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Quasi-equivalence of bases in some Whitney spaces

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Abstract. If the logarithmic dimension of a Cantor-type set K is smaller than 1, then the Whitney space $\mathcal{E}(K)$ possesses an interpolating Faber basis. For any generalized Cantor-type set K, a basis in $\mathcal{E}(K)$ can be presented by means of functions that are polynomials locally. This gives a plenty of bases in each space $\mathcal{E}(K)$. We show that these bases are quasi-equivalent.

1 Introduction

If X is a Banach space with unconditional topological basis $(f_k)_{k=1}^{\infty}$, then any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence of nonzero scalars $(\rho_k)_{k=1}^{\infty}$ generate another basis $(e_k)_{k=1}^{\infty}$ with $e_k = \rho_k f_{\sigma(k)}$. In the case when any unconditional basis $(e_k)_{k=1}^{\infty}$ of X can be represented in this form, we say that X has unique unconditional basis (up to permutation). The property of unicity of basis is quite exceptional in the context of Banach spaces [11].

We observe the opposite situation in the class of nuclear Fréchet (NF) spaces, where, by T.9 in [14], all bases are absolute. In the field, similar in the above sense, bases are called quasi-equivalent. Namely, two bases $(e_k)_{k=1}^{\infty}$ and $(f_k)_{k=1}^{\infty}$ in an NF space X are *quasi-equivalent* if there exist a permutation σ, scalars $\rho_k \neq 0$, and an isomorphism $T : X \to X$ such that $Te_k = \rho_k f_{\sigma(k)}$ for each k.

Let us note some results for NF spaces on this issue. The first achievement is due to Dragilev who proved in [5] that all bases in the space of analytic functions on the unit disc are quasi-equivalent. In [12–14], Mityagin generalized this to the case of Hilbert scales. All bases in X are quasi-equivalent if X has a regular basis [4, 10] or a basis of type G_1 or G_∞ [1]. Zaharjuta et al. showed the quasi-equivalence of bases in special Köthe power spaces [3, 15], in tensor products of power series spaces with their duals [8] and in some other cases. The conjecture on the quasi-equivalence of bases in NF spaces was implicitly stated in [13] (Problem 12) and in [12], see also Problem 8 in [2]. We can quote Zobin [16], who wrote 20 years ago: "There was a common belief at that time that the conjecture will be proven very soon. To everybody's great surprise, it is still an open question." Two decades later, the status of the conjecture remains the

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same. In our opinion, the quasi-equivalence problem is one of the most important open problems in the structure theory of NF spaces.

Here, we discuss this problem for the class of Whitney spaces. In [6], the first author constructed bases in the spaces $\mathcal{E}(K^{(\alpha_n)})$, see Section 2 for the definition of the set $K^{(\alpha_n)}$. Recently, in [7], bases were presented in the Whitney spaces for more general Cantor-type sets $K = K^{(\alpha_n)}_{(N_n)}$. In the case of a "small" set K (here, this means that the logarithmic dimension $\lambda_0(K)$ of the set is smaller than 1), an interpolating Faber basis was presented in $\mathcal{E}(K)$. Consequently, each space has many bases, and for small sets, at least one basis consists of polynomials. We answer in the affirmative to the question posed in [6] about quasi-equivalence of these bases. The proof is given for the space $\mathcal{E}(K^{(\alpha_n)})$ with $\alpha_n \ge 2$, which implies $\lambda_0(K^{(\alpha_n)}) \le 1$. We think that a slight modification of the proof can be applied to other cases as well.

Let X be an NF space with a topology given by an increasing sequence of seminorms $(\|\cdot\|_p)_{p=1}^{\infty}$ and a basis $(e_n)_{n=1}^{\infty}$. Then, X is isomorphic to the Köthe echelon space $K(A)$ with the matrix $A = (\|e_n\|_p)_{n,p=1}^{\infty}$. Beginning with his first papers on this topic, Mityagin proposed a combinatorial technique based on modifications of the Hall–König theorem to compare Köthe matrices for different bases. This method was widely used in the works cited above. In our paper, we explicitly construct the desired isomorphism.

2 Uniform distribution of points on a Cantor-type set

Here, we follow [6, 9]. Let $(\ell_s)_{s=0}^{\infty}$ be a sequence such that $\ell_0 = 1$ and $0 < 3\ell_{s+1} \le \ell_s$ for $s \in \mathbb{N}_0 := \{0, 1, \ldots\}$. For $E_0 = I_{0,1} = [0,1]$ and given E_s, a union of 2^s closed *basic* intervals $I_{j,s}$ of length ℓ_s, we obtain E_{s+1} by replacing each $I_{j,s}$ for $1 \le j \le 2^s$ by two *adjacent* subintervals $I_{2j-1,s+1}$ and $I_{2j,s+1}$. Then, $h_s = \ell_s - 2\,\ell_{s+1}$ is the distance between them. By assumption, $h_s \ge \ell_{s+1}$ for each s. We consider a Cantor-type set $K = \bigcap_{s=0}^{\infty} E_s$, defined by the sequence $(\ell_s)_{s=0}^{\infty}$.

Denote $\alpha_1 = 1$ and $\alpha_s = \frac{\log \ell_s}{\log \ell_{s-1}}$ for $s \ge 2$. Thus, $\ell_s = \ell_1^{\alpha_1 \cdots \alpha_s}$. By $K^{(\alpha_s)}$, we denote a set associated with the sequence $(\alpha_s)_{s=1}^{\infty}$.

Let x be an endpoint of some basic interval. Then, there exists a minimal number s (the *type* of x) such that x is the endpoint of some $I_{j,m}$ for every $m \ge s$. By X_k, we denote all endpoints of type k. Hence, $X_0 := \{0, 1\}$, $X_1 := \{\ell_1, 1 - \ell_1\}$, $X_2 := \{\ell_2, \ell_1 - \ell_2, 1 - \ell_1 + \ell_2, 1 - \ell_2\}$, etc. Set $Y_s = \cup_{k=0}^{s} X_k$. Then, $\#(Y_s) = 2^{s+1}$. Here and below, $\#(Z)$ denotes the cardinality of a finite set Z. Given such set, let $m(j, s, Z) := \#(Z \cap I_{j,s})$. In addition, for each $x \in \mathbb{R}$, by $d_k(x, Z)$, $k = 1, 2, \ldots, \#(Z)$, we denote the distances $|x - z_{j_k}|$ from x to points of Z arranged in the nondecreasing order. In what follows, we omit the parameter Z in $m(\cdot,\cdot,Z)$, $d_k(\cdot, Z)$ if this does not cause misunderstanding.

We put all points from $\cup_{k=0}^{\infty} X_k$ in order by means of *the rule of increase of type*. For the points from $X_0 \cup X_1$, we take $x_1 = 0$, $x_2 = 1$, $x_3 = \ell_1$, $x_4 = 1 - \ell_1$. To put the points from X_2 in order, we increasingly arrange the points from Y_1, so $Y_1 = \{x_1, x_3, x_4, x_2\}$. After this, we increase the index of each point by 4. This gives the ordering $X_2 = \{x_5, x_7, x_8, x_6\}$. Similarly, indices of increasingly arranged points from $Y_{k-1} = \{x_{i_1}, x_{i_2}, \cdots, x_{i_{2^k}}\}$ define the ordering $X_k = \{x_{i_1+2^k}, x_{i_2+2^k}, \ldots, x_{i_{2^k}+2^k}\}$. We see that $x_{j+2^k} = x_j \pm \ell_k$, where the sign is uniquely defined by j. More precisely, every

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natural number N greater than 1 can be uniquely written as $N = 1 + 2^{p_1} + 2^{p_2} + \cdots + 2^{p_n}$ with $0 \le p_1 < p_2 < \cdots < p_n$. Then, $x_N = \ell_{p_1} - \ell_{p_2} + \cdots + (-1)^{n-1}\ell_{p_n}$. Thus, for $j = N - 2^{p_n}$, we have $x_{j+2^{p_n}} = x_j + (-1)^{n-1}\ell_{p_n}$, where the degree of -1 is the number of terms except 1 in the representation of j. For example, $x_{12} = x_{4+2^3} = x_4 + \ell_3$, because $4 = 1 + 2^0 + 2^1$.

A useful property of this order is that for each k, the points $(x_p)_{p=1}^k$ are distributed uniformly on $K^{(\alpha_s)}$ in the following sense.

Suppose we are given a set $U = (z_p)_{p=1}^k$ of points from a Cantor-type set K. We say that points are *uniformly distributed* on K if, for each $s \in \mathbb{N}$ and $i, j \in \{1, 2, \ldots, 2^s\}$, we have

$$|m(i,s,U) - m(j,s,U)| \le 1, \tag{2.1}$$

so any two intervals of the same level contain the same number of points from U or, perhaps, one of the intervals contains one extra point z_p, compared to another interval.

Suppose points $Z = (x_p)_{p=1}^k$ are chosen by the rule of increase of type, $2^q \le k < 2^{q+1}$. Then, the binary representation

$$k = 2^q + 2^r + \cdots + 2^t + \cdots + 2^w \quad \text{with} \quad 0 \le w < \cdots < t < \cdots < r < q \tag{2.2}$$

generates the decomposition $Z = Z_q \cup Z_r \cup \cdots \cup Z_w$ with $Z_q = Y_{q-1}$ and $Z_r \cup \cdots \cup Z_w \subset X_q$. Let $\mathcal{K} := \{w, \cdots, r, q\}$ be the set of exponents in (2.2). For fixed $i \in \mathcal{K}$, each basic interval of ith level contains just one point from Z_i.

Moreover, for each basic interval $I = I_{j,s}$, the points $(x_{i,j,s})_{i=1}^m = Z \cap I$ coincide with points selected on I by the rule of increase of type or mirror symmetric to them.

We say that a Cantor-type set K is ℓ- *regular* if

$$\ell_{s+1}^2 \ge \ell_s \ell_{s+2} \quad \text{for} \quad s \in \mathbb{N}. \tag{2.3}$$

It is easily seen that the following Cantor-type sets have this property.

(1) K_β, where $\ell_s = \beta^s$ for $s \in \mathbb{N}_0$ with $0 < \beta < 1/2$. In particular, the Cantor ternary set $K_{1/3}$ is ℓ- *regular*.
(2) K^α, where $\alpha_s = \alpha$ for $s \ge 2$ with $\alpha > 1$.
(3) $K^{(\alpha_s)}$ with $\alpha_s(2 - \alpha_{s+1}) \le 1$ for $s \ge 2$. Thus, a set $K^{(\alpha_s)}$ with $\alpha_s \ge 2$ for $s \ge 2$ is ℓ-regular.

Lemma 2.1 *If a Cantor-type set K is ℓ-regular, then the value $\frac{h_s}{\ell_s}$ increases, so* $\min_s \frac{h_s}{\ell_s} = h_0$.

Proof It is evident, because $\frac{h_s}{\ell_s} = 1 - 2\frac{\ell_{s+1}}{\ell_s}$. ∎

We consider the space $\mathcal{E}(K^{(\alpha_s)})$ of Whitney functions on s $K^{(\alpha_s)}$ with the topology defined by the norms

$$||f||_q = |f|_q + \sup\left\{\frac{|(R_y^q f)^{(i)}(x)|}{|x-y|^{q-i}} : x, y \in K^{(\alpha_s)}, x \ne y, i = 0, 1, \ldots q\right\}, q \in \mathbb{N}_0,$$

where $|f|_q = \sup\{|f^{(i)}(x)| : x \in K^{(\alpha_s)}, i \le q\}$ and $R^q_y f(x) = f(x) - T^q_y f(x)$ is the Taylor remainder.

3 Correspondence between bases

Recall two theorems from [6]. Let $(x_i)_{i=1}^\infty$ be chosen by the rule of increase of type. Define $e_0 \equiv 1$ and $e_k(x) = \prod_{j=1}^k (x - x_j)$. In the case of rarefied sets, these functions form an interpolating Faber basis, that is, basis of polynomials such that degrees of polynomials coincide with numbers of basis elements. Biorthogonal functionals are given as divided differences $\xi_k(f) = [x_1, x_2, \dots, x_{k+1}]f$.

Theorem 3.1 [6, T.1] *If $\alpha_s \ge 2$ for $s \ge 2$, then the sequence $(e_k)_{k=0}^\infty$ is a Schauder basis in the space $\mathcal{E}(K^{(\alpha_s)})$.*

On the other hand, in any space $\mathcal{E}(K^{(\alpha_s)})$, we can present a variety of bases consisting of local polynomials. Let us fix a nondecreasing sequence of natural numbers $(n_s)_{s=0}^\infty$ with $n_s \nearrow \infty$ such that

$$\overline{\lim}_s [n_s - \log_2(\alpha_1 \alpha_2 \cdots \alpha_s)] < \infty. \tag{3.1}$$

Let $N_s = 2^{n_s}$. The condition above implies that for some $Q > 0$ the sequence $(2^{N_s} \ell_s^Q)_{s=0}^\infty$ is bounded and we can apply

Theorem 3.2 [6, T.2] *If a nondecreasing unbounded sequence $(N_s)_{s=0}^\infty$ of natural numbers of the form $N_s = 2^{n_s}$ is such that for some Q the sequence $(2^{N_s} l_s^Q)_{s=0}^\infty$ is bounded, then the functions $(e_{n,j,s})_{s=0,\, j=1,\, n=M_s}^{\infty,\ j=2^s,\ n=N_s}$ form a basis in the space $\mathcal{E}(K^{(\alpha_s)})$.*

Here, $e_{n,j,s} = \prod_{i=1}^n (x - x_{i,j,s})$ if $x \in K^{(\alpha_s)} \cap I_{j,s}$ and $e_{n,j,s} = 0$ otherwise on $K^{(\alpha_s)}$ with the points $(x_{i,j,s})_{i=1}^n$ chosen on $I_{j,s}$ by the rule of increase of type if j is odd or mirror symmetric to them if j is even.

Our next goal is to number these functions accordingly. We will denote the resulting sequence by $(f_k)_{k=0}^\infty$. First, for $s = 0$ and $0 \le k \le N_0$, we take $f_k = e_k = e_{k,1,0}$.

Let us consider functions corresponding to $s = 1$. The last functional of the previous level $\xi_{N_0}(f)$ is defined by the points $(x_i)_{i=1}^{N_0+1}$, where the left subinterval $I_{1,1}$ contains $M_1^{(l)} = N_0/2 + 1$ of them, whereas the right $I_{2,1}$ does $M_1^{(r)} = N_0/2$. On both intervals, we consider $e_{n,j,1}$ for n starting with the corresponding M_1 and ending with the same value N_1. In order to number them, for each k, we look at the position of x_{k+1}. For $k = N_0 + 1$, the point x_{N_0+2} belongs to $I_{2,1}$, so we take $f_{N_0+1} = e_{N_0/2,2,1}$. It is important to notice that the functions e_{N_0+1} and f_{N_0+1} have the same zeros on $I_{2,1}$. For $k = N_0 + 2$, the point x_{N_0+3} belongs to $I_{1,1}$ and we take $f_{N_0+2} = e_{N_0/2+1,1,1}$. Here, as above, e_{N_0+2} and f_{N_0+2} have the same zeros on $I_{1,1}$. We continue in this fashion to obtain f_k as a certain $e_{n,j,1}$, where $j \in \{1, 2\}$ is defined by the condition $x_{k+1} \in I_{j,1}$. The functions f_k and e_k have the same zeros on $I_{j,1}$. The total number of functions $e_{n,j,1}$ is $\#\{\frac{N_0}{2} + 1, \dots, N_1\} + \#\{\frac{N_0}{2}, \dots, N_1\} = 2N_1 - N_0 + 1$. Let us calculate the number V_1 of k corresponding to $s \le 1$. It is a sum of $N_0 + 1$ functions of the zero level and $2N_1 - N_0 + 1$ functions of the first level, that is, $V_1 = 2N_1 + 2$. We start enumeration from 0. Therefore, the value $s = 1$ corresponds to the functions $(f_k)_{k=N_0+1}^{2N_1+1}$.

Continuing in this manner, we obtain the number $V_s = 2^s(N_s+1)$ of k corresponding to levels $0, 1, \dots, s$. Then, f_k for $V_{s-1} \le k < V_s$ correspond to the levels s. Let us consider these values in detail and, for each k, find $\deg \tilde{f}_k$, where $\tilde{f}_k$ is the analytic continuation of the corresponding $e_{n,j,s}$ from $I_{j,s}$ to $\mathbb{R}$.

As in [6], we consider $M_s^{(l)} = N_{s-1}/2 + 1$, $M_s^{(r)} = N_{s-1}/2$. Each $I_{i,s-1}$ contains subintervals $I_{2i-1,s}$ and $I_{2i,s}$. For $j = 2i-1$, we consider $e_{n,j,s}$ with $M_s^{(l)} \le n \le N_s$, while $M_s^{(r)} \le n \le N_s$ for $j = 2i$. Hence, there are 2^{s-1} right subintervals $I_{2i,s}$ and for $V_{s-1} \le k < V_{s-1} + 2^{s-1}$, we get $\deg \tilde{f}_k = N_{s-1}/2$. After this, each of extra 2^s zeros of e_k will add one to degrees of $\tilde{f}_k$. Hence, for k with

$$V_{s-1} + 2^{s-1} + (i-1)2^s \le k < V_{s-1} + 2^{s-1} + i\ 2^s,$$

we have $\deg \tilde{f}_k = N_{s-1}/2 + i$. Here, $1 \le i \le N_s - N_{s-1}/2$. Indeed, the maximal value of k on sth level is $V_{s-1} + 2^{s-1} + (N_s - N_{s-1}/2)\,2^s - 1 = V_s - 1$ and $\max \deg \tilde{f}_k = N_s$, as expected.

Thus, all functions $e_{n,j,s}$ are included in the sequence $(f_k)_{k=0}^{\infty}$ and, for each $f_k = e_{n,j,s}$, the functions e_k and $e_{n,j,s}$ have the same zeros on $I_{j,s}$. The bijection $f_k \leftrightarrow e_k$ will define the desired isomorphism T. We need to find appropriate coefficients ρ_k.

4 Estimations of norms

For each $n \in \mathbb{N}$, let $\pi_n = \pi_{n,0} := \ell_n\, \ell_{n-1}\, \ell_{n-2}^2\, \ell_{n-3}^4 \cdots \ell_j^{2^{n-j-1}} \cdots \ell_0^{2^{n-1}}$. Similarly, for $n, s \in \mathbb{N}$, set $\pi_{n,s} := \ell_{n+s}\, \ell_{n+s-1}\, \ell_{n+s-2}^2 \cdots \ell_{j+s}^{2^{n-j-1}} \cdots \ell_s^{2^{n-1}}$. Each $\pi_{n,s}$ has 2^n terms.

We proceed to estimate $\|e_k\|_p$. Recall that the set $Z = (x_i)_{i=1}^k$ of zeros of e_k is chosen by the rule of increase of type. Suppose $2^q \le k < 2^{q+1}$. We use the representation (2.2), the corresponding set $\mathcal{K}$, and the product $\prod_{i \in \mathcal{K}} \pi_i$.

Given any product $\prod_{j=1}^N \lambda_j$ with $\lambda_j \ge 0$ and $p < N$, by $(\prod_{j=1}^N \lambda_j)_p$, we denote this product without p smallest terms.

Lemma 4.1 *Let $1 \le p < k$. Then, $h_0^k\, (\prod_{i\in\mathcal{K}} \pi_i)_{p-1} \le \|e_k\|_p \le 5(3\,k)^p\, (\prod_{i\in\mathcal{K}} \pi_i)_p$.*

Proof Fix $x \in K^{(\alpha_s)}$. Then, $|e_k(x)| = \prod_{j=1}^k d_j(x, Z) = \prod_{i\in\mathcal{K}} \prod_{j=1}^{2^i} d_j(x, Z_i)$. Points from Z_i are distributed uniformly on $K^{(\alpha_s)}$. Hence, $d_1(x, Z_i) \le \ell_i$, $d_2(x, Z_i) \le \ell_{i-1}$. For the next two indices, we have $d_j(x, Z_i) \le \ell_{i-2}$, etc. Thus, $\prod_{j=1}^{2^i} d_j(x, Z_i) \le \pi_i$ and $|e_k(x)| \le \prod_{i\in\mathcal{K}} \pi_i$.

The pth derivative of e_k at x is a sum of $k!/(k-p)!$ products, where each product contains $k-p$ terms of the type $x - x_j$. Hence,

$$|e_k^{(p)}(x)| \le k^p \prod_{j=p+1}^{k} d_j(x, Z) = k^p \left(\prod_{i\in\mathcal{K}} \pi_i\right)_p .$$

This gives

$$|e_k|_p \le k^p \left(\prod_{i\in\mathcal{K}} \pi_i\right)_p .$$

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As for the norms $||e_k||_p$, we completely repeat the reasoning from the proof of Theorem 1 in [6], see page 354:

$$||e_k||_p \le 5(3\,k)^p \left(\prod_{i\in\mathcal{K}} \pi_i\right)_p . \tag{4.1}$$

We now turn to lower bound of $||e_k||_p$. Clearly, $||e_k||_p \ge |e_k^{(p)}(0)|$. As above, $|e_k^{(p)}(0)|$ is a sum of $k!/(k-p)!$ products, each of $k-p$ terms. Here, all products have the same sign, so we can neglect all products except one with largest terms:

$$||e_k||_p \ge \left(\prod_{j=1}^{k} d_j(0,Z)\right)_p . \tag{4.2}$$

Of course, $d_1(0,Z) = 0$. To estimate $\prod_{j=2}^k d_j(0,Z)$ from below in terms of π_i, let us consider (2.2). We use the method from Lemma 2.2 in [9]. First, consider distances from 0 to the points from Z_q : $d_2(0,Z_q) = \ell_{s-1} > h_{s-1}$, $d_3(0,Z_q) = \ell_{q-2} - \ell_{q-1} > h_{q-2}, d_4(0,Z_q) = \ell_{q-2} > h_{q-2}, \dots$ Continuing in this fashion, we get

$$\prod_{j=2}^{2^q} d_j(0,Z_q) \ge h_{q-1}\, h_{q-2}^2\, h_{q-3}^4 \cdots h_0^{2^{q-1}} = \ell_q^{-1}\, \pi_q \prod_{i=0}^{q-1} \left(\frac{h_i}{\ell_i}\right)^{2^{q-1-i}} \ge \ell_q^{-1}\, \pi_q\, h_0^{2^q-1}, \tag{4.3}$$

by Lemma 4.5.

By the rule of increase of type, the first 2^q points are all endpoints of basic intervals of $(q-1)$st type. The next 2^r points (in increasing order) are: $\ell_q, \ell_{r-1} - \ell_q, \ell_{r-2} - \ell_{r-1} + \ell_q, \ell_{r-2} - \ell_q, \dots$ They define $d_j(0,Z_r)$. As above, we get

$$\prod_{j=1}^{2^r} d_j(0,Z_r) \ge \frac{\ell_q}{\ell_r}\, \pi_r\, h_0^{2^r} . \tag{4.4}$$

Other indices from the set $\mathcal{K} = \{w, v, \dots, t, r, q\}$ can be handled in the same way. Combining (4.3) with (4.4) and analogous inequalities for other $i \in \mathcal{K}$ yields

$$\prod_{j=2}^{k} d_j(0,Z) \ge h_0^k\, \ell_q^{-1}\, \pi_q\, \frac{\ell_q}{\ell_r}\, \pi_r \frac{\ell_r}{\ell_t}\, \pi_t \cdots \frac{\ell_v}{\ell_w}\, \pi_w = h_0^k\, \ell_w^{-1} \prod_{i\in\mathcal{K}} \pi_i .$$

We neglect ℓ_w^{-1} and note that $(\prod_{j=1}^k d_j(0,Z))_p \ge (\prod_{i\in\mathcal{K}} \pi_i)_{p-1}$. By (4.2), we have the desired lower bound. ■

In an exactly similar way, we estimate norms of elements of local bases. Let $2^m \le n < 2^{m+1}$, so $n = \sum_{m_i\in\mathcal{M}} 2^{m_i}$, where $\mathcal{M} = \{m_y, \dots, m_2, m_1\}$ with $0 \le m_y < \cdots < m_2 < m_1 = m$. Here, we replace $\frac{h_0}{\ell_0}$ with $\frac{h_s}{\ell_s}$ and, as in [6], note that the points from $K^{(\alpha_s)} \setminus I_{j,s}$ have no influence on the estimation of $||e_{n,j,s}||_p$ for $p < n$, because $dist(I_{j,s}, K^{(\alpha_s)} \setminus I_{j,s}) = h_{s-1}$ is larger than ℓ_s.

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Lemma 4.2 *If $1 \le p < n$, then*

$$\left(\frac{h_s}{\ell_s}\right)^n \left(\prod_{m_i \in \mathcal{M}} \pi_{m_i,s}\right)_{p-1} \le \|e_{n,j,s}\|_p \le 5(3\,n)^p \left(\prod_{m_i \in \mathcal{M}} \pi_{m_i,s}\right)_p .$$

It should be noted that bases above are not regular, so the arguments from [4, 10] cannot be used to show their quasi-equivalence. We recall the definition of regular bases.

A basis $(b_k)_{k=0}^{\infty}$ in an NF space X is *regular* if there is a fundamental system of seminorms $(\|\cdot\|_p)_{p=0}^{\infty}$ in X such that for each p, q with $p < q$, the sequence $\frac{\|b_k\|_p}{\|b_k\|_q}$ monotonically tends to 0 as $k \to \infty$.

Let us show that the ratio $\frac{\|e_k\|_p}{\|e_k\|_{p+1}}$ is not decreasing. The exact calculation of the norm $\|e_k\|_p$, in general, is rather sophisticated. We restrict ourselves to small values of k, p and the case $\ell_1 < \frac{1}{5}$. We proceed to show that

$$\frac{\|e_2\|_0}{\|e_2\|_1} < \frac{\|e_3\|_0}{\|e_3\|_1}. \tag{4.5}$$

For $e_2(x) = x^2 - x$, it is easily seen that $|e_2|_0 = |e_2(\ell_1)| = \ell_1(1-\ell_1)$, $\|e_2\|_0 = 2\ell_1(1-\ell_1)$, and $|e_2|_1 = e_2'(1) = 1$ with

$$\sup\left\{\frac{|(R_y^1 e_2)^{(i)}(x)|}{|x-y|^{1-i}} : x, y \in K^{(\alpha_s)}, x \ne y, i = 0, 1\right\} = (R_0^1 e_2)'(1) = 2.$$

Thus, the left-hand side of (4.5) is $\frac{2}{3}\ell_1(1-\ell_1)$.

On the other hand, for $e_3(x) = x(x-1)(x-\ell_1)$, we have $|e_3|_0 = |e_3(1-\ell_1)| = \ell_1(1-\ell_1)(1-2\ell_1)$ and $\sup_{x,y\in K}|e_3(x) - e_3(y)| = e_3(\ell_2) - e_3(1-\ell_1)$. From this, $\|e_3\|_0 = 2\ell_1(1-\ell_1)(1-2\ell_1) + \ell_2(1-\ell_2)(\ell_1-\ell_2)$. It is straightforward to show that $|e_3|_1 = e_3'(1) = 1-\ell_1$.

We proceed to analyze $(R_y^1 e_3)(x) = e_3(x) - e_3(y) - e_3'(y)(x-y) = (x-y)[x^2 + xy - 2y^2 - (1+\ell_1)(x-y)]$. We will denote by g the function in square brackets. Now, the task is to find the maximum value of $|g(x,y)|$ on the set $K^{(\alpha_s)} \times K^{(\alpha_s)}$, which is a subset of $E_1 \times E_1$, where $E_1 = [0,\ell_1] \cup [1-\ell_1, 1]$. The set $E_1 \times E_1$ is a union of four squares. For the squares $Q_1 = [0,\ell_1] \times [0,\ell_1]$, $Q_2 = [1-\ell_1, 1] \times [0,\ell_1]$, $Q_3 = [1-\ell_1,1] \times [1-\ell_1,1]$, the representation $g(x,y) = (x-y)(x+2y-1-\ell_1)$ implies $|g(x,y)| \le 1-\ell_1$, as is easy to check. Let us consider $(x,y) \in Q_4 = [0,\ell_1] \times [1-\ell_1,1]$. Here, for a fixed x, the function $h(y) := |g(x,y)|$ is an upward parabola $(y-x)(x+2y-1-\ell_1)$ with the vertex below the line $y = 1-\ell_1$. Hence, $\max_{1-\ell_1 \le y \le 1} h(y) = h(1) = -x^2 + x\ell_1 + 1 - \ell_1$. For $x \in [0,\ell_2] \cup [\ell_1 - \ell_2, \ell_1]$, the function $-x^2 + x\ell_1 + 1 - \ell_1$ attains its maximum at $x = \ell_2$. Thus,

$$\sup\left\{\frac{|(R_y^1 e_3)(x)|}{|x-y|} : x, y \in K^{(\alpha_s)}, x \ne y\right\} = |g(\ell_2, 1)| = 1 - \ell_1 + \ell_1\ell_2 - \ell_2^2.$$

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Similarly,

$$\sup\left\{|(R^1_y e_3)'(x)| : x, y \in K^{(\alpha_s)}, x \neq y\right\} = |e_3'(1) - e_3'(\ell_1)| = 1 - \ell_1^2$$

that exceeds $1 - \ell_1 + \ell_1\ell_2 - \ell_2^2$. Therefore, $||e_3||_1 = 2 - \ell_1 - \ell_1^2$. It follows that the right-hand side of (4.5) exceeds $\frac{2\ell_1(1-\ell_1)(1-2\ell_1)}{2-\ell_1-\ell_1^2}$ and (4.5) is proved.

5 Quasi-equivalence of bases

Suppose $\alpha_s \geq 2$ for $s \geq 2$ and a sequence $(n_s)_{s=0}^\infty$ satisfying (3.1) is given. Then, the space $\mathcal{E}(K^{(\alpha_s)})$ has a polynomial basis $(e_k)_{k=0}^\infty$, by Theorem 3.1, and a local polynomial basis $(e_{n,j,s})_{s=0,\, j=1,\, n=M_s}^{\infty,\, j=2^s,\, n=N_s}$, by 3.2. In Section 3, we established a bijection between these bases. We are now in a position to indicate lengths of intervals, which are included in $\pi_{m,s}$ that define norms of the corresponding basis elements. Let $f_k = e_{n,j,s}$ be fixed. For $n = \deg \tilde{f}_k$, by construction, we have $2^m \leq n < 2^{m+1}$ for some $n_{s-1} - 1 \leq m \leq n_s$. Therefore, the product $\prod_{m_i \in \mathcal{M}} \pi_{m_i,s}$ in Lemma 4.2 includes $\pi_{m,s} = \ell_{s+m} \cdots \ell_s^{2^{m-1}}$ and other $\pi_{m_i,s}$ that start from ℓ_t with $s \leq t < s+m$.

For the corresponding e_k, by (2.2), each interval of sth level may contain n or $n+1$ points. It cannot contain $n-1$ points, because, given s, we chose first functions $f_k = e_{n,j,s}$ on intervals with minimal number of points. Hence, $n\,2^s \leq k \leq (n+1)2^s$ and $2^{s+m} \leq k \leq 2^{s+m+1}$, as $n+1 \leq 2^{m+1}$. This means that the product $\prod_{i \in \mathcal{K}} \pi_i$ in Lemma 4.1 includes $\pi_{m+s} = \ell_{s+m} \cdots \ell_0^{2^{m+s-1}}$ and others π_i that start from ℓ_i with $0 \leq i < s+m$.

The feature of the sequence $(x_i)_{i=1}^\infty$ mentioned above is that, given k with $f_k = e_{n,j,s}$, the set $(x_i)_{i=1}^k \cap I_{j,s}$ defines zeros of f_k on this interval. Hence, $\prod_{m_i \in \mathcal{M}} \pi_{m_i,s}$ coincides with the product of n smallest terms of $\prod_{i \in \mathcal{K}} \pi_i$. The rest gives the desired coefficient

$$\rho_k := \prod_{i \in \mathcal{K}} \pi_i / \prod_{m_i \in \mathcal{M}} \pi_{m_i,s} = \left(\prod_{i \in \mathcal{K}} \pi_i\right)_n .$$

Of course,

$$\left(\prod_{i \in \mathcal{K}} \pi_i\right)_q = \rho_k \left(\prod_{m_i \in \mathcal{M}} \pi_{m_i,s}\right)_q , \tag{5.1}$$

for $q < n$, because, by definition, each term in $\prod_{m_i \in \mathcal{M}} \pi_{m_i,s}$ does not exceed each term in ρ_k.

We are able to present the main result of the paper.

Theorem 5.1 *Let $\alpha_s \geq 2$ for $s \geq 2$ and a sequence $(n_s)_{s=0}^\infty$ satisfy (3.1). Then, the bases $(e_k)_{k=0}^\infty$ and $(e_{n,j,s})_{s=0,\, j=1,\, n=M_s}^{\infty,\, j=2^s,\, n=N_s}$ of the space $\mathcal{E}(K^{(\alpha_s)})$ are quasi-equivalent.*

Proof Let $Te_k = \rho_k f_k$, $k \in \mathbb{N}_0$, where $(f_k)_{k=0}^\infty$ is the enumeration of the system $(e_{n,j,s})_{s,j,n}$ given in Section 3. It can be extended linearly to an operator T acting on the whole space. We aim to prove that T is an isomorphism. First, we show that for each p, there are q and C_p such that

$$||Te_k||_p = \rho_k\, ||f_k||_p \leq C_p\, ||e_k||_q \quad \text{for all} \quad k. \tag{5.2}$$

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Without loss of generality, we can consider p in the form $p = 2^u$ with $u \in \mathbb{N}$. Fix such p. Let $q = 2^v$ with $v = u + 10$.

Given $q = q(p)$, there are only finitely many values k with $n \le q$, where $n = \deg \tilde{f}_k$. For such k, (5.2) is valid with an appropriate choice of C_p. Hence, we can assume that k is so large that we can use above lemmas.

By Lemma 4.2, $||f_k||_p = ||e_{n,j,s}||_p \le 5(3n)^p\,(\prod_{m_i \in \mathcal{M}} \pi_{m_i,s})_p$. By (5.1),

$$\rho_k\,||f_k||_p \le 5(3n)^p \left(\prod_{i\in\mathcal{K}} \pi_i\right)_p .$$

On the other hand, by Lemma 4.1, $\ell_1^k\,(\prod_{i\in\mathcal{K}} \pi_i)_{q-1} \le ||e_k||_q$, because $h_0 \ge \ell_1$.

To obtain (5.2), it remains to find C_p such that

$$5(3n)^p \left(\prod_{i\in\mathcal{K}} \pi_i\right)_p \le C_p \ell_1^k \left(\prod_{i\in\mathcal{K}} \pi_i\right)_{q-1} . \tag{5.3}$$

Let $\prod_{i\in\mathcal{K}} \pi_i = \prod_{\gamma=1}^k \lambda_\gamma$ with $\lambda_1 \le \cdots \le \lambda_k$. Then,

$$\left(\prod_{i\in\mathcal{K}} \pi_i\right)_p = \lambda_{p+1}\cdots\lambda_{q-1} \left(\prod_{i\in\mathcal{K}} \pi_i\right)_{q-1} .$$

We note that $\prod_{i\in\mathcal{K}} \pi_i$ contains $\pi_{m+s} = \ell_{s+m}\,\ell_{s+m-1}\,\ell_{s+m-2}^2 \cdots \ell_0^{2^{m+s-1}} = \prod_{\gamma=1}^{2^{s+m}} \lambda'_\gamma$ with $\lambda'_\gamma \uparrow$. The product $\prod_{\gamma=p+1}^{q-1} \lambda_\gamma$ does not exceed $\prod_{\gamma=p+1}^{q-1} \lambda'_\gamma$, because including the terms from other π_i can only reduce the result.

We see that $\lambda'_\gamma = \ell_{s+m-i}$ for $\gamma \in \{2^{i-1}+1, \dots, 2^i\}$ with $2 \le i \le s+m$. For $p = 2^u, q = 2^v$, we have $\prod_{\gamma=p+1}^{q-1} \lambda'_\gamma = \ell_{s+m-u-1}^{2^u} \cdots \ell_{s+m-v}^{2^{v-1}-1} = \ell_1^\kappa$ with

$$\kappa = \sum_{i=u}^{v-2} 2^i\,\alpha_2 \cdots \alpha_{s+m-i-1} + (2^{v-1}-1)\,\alpha_2 \cdots \alpha_{s+m-v} \ge 2^{s+m-2}(v-u-1).$$

Indeed, we neglect the last summand and use the fact that all α_i are bounded below by 2. The problem now reduces to establishing that

$$5(3n)^p\,\ell_1^{2^{s+m-2}(v-u-1)} \le C_p \ell_1^k,$$

for $n < 2^{m+1}$ and $k \le 2^{s+m+1}$. It is easy to check that our choice $v = v + 10$ provides this inequality together with (5.3) and (5.2).

The same reasoning applies to the inverse inequality: for each p, there are q and C_p:

$$\rho_k\,||T^{-1} f_k||_p = ||e_k||_p \le C_p\,||f_k||_q \quad \text{for all} \quad k,$$

but we will not need it.

For each $f = \sum_{k=0}^\infty \xi_k(f)\,e_k \in \mathcal{E}(K^{(\alpha_s)})$, we define $Tf = \sum_{k=0}^\infty \xi_k(f)\rho_k\,f_k$. It is a simple matter to check that T is a bijection. Let us show its continuity.

The basis $(e_k)_{k=0}^\infty$ is absolute. Hence, the norms $|||f|||_q := \sum_{k=0}^\infty |\xi_k(f)|\,||e_k||_q$ are well-defined for each q. It is a standard fact that the topology given by these norms is

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complete. By the open mapping theorem, for any q, there exist r, $C > 0$ such that

$$|||f|||_q \le C\,||f||_r. \tag{5.4}$$

Fix any $p \in \mathbb{N}$ and $f \in \mathcal{E}(K^{(\alpha_s)})$. Then, by (5.2) and (5.4),

$$||Tf||_p \le \sum_{k=0}^{\infty} |\xi_k(f)|\,\rho_k\,||f_k||_p \le C_p \sum_{k=0}^{\infty} |\xi_k(f)|\,||e_k||_q \le C_p\,C||f||_r.$$

Thus, T is an isomorphism, which is our claim. ∎

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