_{1,0}}{\partial u}(u;\sigma, d)\right)^2\, du, \end{aligned} \tag{5.2}$$

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where $V_{1,0}(u;\sigma,d) = A_{1,0}(x;\kappa,d)$, so that $V_{1,0}(\cdot;\sigma,d)$ satisfies

$$\begin{cases} -\dfrac{\partial^2 V_{1,0}}{\partial u^2} + 4d^2\, g_0(\cdot;\sigma,d)^2\, V_0(\cdot;\sigma,d) = 0 \quad \text{in }]-1/2,1/2[, \\ \partial_u V_{1,0}(\pm 1/2;\sigma,d) = 0, \\ V_{1,0}(\pm 1/2;\sigma,d) = 0. \end{cases} \tag{5.3}$$

We proceed now as in the preceding section. We denote by $\bar{\bar{A}}_2$ the right-hand side of (5.2). It is a function of $\sigma \geqslant 0$ and $d > 0$ given by

$$\bar{\bar{A}}_2(\sigma,d) \equiv -\int_{-1/2}^{1/2} (g_0(u;\sigma,d))^4\, du + 2^{-1}\, d^{-2} \int_{-1/2}^{1/2} \left(\frac{\partial V_{1,0}}{\partial u}(u;\sigma,d)\right)^2 du, \tag{5.4}$$

but we note that the Ginzburg–Landau equations parametrized by σ and d can be extended to negative values of σ in such a way that $g_0(\cdot;\sigma,d)$ is an even function of σ. From (5.4) and (5.3), $\bar{\bar{A}}_2(\sigma,d)$ is an even function of σ.

We have $\bar{\bar{h}}_1(\sigma,d) = 0$ if and only if $\bar{\bar{A}}_2(\sigma,d) = 0$, and we are interested in solving the problem near $d = \sqrt{5}/2$. When $\sigma = 0$, we see that

$$V_{1,0}(u;0,d) = \frac{4\sqrt{3}}{3}\, d^2\, u^3 - \sqrt{3}\, d^2\, u,$$

so that

$$\bar{\bar{A}}_2(0,d) = -1 + \frac{4d^2}{5} \quad \text{for } d > 0. \tag{5.5}$$

We now apply the Implicit Function Theorem to $\bar{\bar{A}}_2$ at $(\sigma,d) = (0,\sqrt{5}/2)$ to get the existence of a curve $\sigma \mapsto \bar{d}_1(\sigma)$ defined for σ in a neighbourhood of 0, and such that

$$\begin{cases} \bar{\bar{A}}_2(\sigma,\bar{d}_1(\sigma)) = 0, \\ \bar{d}_1(0) = \frac{\sqrt{5}}{2}\,. \end{cases} \tag{5.6}$$

The regularity of the function $\bar{\bar{A}}_2$ with respect to (σ,d) in $[0,+\infty[\times]0,+\infty[$ will result from the regularity of the functions g_0 and $\partial_u V_{1,0}$. However, $g_0(\cdot;\sigma,d)$ satisfies

$$\begin{cases} -\dfrac{\partial^2 g_0}{\partial u^2} + \sigma^2\, \eta_0(\sigma,d)^2\, u^2\, g_0 = \sigma^2\, g_0 \quad \text{in }]-1/2,1/2[, \\ (\partial_u g_0)(\pm 1/2;\sigma,d) = 0, \\ g_0(\cdot;\sigma,d) > 0, \\ g_0(\cdot;\sigma,d) \in H^2(]-1/2,1/2[), \end{cases} \tag{5.7}$$

and $\partial_u V_{1,0}$ satisfies (5.3). These are Sturm–Liouville type equations, so by standard theorems we get that g_0 and $\partial_u V_{1,0}$ are C^∞ with respect to the parameters σ and d in a neighbourhood of $(0,\sqrt{5}/2)$. Consequently, $\bar{\bar{A}}_2$ is a C^∞ function in (σ,d) for (σ,d) in a neighbourhood of $(0,\sqrt{5}/2)$.

We now just observe that, according to (5.5):

$$\frac{\partial \bar{\bar{A}}_2}{\partial d}(0,\sqrt{5}/2) = \frac{2\sqrt{5}}{5} \neq 0. \tag{5.8}$$

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Thus, the Implicit Function Theorem implies the existence of a C^1 curve $\sigma \mapsto \bar{d}_1(\sigma)$ defined for σ small and satisfying (5.6).

Using now the relation $\sigma = 2\kappa d$, we can proceed as in the proof of Theorem 2.2 to get the existence of a C^1 curve $\kappa \mapsto d_1(\kappa)$ defined for κ small and satisfying Theorem 2.4. The last part of the theorem and Corollary 2.5 follows from the estimate, proved for κd small, Bolley & Helffer [1, Formula (5.24)]:

$$h(\kappa, d, \epsilon) = \bar{h}(\kappa, d) - \frac{\sqrt{3}}{4d^2}\left(1 - \frac{4d^2}{5} + (1+4d^2)\,\mathcal{O}(\kappa^2 d^2)\right)\epsilon^2 + \mathcal{O}_{\kappa,d}(\epsilon^3). \tag{5.9}$$

6 Remarks on the stability of the symmetric bifurcating solutions

6.1 Local stability for the problem $(GL)_d$

Let (κ, d) be in $(]0, +\infty[)^2$. A solution $(\hat{f}, \hat{A}; \hat{h})$ of the Ginzburg–Landau equations $(GL)_d$ is said to be *locally stable* if, at fixed $\hat{h}$, it gives a local minimum of the GL functional $(\Delta G)_h(f, A)$ attached to $(GL)_d$, with respect to the pairs (f, A). Otherwise, it is said to be *unstable*. We recall that the GL functional $(\Delta G)_h(f, A)$ is defined on $H^1(]-d, d[)^2$ by (see, for example, Bolley & Helffer [1])

$$(f, A) \mapsto (\Delta G)_h(f, A) = \int_{-d}^{d}\left[\kappa^{-2} f'^2 - f^2 + \frac{1}{2}f^4 + A^2 f^2 + (A' - h)^2\right] dx. \tag{6.1}$$

From Bolley & Helffer [1, Theorem 4.5], we see that, for κd large, the symmetric bifurcating solutions are unstable. This instability is completely determined for κd large enough, whatever the sign of $h_1(\kappa, d)$. So, the preceding sections are not useful for this study. But, when κd is small, Theorem 2.4 allows us to improve Theorem 4.3 of Bolley & Helffer [1] by the following theorem:

Theorem 6.1 *There exist strictly positive constants a_0 and a_1 such that, for (κ, d) in $]0, +\infty[)^2$ satisfying $\kappa d \leqslant a_0$ and $0 < \kappa \leqslant a_1$, and for h_0 satisfying (2.4), there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, the following properties are satisfied:*

(i) *When $d < d_1(\kappa)$, the bifurcating solutions starting from the normal solution $(0, h_0\, x; h_0)$ are locally stable.*

(ii) *When $d > d_1(\kappa)$, the bifurcating solutions starting from the normal solution $(0, h_0\, x; h_0)$ are unstable,*

where $d_1(\kappa)$ is the function defined in Theorem 2.4.

Proof We come back to the proof of Theorem 4.3 in Bolley & Helffer [1]. We consider here the local stability of symmetric bifurcating solutions in $\left(H^1(]-d, d[)\right)^2$, so that the perturbations are not restricted to symmetric functions. In that case, the eigenvalue $\tau = 0$ of the spectral problem

$$\left\{\begin{array}{ll} \text{(a)} & \left\{\begin{array}{l} -\kappa^{-2}\phi'' + h_0^2\, x^2\, \phi - \phi = \tau\, \phi \quad \text{in }]-d, d[\\ \phi'(\pm d) = 0, \end{array}\right. \\ \text{(b)} & \left\{\begin{array}{l} -v'' = \tau\, v \quad \text{in }]-d, d[\\ v'(\pm d) = 0, \end{array}\right. \end{array}\right. \tag{6.2}$$

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with $(\phi, v) \in (H^2(]-d,d[))^2$, associated to the normal solution $(0, h_0 x; h_0)$ has a multiplicity of *two* (with $\binom{0}{c}$, where $c \in \mathbb{R}$, as an eigenvector). The study of the stability of the bifurcating solutions is then reduced to the study of the sign of the two lowest eigenvalues, denoted $\lambda^{(1)}(\varepsilon, \kappa, d)$ and $\lambda^{(2)}(\varepsilon, \kappa, d)$ of the operator

$$\left(H^2(]-d,d[)\right)^2 \ni \begin{pmatrix} g \\ b \end{pmatrix} \longmapsto \begin{pmatrix} -\kappa^{-2}g'' + (A^2 + 3f^2 - 1)\,g + 2\,A\,f\,b \\ -b'' + f^2\,b + 2A\,f\,g \end{pmatrix}, \tag{6.3}$$

with $A = A(\cdot, \kappa, d, \varepsilon)$, $f = f(\cdot, \kappa, d, \varepsilon)$ and Neumann conditions, when ε is small. When $\varepsilon = 0$, 0 is the lowest eigenvalue.

When $|\varepsilon| > 0$, using a perturbation analysis, we have seen that the sign of the eigenvalues $\lambda^{(1)}(\varepsilon, \kappa, d)$ and $\lambda^{(2)}(\varepsilon, \kappa, d)$, is given, for ε small, by the sign of two reals denoted $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ which depend upon κ and d.

We have proved that $\lambda_2^{(2)}(\kappa, d) > 0$ when κd is small (see Bolley & Helffer [1, Lemma 7.3 i]), and that the sign of $\lambda_2^{(1)}(\kappa, d)$ is given by the sign of $h_1(\kappa, d)$ (see Bolley & Helffer [1, (7.6)]). Therefore, Theorem 6.1 results from Theorem 2.4. □

6.2 Local stability for a reduced symmetric problem

When κd is large, the problem is different. Our study (see Bolley & Helffer [1, Theorem 4.5]) of the stability of the bifurcating solutions with respect to the GL functional $(\Delta G)_h$, has proved that the symmetric bifurcating solutions are unstable for κd large enough. If we now restrict ourselves to symmetric solutions (as was done elsewhere [17–20]), we restrict the domain of the GL functional to a subset of $\left(H^1(]-d,d[)\right)^2$ corresponding to these solutions. This subset is defined, for fixed κ and d, by

$$\begin{aligned} \mathscr{H}_{sym} = \{ &(f, A) \in (H^1(]-d,d[))^2 \ ; \\ &\forall x \in]-d,d[,\ (f(-x), A(-x)) = (f(x), -A(x)) \}. \end{aligned}$$

By considering this problem, the stability of a symmetric bifurcating solution is different when κd is large. We prove the following theorem:

Theorem 6.2 *There exist constants $d_0 > 0$ and $a_0 > 0$ such that, for (κ, d) in $(]0, +\infty[)^2$ satisfying $\kappa d \geqslant a_0$ and $d \geqslant d_0$, and for $h_0(\kappa, d)$ satisfying* (2.4)*, the following properties are satisfied.*

(i) *When $\kappa > \kappa_0(d)$, the bifurcating solutions starting from $(0, h_0\,x; h_0)$ are locally stable by respect to the functional $(\Delta G)_h$, in restriction to the set $\mathscr{H}_{sym}$.*

(ii) *When $\kappa < \kappa_0(d)$, they are unstable,*
where $\kappa_0\ :\ d \mapsto \kappa_0(d)$ is the map defined in Theorem 2.2.

Proof It is sufficient to prove that the bifurcating solutions give local minima for $(\Delta G)_h$, when h is fixed. So, as in the proof of Bolley & Helffer [1, Theorem 4.5], we study the Hessian of the functional in restriction to $\mathscr{H}_{sym}$ corresponding to the symmetric solutions.

The study of the local stability is then reduced to an analysis of the spectrum of the self-adjoint operator attached to this problem. This leads us to study the corresponding linearized GL equations (6.3) at the triple $(f(.,\varepsilon,\kappa,d), A(\cdot,\varepsilon,\kappa,d); h(\varepsilon,\kappa,d))$, but when the solutions (g,b) are sought in $\mathscr{H}_{sym} \bigcap \left(H^2(]-d,d[)\right)^2$. □

By the choice $h_0(\kappa,d) = \bar{h}(\kappa,d)$, the lowest eigenvalue for this problem, when $\varepsilon = 0$, is equal to *zero* with multiplicity *one* (but *two* for the full problem). The corresponding eigenspace is generated by

$$\bar{u}_1(\cdot;\kappa,d) = \begin{pmatrix} f_0(\cdot;\kappa,d) \\ 0 \end{pmatrix},$$

where $f_0(\cdot;\kappa,d)$ is the normalized eigenfunction of (2.1) where $\tau = 1$. By regular perturbation analysis, we get,

Lemma 6.3 *Let $d > 0$, $\kappa > 0$ and h_0 satisfying* (2.4). *Then, there exists ε_0 such that, for $0 < |\varepsilon| \leqslant \varepsilon_0$, the eigen-elements corresponding to the first lowest eigenvalue of* (6.3) *with (g,b) in $\mathscr{H}_{sym} \bigcap \left(H^2(]-d,d[)\right)^2$, are described by*

$$\begin{cases} \mu^{(1)}(\varepsilon,\kappa,d) = \varepsilon^2 \mu_2^{(1)}(\kappa,d) + o(\varepsilon^2), \\ g^{(1)}(x,\varepsilon,\kappa,d) = f_0(x;\kappa,d) + o(\varepsilon) \quad \text{in } H^2(]-d,d[), \\ b^{(1)}(x,\varepsilon,\kappa,d) = 2\,\varepsilon\, A_{1,0}(x;\kappa,d) + o(\varepsilon^2), \end{cases} \tag{6.4}$$

with

$$\mu_2^{(1)}(\kappa,d) = -4\,\frac{h_1(\kappa,d)}{h_0(\kappa,d)} \int_{-d}^{d} (A_0(x;\kappa,d))^2\,(f_0(x;\kappa,d))^2\,dx. \tag{6.5}$$

To prove Theorem 6.2, it is then sufficient to note that, using the symmetry of $f_0(\cdot;\kappa,d)$ and $A_0(\cdot;\kappa,d)$, we have

$$\lambda_2^{(1)}(\kappa,d) = \mu_2^{(1)}(\kappa,d),$$

and then to apply Theorem 2.2. We have, indeed, $h_1(\kappa,d) < 0$ when $\kappa > \kappa_0(d)$ and $h_1(\kappa,d) > 0$ when $\kappa < \kappa_0(d)$.

7 Conclusion

This study gives a complete answer to the stability of bifurcating symmetric solutions when κd is large or small. However, we still have no information on the sign of h_1 in domains of the type $\{(\kappa,d)\ ;\ a_1 < \kappa d < a_2\}$ for $(a_1,a_2) \in (I\!R^{+*})^2$. Let us note that alternative proofs of some of the results (see Theorem 2.2 and Theorem 2.4) can also be obtained by other techniques, as is shown by Hastings [20].

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