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Absolute continuity, Lyapunov exponents, and rigidity II: systems with compact center leaves

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Abstract. We explore new connections between the dynamics of conservative partially hyperbolic systems and the geometric measure-theoretic properties of their invariant foliations. Our methods are applied to two main classes of volume-preserving diffeomorphisms: fibered partially hyperbolic diffeomorphisms and center-fixing partially hyperbolic systems. When the center is one-dimensional, assuming the diffeomorphism is accessible, we prove that the disintegration of the volume measure along the center foliation is either atomic or Lebesgue. Moreover, the latter case is rigid in dimension three (this does not require accessibility): the center foliation is actually smooth and the diffeomorphism is smoothly conjugate to an explicit rigid model. A partial extension to fibered partially hyperbolic systems with compact fibers of any dimension is also obtained. A common feature of these classes of diffeomorphisms is that the center leaves either are compact or can be made compact by taking an appropriate dynamically defined quotient. For volume-preserving partially hyperbolic diffeomorphisms whose center foliation is absolutely continuous, if the generic center leaf is a circle, then every center leaf is compact.

Key words: Lyapunov exponent, rigidity, absolute continuity, invariance principle, partial hyperbolicity
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Contents

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1. *Introduction*

Consider the volume-preserving linear map defined on the 3-torus $\mathbb{T}^3$ by $f_0 : (x, y, z) \mapsto (2x + y, x + y, z)$. It admits an invariant foliation by circles, namely the vertical circles $\{(x, y) = \text{const.}\}$, and this foliation is normally hyperbolic: there is an invariant normal bundle to the foliation on which the dynamics is hyperbolic. Indeed, f_0 is one of the simplest examples of a partially hyperbolic diffeomorphism and one whose properties have been analyzed thoroughly.

It follows from the general theory of normally hyperbolic manifolds (Hirsch *et al* [19]) that every map in a C^1 neighborhood of f_0 also admits an invariant foliation $\mathcal{W}^c$ whose leaves are smoothly embedded circles and which is the image of the vertical foliation by a global homeomorphism. However, this *center foliation* is usually not transversely smooth.

Indeed, Shub and Wilkinson [36], and later Ruelle and Wilkinson [33, 34], considered volume-preserving perturbations of f_0 and found open sets of maps whose center foliations $\mathcal{W}^c$ are not smooth and, in fact, exhibit the following bizarre behavior: there are full-volume subsets of $\mathbb{T}^3$ that intersect every leaf in a finite (bounded) number of points.

The mechanism in these papers behind this phenomenon of *atomic disintegration* of the volume along the center leaves is non-vanishing of the center Lyapunov exponent. In brief, for almost every point $x \in \mathbb{T}^3$, the tangent direction to the center leaf is either exponentially expanded or exponentially contracted by the dynamics:

$$\lambda^c(x) := \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n(x)|_{T_x\mathcal{W}^c}\| \neq 0.$$

However, the center foliation may have atomic disintegration even when the center Lyapunov exponent λ^c does vanish almost everywhere. Such an example has been given by Katok (see [25]), and we also describe some generalizations in §10.

The purpose of this paper, a follow-up to [6], is to investigate the measure-theoretical properties of center foliations and, in particular, to understand when this and other forms of pathological behavior may occur, within a general context of partially hyperbolic dynamics.

One property that is of special interest to us is *absolute continuity* which, in this paper, we take to mean that the volume has *Lebesgue disintegration* along the leaves, meaning that a subset of $\mathbb{T}^3$ has full volume if and only if its intersection with almost every center leaf has full volume inside the leaf. This is implied by (but somewhat weaker than) the usual definition of absolute continuity, which requires that holonomy maps between cross-sections to the foliation preserve the class of zero measure sets. When the leaves are circles, vanishing of the center Lyapunov exponent is a necessary condition for absolute continuity [36].

Our first main result applies, in particular, to every volume-preserving perturbation of f_0. More generally, it applies to partially hyperbolic diffeomorphisms in dimension 3 preserving a foliation by circles.

A diffeomorphism f is *partially hyperbolic* if the tangent bundle TM admits Df-invariant splitting $E^s \oplus E^c \oplus E^u$ such that $Df|_{E^s}$ is a uniformly contracting, $Df|_{E^u}$ is uniformly expanding, and $Df|_{E^c}$ is dominated by both: vectors in E^c are neither as contracted as vector in E^s, nor as expanded as vectors in E^u. The stable and unstable subbundles, E^s and E^u, are always uniquely integrable, that is, there exist unique foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ whose leaves are smoothly embedded manifolds tangent to E^s and E^u at every point. Moreover, these foliations are f-invariant. A *center foliation*, tangent to E^c, need not exist although many interesting examples do have such foliations. *A priori*, such a foliation need not be unique or invariant under the dynamics.

By a *rotation extension* we mean a diffeomorphism that acts by isometries on the fibers of an invariant C^∞ circle bundle.

THEOREM 1.1. *Let M be a 3-manifold and let $f : M \to M$ be a partially hyperbolic, volume-preserving diffeomorphism. Assume that there exists an f-invariant foliation $\mathcal{W}^c$ with C^1 leaves, the leaves of which are all circles.*

If $\mathcal{W}^c$ is absolutely continuous, then $\mathcal{W}^c$ is C^∞; moreover, up to finite covering, f is C^∞ conjugate to a rotation extension of a volume-preserving Anosov diffeomorphism on $\mathbb{T}^2$. That is, there exists a C^∞ $\mathbb{T}$-bundle

$$\mathbb{T} \hookrightarrow B \xrightarrow{\pi} \mathbb{T}^2,$$

a lift of f to a finite cover (at most fourfold)

$$\hat{f} : \widehat{M} \to \widehat{M}$$

and a C^∞ diffeomorphism $h : \widehat{M} \to B$ sending the leaves of $\widehat{\mathcal{W}}^c$ to fibers of B and such that $h \circ \hat{f} \circ h^{-1} : B \to B$ is a bundle isomorphism, rotating the fibers and covering an area-preserving diffeomorphism of $\mathbb{T}^2$.

In fact, it suffices to suppose that the *generic* leaf of the center foliation is a circle: we show that in this and more general contexts, this condition implies that *all* the leaves are circles (see Theorem D).

To state the next result, we need to discuss the notion of accessibility. A partially hyperbolic diffeomorphism $f : M \to M$ is *accessible* (or *has the accessibility property*) if any two points in M can be joined by an *su-path*, which is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s$ or a single leaf of $\mathcal{W}^u$.

The next result shows that for accessible circle extensions in dimension 3, the only way for the center foliation of a perturbation to *fail* to be absolutely continuous is to have atomic disintegration of volume.

THEOREM 1.2. *Let M be a 3-manifold and let $f : M \to M$ be a partially hyperbolic, volume-preserving diffeomorphism. Assume that f is accessible and that it admits an f-invariant foliation $\mathcal{W}^c$ with C^1 leaves, the leaves of which are all circles.*

If $\mathcal{W}^c$ is not absolutely continuous then there exists $k \geq 1$ and a full-volume subset of M that intersects every leaf of $\mathcal{W}^c$ in exactly k points.

In dimension three, any perturbation of a circle extension of an Anosov diffeomorphism is accessible unless it (or some finite-order quotient) is smoothly conjugate to the product of an Anosov diffeomorphism with a rotation [**12**].

One ingredient in the proofs of Theorems 1.1 and 1.2 is a general result about fibered partially hyperbolic diffeomorphisms with circle fibers, Theorem 2.2, which we state in the next section. In this section, we also state a result (Theorem 2.3) that applies to skew products with higher-dimensional compact leaves. In Theorem 2.5, we show that for partially hyperbolic diffeomorphisms preserving an absolutely continuous center foliation $\mathcal{W}^c$, if the generic leaf is compact, then every leaf is compact.

Finally, we describe a result (Theorem 2.7) that applies to partially hyperbolic diffeomorphisms fixing the leaves of a one-dimensional foliation.

2. *Further results*

Throughout this paper, unless otherwise mentioned, M is a compact Riemannian manifold without boundary and all diffeomorphisms are assumed to be partially hyperbolic and C^∞ (C^2 will suffice in most cases, but we restrict to C^∞ to keep the statements clean) and to preserve a C^∞ volume measure, usually denoted by m.

When we say that 'every perturbation' of a volume-preserving diffeomorphism $f : M \to M$ satisfies some property, we mean that there exists a C^1-open neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ satisfies this property.

A *dominated splitting* for a C^∞ diffeomorphism $h : M \to M$ is a direct sum decomposition of the tangent bundle

$$TM = E^1 \oplus E^2 \oplus \cdots \oplus E^k$$

such that:

- the bundles E^i are *Dh-invariant*; for every $i \in \{1, \ldots, k\}$ and $x \in M$, we have $D_x h(E^i(x)) = E^i(h(x))$; and
- *$Dh|_{E^i}$ dominates $Dh|_{E^{i+1}}$*; there exists $N \geq 1$ such that for any $x \in M$ and any unit vectors $u \in E^{i+1}(x)$, and $v \in E^i(x)$

$$\|D_x h^N(u)\| \leq \tfrac{1}{2}\, \|D_x h^N(v)\|.$$

The property of a splitting being dominated is independent of choice of metric and is always continuous. If h' is C^1 close to h with a dominated splitting, then h' also has a dominated splitting, which varies continuously with h' in the C^1 topology.

A C^1 diffeomorphism $f : M \to M$ of a complete Riemannian manifold M is *partially hyperbolic* if there is a dominated splitting $TM = E^u \oplus E^c \oplus E^s$ and $N \geq 1$ such that for any $x \in M$, and any choice of unit vectors $v^s \in E^s(x)$ and $v^u \in E^u(x)$, we have

$$\max\{\|D_x f^N(v^s)\|, \|D_x f^{-N}(v^u)\|\} < 1/2.$$

We always assume the bundles E^s and E^u are non-trivial. If E^c is trivial, then f is *Anosov*. As mentioned previously, the bundles E^s and E^u are uniquely integrable, tangent to foliations $\mathcal{W}^s$, $\mathcal{W}^u$ with C^∞ leaves. The leaves of these foliations are always contractible.

A partially hyperbolic diffeomorphism f is *dynamically coherent* if there exist f-invariant center stable and center unstable foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, tangent to the bundles $E^{cs} := E^c \oplus E^s$ and $E^{cu} := E^c \oplus E^u$, respectively; intersecting their leaves gives an invariant center foliation $\mathcal{W}^c$ tangent to E^c.

The foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ of a partially hyperbolic diffeomorphism $f : M \to M$ induce an equivalence relation on M: we say that $x, y \in M$ are in the same *accessibility class* if they can be joined by an *su*-path, that is, a piecewise C^1 path such that every piece

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is contained in a single leaf of $\mathcal{W}^s$ or a single leaf of $\mathcal{W}^u$. Then f is accessible if M is an accessibility class.

We say that a partially hyperbolic diffeomorphism $f\colon M \to M$ is *center bunched* if there exists an integer $k \geq 1$ such that for every $p \in M$:

$$\|D_p f^k|E^s\| \cdot \|(D_p f^k|E^c)^{-1}\| \cdot \|D_p f^k|E^c\| < 1$$

and

$$\|(D_p f^k|E^u)^{-1}\| \cdot \|D_p f^k|E^c\| \cdot \|(D_p f^k|E^c)^{-1}\| < 1.$$

In words, center bunching requires that the non-conformality of $Df \mid E^c$ be dominated by the hyperbolicity of $Df \mid E^u \oplus E^s$. Center bunching holds automatically if the restriction of Df to E^c is conformal in some continuous metric; in particular, if E^c is one-dimensional, then f is center bunched. Center bunching is a hypothesis in all results in this paper but for this reason appears explicitly only the theorems where E^c is potentially higher-dimensional. In §3.9 we discuss a generalization of center bunching called r-bunching.

In what follows, $\mathcal{P}(M)$ denotes the space of C^∞, volume-preserving, partially hyperbolic, dynamically coherent, and center-bunched diffeomorphisms of M, and $\mathcal{P}^j(M)$ denotes the set of all $f \in \mathcal{P}(M)$ with j-dimensional center distribution E^c.

Burns and Wilkinson [13] have shown that any $f \in \mathcal{P}(M)$ that is accessible is ergodic with respect to m. More generally, if U is an open accessibility class for a C^2, volume-preserving, center-bunched, partially hyperbolic diffeomorphism $f\colon M \to M$, then there exists an $\ell \geq 1$ such that $f^\ell(U) = U$ and the restriction of f^ℓ to U is ergodic with respect to volume on U.

2.1. *Fibered partially hyperbolic systems.* Our strategy for proving Theorems 1.1 and 1.2 is to establish corresponding facts for a special class of dynamics that we call *fibered partially hyperbolic systems*, a class of systems that we define in the sequel and that includes an arbitrary perturbation of the map f_0 in the introduction.

The manifolds that we consider will be endowed with a continuous fiber bundle structure, a generalization of the familiar smooth fiber bundle structure. A *continuous fiber bundle with C^1 fiber* is a continuous surjection $\pi : M \to B$, where M and B are smooth manifolds, with the following properties. There exists a Riemannian manifold N, an open cover $\{U_\alpha\}$ of the base B, and a family of homeomorphisms $h_\alpha : U_\alpha \times N \to \pi^{-1}(U_\alpha)$ such that:

(1) h_α maps each $\{b\} \times N$ to the fiber $\pi^{-1}(b)$;
(2) if $U_\alpha \cap U_\beta$ is non-empty, then

$$h_\beta \circ h_\alpha^{-1} : (U_\alpha \cap U_\beta) \times N \to (U_\alpha \cap U_\beta) \times N$$

has the form $h_\beta \circ h_\alpha^{-1}(b, x) = (b, \phi_b(x))$, where $\phi_b : N \to N$ is a C^1 diffeomorphism of N depending continuously on the base point b in the uniform C^1 topology, and such that $\|D\phi_b^{\pm 1}\|$ are uniformly bounded.

There is a natural notion of morphism between continuous fiber bundles with C^1 fiber: a morphism between $\pi\colon M \to B$ and $\pi'\colon M' \to B'$ is a homeomorphism $f\colon M \to M'$ that sends the fibers of π to the fibers of π', and whose restriction to each fiber is a C^1

diffeomorphism, varying uniformly continuously with the fiber. In the case where $\pi = \pi'$, we say that π is *f-invariant*. Two bundles $\pi : M \to B$ and $\pi' : M \to B$ are *isomorphic* if there is a morphism between them covering the identity on B.

A diffeomorphism $f : M \to M$ is a *fibered partially hyperbolic system* if it is partially hyperbolic, with Df-invariant splitting $E^s \oplus E^c \oplus E^u$, and M admits an f-invariant structure $\pi : M \to B$ of continuous fiber bundle with C^1 fiber, such the fibers of π are tangent to E^c.

Remark 2.1. If f is a fibered partially hyperbolic system, and g is a C^1 perturbation of f, then g is also a fibered partially hyperbolic system. More precisely, if f preserves the bundle structure $\pi : M \to B$ with fibers tangent to $E^c(f)$, then there is a g-invariant bundle structure, $\pi_g : M \to B$ and a morphism h between π and π_g such that $\pi_g \circ h \circ f = \pi_g \circ g \circ h$ (the fibers of π_g are then necessarily tangent to $E^c(g)$).

This follows immediately from the main structural stability result of [19] assuming that the center foliation for f is plaque expansive. This plaque expansivity was proved in [29] and also implies that if f is a fibered partially hyperbolic system, then the f-invariant fiber bundle structure tangent to E^c is unique: any two such f-invariant structures must be isomorphic.

A fibered partially hyperbolic system is dynamically coherent [7, Theorem 1.26].

To summarize, the set of fibered partially hyperbolic systems form a C^1-open subset of the partially hyperbolic, dynamically coherent diffeomorphisms, and $g \mapsto \mathcal{W}^c_g$ is continuous on this set. We denote by $\mathcal{P}_{\text{fib}}(M)$ the set of C^∞ volume-preserving fibered partially hyperbolic, center-bunched systems, and by $\mathcal{P}^j_{\text{fib}}(M)$ the set of $f \in \mathcal{P}_{\text{fib}}(M)$ with j-dimensional fiber. We note that $\mathcal{P}_{\text{fib}}(M) \subset \mathcal{P}(M)$ and $\mathcal{P}^j_{\text{fib}}(M) \subset \mathcal{P}^j(M)$.

In higher dimension, there is still a strong result for fibered systems if we assume the fibers have dimension one. We can also relax the accessibility assumption.

THEOREM 2.2. *Let M be a manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}^1_{\text{fib}}(M)$.*

(1) *If $\mathcal{W}^c$ is absolutely continuous, then there exists a continuous, volume-preserving flow ψ_t on M commuting with f, and with the property that $\psi_1 = \text{id}$. The continuous vector field X generating ψ_t is tangent to the leaves of $\mathcal{W}^c$. If f is accessible, then X is C^∞ along the leaves of $\mathcal{W}^c$.*

(2) *Suppose that f has a non-empty open accessibility class $U \subseteq M$.*
Then either:
   (a) *$m|U$ has atomic disintegration along the leaves of $\mathcal{W}^c$; or*
   (b) *f is accessible and $\mathcal{W}^c$ is absolutely continuous.*

We emphasize that Theorem 2.2 says that, although it is *possible* to be accessible and have atomic disintegration, if there is a non-trivial accessibility class $U \notin \{\emptyset, M\}$, then the disintegration of $m|U$ *must* be atomic.

Part of Theorem 1.2 generalizes to fibered systems with higher-dimensional compact fiber. Here we need to add the hypotheses of vanishing of center Lyapunov exponents. Let $f : M \to M$ be a partially hyperbolic diffeomorphism preserving the volume m, with splitting $TM = E^u \oplus E^c \oplus E^s$. We say that *the center Lypapunov exponents of f vanish*

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if, for m-almost every $x \in M$ and every $v \in E^c(x) \setminus \{0\}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\| = 0.$$

Interestingly, the presence of vanishing center exponents forces a rather rigid structure upon the disintegration of Lebesgue measure.

THEOREM 2.3. *Let M be a manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}_{\text{fib}}(M)$. Assume that f is accessible and that the center Lyapunov exponents of f vanish almost everywhere. Then either:*

(1) *the disintegration of volume is atomic along the leaves of $\mathcal{W}^c$; or*

(2) *There exists an absolutely continuous foliation $\mathcal{W}^{cc}$ with C^1 leaves that is f-invariant, holonomy invariant, subfoliates $\mathcal{W}^c$, and all of whose leaves are compact and diffeomorphic; in particular, if $\mathcal{W}^{cc} = \mathcal{W}^c$, then $\mathcal{W}^c$ is absolutely continuous.*

If f is center r-bunched, for some $r \geq 2$, then $\mathcal{W}^{cc}$ is a C^{r-1} subfoliation of $\mathcal{W}^c$.

Topological considerations sometimes rule out possibilities in the conclusion of this theorem. For example, if the leaves of $\mathcal{W}^c$ are homeomorphic to a surface other than the torus, it follows that, under the hypotheses of Theorem 2.3, either the disintegration of volume is atomic or $\mathcal{W}^c$ is absolutely continuous. On the torus, other possibilities may occur.

Example 2.4. Consider $g : M \times \mathbb{R}/\mathbb{Z} \to M \times \mathbb{R}/\mathbb{Z}$ a volume-preserving, accessible, C^∞ perturbation of an Anosov skew product with $\mathbb{R}/\mathbb{Z}$ fiber for which the disintegration of Lebesgue measure is atomic. Now construct a C^∞ skew product (isometric extension) on $M \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ over g of the form $g_\phi(x, t, u) = (g(x, t), u + \phi(x, t))$. One can choose ϕ so that g_ϕ is accessible. In this case, the leaves of $\mathcal{W}^{cc}$ are circles.

2.2. *Systems with mostly compact leaves.* As mentioned in the introduction, the hypotheses of Theorem 1.1 can be weakened in another direction. Rather than assuming that every leaf of the f-invariant foliation $\mathcal{W}^c$ is compact, it suffices to assume that the *generic center leaf is compact*. By this we mean that for all points x in a dense G_δ in M, the leaf $\mathcal{W}^c(x)$ through x is compact. The following theorem applies to all partially hyperbolic diffeomorphisms admitting an invariant center foliation with generic leaf compact.

THEOREM 2.5. *Let M be a closed manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}(M)$. Assume the center foliation $\mathcal{W}^c$ is leafwise absolutely continuous, the center Lyapunov exponents vanish, and the generic center leaf is compact.*

Then every center leaf is compact, with uniformly bounded volume, and the center foliation $\mathcal{W}^c$ has finite holonomy. Moreover, if $\dim \mathcal{W}^s = \dim \mathcal{W}^u = 1$, then $\mathcal{W}^c$ has finitely many non-regular leaves, and f is finitely covered by a fibered partially hyperbolic system.

COROLLARY 2.6. *In Theorems 1.1 and 1.2 the hypothesis 'there exists an f-invariant foliation $\mathcal{W}^c$ with C^1 leaves, all whose leaves are circles' can be replaced by 'there exists an f-invariant foliation $\mathcal{W}^c$ tangent to E^c, whose generic leaf is a circle'.*

2.3. *Center fixing dynamical systems.* Our final series of main results concern a generalization of the setting in our previous paper [6].

We say that a partially hyperbolic diffeomorphism $f : M \to M$ is *center fixing* if it is dynamically coherent and $f(\mathcal{W}^c(x)) = \mathcal{W}^c(x)$, for each $x \in M$. Center fixing diffeomorphisms arise naturally as elements of partially hyperbolic Lie group actions—to name two examples, the $\mathbb{R}$ action of an Anosov flow and the $\mathbb{R}^{n-1}$ action of the diagonal subgroup on a homogeneous space of $SL(n, \mathbb{R})$. We denote by $\mathcal{P}^j_{\text{fix}}(M)$ the set of all center fixing elements of $\mathcal{P}^j(M)$.

There is an analog to Theorem E for center fixing diffeomorphisms.

THEOREM 2.7. *Let M be a manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}^1_{\text{fix}}(M)$.*

(1) *If $\mathcal{W}^c$ is absolutely continuous, then there exists a continuous, volume-preserving flow ψ_t on M such that $f = \psi_1$. Orbits of ψ_t are tangent to the leaves of $\mathcal{W}^c$. If f is accessible, then ψ_t is C^∞ along the leaves of $\mathcal{W}^c$. If $\dim(M) = 3$ (without the assumption of accessibility), then ψ_t is a C^∞, volume-preserving Anosov flow.*
(2) *Suppose that f has an non-empty open accessibility class $U \subset M$.*
 Then:
 (a) *$m|U$ has atomic disintegration along the leaves of $\mathcal{W}^c$; or*
 (b) *$\mathcal{W}^c$ is absolutely continuous; or*
 (c) *f is accessible.*

Theorem 2.7 generalizes the main results in our previous paper [6], in which we considered perturbations of the time-one map of geodesic flows over negatively curved surfaces.

COROLLARY 2.8. *Let M be a 3-manifold, and let $\varphi_t : M \to M$ be a C^∞, volume-preserving Anosov flow. Assume that φ_t is not the constant time suspension of an Anosov diffeomorphism. Suppose that $f \in \text{Diff}^\infty_m(M)$ is C^1-close to φ_1. Then either:*

(1) *m has atomic disintegration along the leaves of $\mathcal{W}^c$; or*
(2) *f embeds in a C^∞, volume-preserving Anosov flow.*

Proof. According to [12], because φ_t is not a constant time suspension, it is stably accessible; hence, f is accessible. Theorem 2.7 implies that either m has atomic disintegration, or f embeds in a C^∞, volume-preserving Anosov flow. □

We remark that Bonatti and Wilkinson [9] showed that in dimension three, under the weaker assumption that $f : M \to M$ is partially hyperbolic, transitive, and dynamically coherent and every closed center manfold is periodic, one obtains that there exists an n such that $f^n \in \mathcal{P}^1_{\text{fix}}(M)$ and the center foliation admits an expansive continuous flow. Recently, Mario Shannon [35] has announced that transitive topologically Anosov flows in dimension three are orbit conjugate to Anosov flows. Little is known about center-fixing systems in higher dimension.

2.4. *Structure of the paper.* In §3, we give background on foliations, disintegration of measure, absolute continuity, and normal hyperbolicity. Section 3.10 is devoted to the main technical result we use, an invariance principle of Avila *et al* [4] whose origins go back to work of Ledrappier [22, 23]. In §4 we sharpen this invariance principle so that it can be applied to analyze the disintegration of measures along center foliations. Section 5 presents a result of Repovš *et al* [31] that we use, as in [37], to establish regularity of holonomy-invariant objects such as vector fields and foliations.

The proofs of Theorems 1.1, 1.2, and 2.2 concerning systems with compact one-dimensional center foliation are in §6. Fibered systems with higher-dimensional compact center are discussed in §7, in which Theorem 2.3 is proved. In §8, we prove Theorem 2.5, the main result about center foliations with mostly compact leaves. Section 9 is devoted to center-fixing systems and is where we prove Theorem 2.7.

Finally, in §10, we discuss some open questions and construct examples.

3. *Background and preliminaries*

3.1. *Topological preliminaries.*

3.1.1. *Foliations.*

Let M be a manifold of dimension $d \geq 2$. A *foliation (with C^r leaves)* is a partition $\mathcal{F}$ of the manifold M into C^r submanifolds of dimension k, for some $0 < k < d$ and $1 \leq s \leq \infty$, such that for every $p \in M$ there exists a continuous local chart

$$\Phi : B_1^k \times B_1^{d-k} \to M \quad (B_1^m \text{ denotes the unit ball in } \mathbb{R}^m)$$

with $\Phi(0, 0) = p$ and such that the restriction to every horizontal $B_1^k \times \{\eta\}$ is a C^r embedding depending continuously on η and whose image is contained in some $\mathcal{F}$-leaf. The image of such a chart Φ is a *foliation box* and the $\Phi(B_1^k \times \{\eta\})$ are the corresponding *local leaves* or *plaques*.

A foliation $\mathcal{F}$ has *uniformly compact leaves* if there exists a constant $C > 0$ such that the restricted Riemannian volume of every leaf $\mathcal{F}$ is bounded by C, with respect to some (any) Riemannian metric on M. If f is a fibered partially hyperbolic system, then the leaves of $\mathcal{W}^c$ are uniformly compact.

To study the precise smoothness of the leaves of a normally hyperbolic foliation, we refine the definition of normal hyperbolicity. For $r \geq 1$ we say that $(f, \mathcal{F})$ is *r-normally hyperbolic* if there exists $k \geq 1$ such that

$$\sup_p \|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{T\mathcal{F}})^{-1}\|^r < 1 \quad \text{and} \quad \sup_p \|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{T\mathcal{F}}\|^r < 1.$$

Note that 1-normally hyperbolic = normally hyperbolic, and r-normal hyperbolicity is a C^1-open condition.

3.1.2. *Normal hyperbolicity.*

Suppose M is a closed manifold, and let $f_1, f_2 \in \mathrm{Diff}(M)$. Assume that $\mathcal{F}_1, \mathcal{F}_2$ are foliations of M with C^1 leaves and that f_1 and f_2 preserve $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

Definition 3.1. A *leaf conjugacy* from $(f_1, \mathcal{F}_1)$ to $(f_2, \mathcal{F}_2)$ is a homeomorphism $h : M \to M$ sending $\mathcal{F}_1$ leaves diffeomorphically onto $\mathcal{F}_2$ leaves, equivariantly in the sense that

$$h(f_1(\mathcal{F}_1(p))) = f_2(\mathcal{F}_2(h(p))) \quad \text{for all } p \in M.$$

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Definition 3.2. Suppose $f \in \mathrm{Diff}(M)$ and $\mathcal{F}$ is an f-invariant foliation of M with C^1 leaves. Here $\mathcal{F}$ is *normally hyperbolic* if there exists a Df-invariant dominated splitting $TM = E^u \oplus E^c \oplus E^s$, with at least two of the bundles non-trivial, such that Df uniformly expands E^u, uniformly contracts E^s, and such that $T\mathcal{F} = E^c$.

Note that a diffeomorphism with a normally hyperbolic foliation is partially hyperbolic, with $E^c = T\mathcal{F}$, but, as remarked previously, the converse does not hold in general: the center bundle of a partially hyperbolic diffeomorphism is not necessarily tangent to a foliation, let alone an invariant foliation.

3.1.3. *Dynamical coherence.* Throughout this section, f denotes a partially hyperbolic diffeomorphism. Recall that f is *dynamically coherent* if there exist f-invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to the bundles E^{cs} and E^{cu}, respectively. Intersecting the leaves of $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ gives an f-invariant foliation $\mathcal{W}^c$ tangent to E^c. Most of the facts here are proved in [**19**]. A more detailed discussion can be found in [**13**].

We first discuss the stability of dynamical coherence under perturbation. It is not known whether every perturbation of a dynamically coherent diffeomorphism is dynamically coherent, but in systems that are *plaque expansive*, dynamical coherence is stable. Plaque expansiveness was introduced by Hirsch *et al* [**19**], who proved among other things that any perturbation of a plaque expansive diffeomorphism is dynamically coherent. Roughly, f is plaque expansive if pseudo-orbits that respect local leaves of the center foliation cannot shadow each other too closely (in the case of Anosov diffeomorphisms, plaque expansiveness is the same as expansiveness, which is automatic). Plaque expansiveness holds in a variety of natural settings; in particular, we have the following, whose proof can be found in [**15**].

THEOREM 3.3. (Foliation stability and Hölder continuity of the leaf conjugacy) *Let M be a closed manifold, and let $(f, \mathcal{F})$ be an r-normally hyperbolic foliation of M, for some $r \geq 1$, with Df-invariant splitting $E^u \oplus (T\mathcal{F} = E^c) \oplus E^s$. Then the leaves of $\mathcal{F}$ are uniformly C^r, and we have the following.*

(1) *Suppose that one of the following holds:*
 (a) *the restriction $Df|_{T\mathcal{F}}$ is an isometry; or*
 (b) *the bundles E^{cu} and E^{cs} are C^1; or*
 (c) *$\mathcal{F}$ is uniformly compact.*

 Then f is dynamically coherent, plaque expansive, and r-normally hyperbolic with respect to the foliations $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$, and $\mathcal{F} = \mathcal{W}^{cu} \cap \mathcal{W}^{cs}$.

(2) *If $(f, \mathcal{F})$ is plaque expansive, then it is structurally stable in the following sense. For each diffeomorphism g that C^1-approximates f, there exists a unique g-invariant foliation $\mathcal{F}_g$ (with C^1-leaves) near $\mathcal{F}$. The foliation $\mathcal{F}_g$ is normally hyperbolic, plaque expansive, and $(f, \mathcal{F})$ is leaf conjugate to $(g, \mathcal{F}_g)$ by a homeomorphism $h^c : M \to M$ close to the identity.*

Problem 3.4. Is every diffeomorphism $f \in \mathcal{P}^1_{\mathrm{fix}}(M)$ plaque expansive?

3.2. *Local and global holonomy maps.* If f is dynamically coherent, then each leaf of $\mathcal{W}^{cs}$ is simultaneously subfoliated by the leaves of $\mathcal{W}^c$ and by the leaves of $\mathcal{W}^s$. Similarly $\mathcal{W}^{cu}$ is subfoliated by $\mathcal{W}^c$ and $\mathcal{W}^u$. This implies that for any two points $x, y \in M$ with $y \in \mathcal{W}^s_x$ there is a neighborhood U_x of x in the leaf $\mathcal{W}^c_x$ and a homeomorphism $h^s_{x,y}$: $U_x \to \mathcal{W}^c_y$ with the property that $h^s_{x,y}(x) = y$ and, in general, $h^s_{x,y}(z) \in \mathcal{W}^s_z \cap \mathcal{W}^{c,\mathrm{loc}}_y$. We refer to $h^s_{x,y}$ as a *(local) stable holonomy map*. We similarly define unstable holonomy maps between local center leaves. We note that, because the leaves of stable and unstable foliation are contractible, the local holonomy maps $h^*_{x,y}$ for $* \in \{s, u\}$ are well-defined and are uniquely defined as germs by the endpoints x, y. An important fact that will be used repeatedly is that if f is center bunched, then $h^*_{x,y}$ is C^1, locally uniformly in x, y. See [28, 37] and §3.9.

We say that f *admits global stable holonomy maps* if, for every $x, y \in M$ with $y \in \mathcal{W}^s_x$, there exists a homeomorphism $h^s_{x,y} : \mathcal{W}^c_x \to \mathcal{W}^c_y$ with the property that $h^s_{x,y}(x) = y$ and, in general, $h^s_{x,y}(z) \in \mathcal{W}^s_z \cap \mathcal{W}^c_y$. As global stable holonomy maps must agree locally with local stable holonomy, we use the same notation $h^s_{x,y}$ for both local and global. We similarly define global unstable holonomy maps and say that f *admits global su-holonomy maps* if it admits both global stable and unstable holonomy. Note that if f admits global su-holonomy, then all leaves of $\mathcal{W}^c$ are homeomorphic.

LEMMA 3.5. *Fibered partially hyperbolic systems have global su-holonomies.*

Proof. Let $f: M \to M$ be a fibered partially hyperbolic system. Dynamical coherence implies that the foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ project to topological foliations $\bar{\mathcal{W}}^u$, $\bar{\mathcal{W}}^s$ on the leaf space $B = M/\mathcal{W}^c$ and the restriction of the projection $M \to B$ to any $\mathcal{W}^*$-leaf is a homeomorphism (and these homeomorphisms vary continuously from leaf to leaf).

Dynamical coherence and unique integrability of the restriction of $\mathcal{W}^*$ to $\mathcal{W}^{c*}$ for $* \in \{u, s\}$ imply that for any $\bar{\mathcal{W}}^*$-path $\bar{\gamma}: [0, 1] \to B$, and any $x \in M$ that projects to $\bar{\gamma}(0)$, there is a unique lift of $\bar{\gamma}$ to a $\mathcal{W}^*$-path $\gamma_x: [0, 1] \to M$ with $\gamma_x(0) = x$. These lifts γ_y vary continuously over $y \in \mathcal{W}^c_x$.

Given $x, x' \in M$ with $x, x' \in \mathcal{W}^*_x$, any path $\gamma_x: [0, 1] \to \mathcal{W}^*_x$ connecting x to x' projects to a $\bar{\mathcal{W}}^*$-path $\bar{\gamma}$ in B. Fixing such a path and taking lifts γ_y over $y \in \mathcal{W}^c_x$ defines a $*$-holonomy map from $\mathcal{W}^c_x$ to $\mathcal{W}^c_{x'}$ by $y \mapsto \gamma_y(1)$. □

In contrast to the fiber-bunched maps, time-one maps of Anosov flows do *not* have global su-holonomies, because their center leaves are not all homeomorphic.

3.3. *Measure-theoretic preliminaries.* We expand here on the discussion in §3 of our previous paper [6].

We begin with a general discussion of disintegration of measures. Let Z be a polish metric space, let μ be a finite Borel measure on Z, and let $\mathcal{P}$ be a partition of Z into measurable sets. Denote by $\hat{\mu}$ the induced measure on the σ-algebra generated by $\mathcal{P}$, which may be naturally regarded as a measure on $\mathcal{P}$.

A *system of conditional measures* (or a *disintegration*) of μ with respect to $\mathcal{P}$ is a family $\{\mu_P\}_{P\in\mathcal{P}}$ of probability measures on Z such that:

(1) $\mu_P(P) = 1$ for μ-almost every $P \in \mathcal{P}$;
(2) given any continuous function $\psi : Z \to \mathbb{R}$, the function $P \mapsto \int \psi \, d\mu_P$ is measurable, and

$$\int_M \psi \, d\mu = \int_{\mathcal{P}} \left(\int \psi \, d\mu_P \right) d\hat{\mu}(P).$$

3.4. *Measurable partitions and disintegration of measure.* It is not always possible to disintegrate a probability measure with respect to a partition (we discuss examples in the following), but disintegration is always possible if $\mathcal{P}$ is a measurable partition. We say that $\mathcal{P}$ is a *measurable partition* if there exist measurable subsets $E_1, E_2, \ldots, E_n \ldots$ of Z such that

$$\mathcal{P} = \{E_1, Z \setminus E_1\} \vee \{E_2, Z \setminus E_2\} \vee \cdots \mod 0. \tag{1}$$

In other words, there exists a full μ-measure subset $F_0 \subset Z$ such that, for any atom P of $\mathcal{P}$, we have

$$P \cap F_0 = E_1^* \cap E_2^* \cap \cdots \cap F_0$$

where E_i^* is either E_i or $Z \setminus E_i$, for $i \geq 1$. Our interest in measurability of a partition derives from the following fundamental result.

THEOREM 3.6. (Rokhlin [32]) *If $\mathcal{P}$ is a measurable partition, then there exists a system of conditional measures relative to $\mathcal{P}$. It is essentially unique in the sense that two such systems coincide in a set of full $\hat{\mu}$-measure.*

A basic family of examples of measurable partition is given by the following proposition.

PROPOSITION 3.7. *Let $\mathcal{F}$ be a foliation of M, and let μ be a Borel probability measure on M. Suppose for μ-almost every $x \in M$, the leaf $\mathcal{F}_x$ is compact. Then $\mathcal{F}$ is a measurable partition.*

Proof. (A related result is proved in [5, §4.3].) Replacing M by some full μ-measure subset if necessary, we may suppose that every leaf is compact. Let X be a countable dense subset of M. For each $x \in X$ and $n \geq 1$, define $V(x, k)$ to be the points $y \in M$ such that the leaf $\mathcal{F}_y$ intersects the closed ball of radius $1/k$ around x. We claim that $V(x, k)$ is closed and, hence, measurable. Indeed, let y_n be any sequence in $V(x, k)$ converging to some $y \in M$, and let $z_n \in \mathcal{F}_{y_n} \cap \overline{B}(x, 1/k)$. By compactness and continuity of the leaves, $\mathcal{F}_{y_n}$ converges to $\mathcal{F}_y$ in the Hausdorff topology and then $z_n \in \mathcal{F}_{y_n}$ must accumulate on some $z \in \mathcal{F}_y$. As z also belongs to $\overline{B}(x, 1/k)$, this implies that $y \in V(x, k)$. That proves the claim. It is clear from the definition that each $V(x, k)$ consists of entire leaves. It is easy to see that for any two different leaves $\mathcal{F}_1$ and $\mathcal{F}_2$ there exists (x, k) such that $V(x, k)$ contains one of the leaves but not the other. First, take k large enough so that $2/k$ is smaller than the distance from $\mathcal{F}_1$ to $\mathcal{F}_2$. By density, we may find $x \in X$ such that $\overline{B}(x, 1/k)$ intersects $\mathcal{F}_1$; clearly, it cannot intersect $\mathcal{F}_2$. This proves that the countable family of partitions $\{V(x, k), M \setminus V(x, k)\}$ generates the foliation. □

The lack of measurability of a partition can be just as interesting as the measurability. A typically invoked example of a non-measurable partition is the partition of the 2-torus into lines of irrational slope. More generally, the following is true.

PROPOSITION 3.8. *Let $(\varphi_t)_t$ be a μ-preserving flow on Z and $\mathcal{O}$ be the partition of Z into flow lines. If the flow is ergodic and μ does not give full weight to a single orbit, then $\mathcal{O}$ is not measurable. More generally, if $\mathcal{O}$ is measurable, then μ-almost every orbit of the flow is periodic.*

Proof. Suppose measurable subsets E_j, $j \geq 1$, as in (1) do exist. Each E_j coincides mod 0 with a union of partition atoms, that is, with a φ_t-invariant subset. Then, by ergodicity, every E_j has either zero or full measures. This implies that some partition atom (that is, some orbit) has full measure, contradicting the hypothesis. This proves the first statement. Now, assume $\mathcal{O}$ is measurable and let $\{\mu_O : O \in \mathcal{O}\}$ be a disintegration. Then almost every μ_O is a probability measure supported on the orbit O and flow-invariant. As there are no flow-invariant finite measures on open orbits, it follows that almost every orbit is closed, as stated. This completes the proof. □

In light of this, it is notable that it is possible to construct a foliation $\mathcal{F}$ with a dense set of non-compact leaves that is a measurable partition with respect to volume.

Example 3.9. Let $f : M \to M$ be a perturbation of the time-one map of an Anosov flow on a 3-manifold so that volume has atomic center disintegration along $\mathcal{W}^c_f$. Consider the product $f \times f$. The disintegration of volume along $\mathcal{W}^c_{f\times f}$ is again atomic, with atoms at points $(f^k(x), f^\ell(x))$, where x is an atom for $\mathcal{W}^c_f$ and $k, \ell \in \mathbb{Z}$. Take any smooth foliation of $M \times M$ with five-dimensional leaves and intersect with $\mathcal{W}^c_{f\times f}$. Typical choices are 'irrational' with respect to the lattice of atoms and, thus, the intersection gives a one-dimensional foliation with dense leaves and atomic disintegration. This is a measurable partition: just take a sequence of partitions nesting to points; at stage n take all leaves in a partition element whose atom is contained in that element.

3.5. *Disintegration of measure along foliations with non-compact leaves.* The disintegration theorem of Rokhlin [32] does not apply directly when a foliation has a positive measure set of non-compact leaves. Instead, one must consider disintegrations into *measures defined up to scaling*, that is, equivalence classes where one identifies any two (possibly infinite) measures that differ only by a constant factor. Here we present this theory in a fairly general setting. See also [21, §4] and [24, §3].

Let M be a manifold of dimension $d \geq 2$, and let m be a locally finite measure on M. Let $\mathcal{B}$ be any (small) foliation box. Rokhlin [32] proved that there is a disintegration $\{m^{\mathcal{B}}_x : x \in \mathcal{B}\}$ of the restriction of m to the foliation box into conditional probabilities along the local leaves, and this disintegration is essentially unique. The crucial observation is that conditional measures corresponding to different foliation boxes coincide on the intersection, up to a constant factor.

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LEMMA 3.10. [6, Lemma 3.2] *For any foliation boxes $\mathcal{B}$ and $\mathcal{B}'$ and for m-almost every $x \in \mathcal{B} \cap \mathcal{B}'$, the restrictions of $m_x^{\mathcal{B}}$ and $m_x^{\mathcal{B}'}$ to $\mathcal{B} \cap \mathcal{B}'$ coincide up to a constant factor.*

This implies that there exists a family $\{\mathfrak{m}_x : x \in M\}$ where each $\mathfrak{m}_x$ is a measure defined up to scaling with $\mathfrak{m}_x(M \setminus \mathcal{F}_x) = 0$, the function $x \mapsto \mathfrak{m}_x$ is constant on the leaves of $\mathcal{F}$, and the conditional probabilities $m_x^{\mathcal{B}}$ along the local leaves of any foliation box $\mathcal{B}$ coincide almost everywhere with the normalized restrictions of the $\mathfrak{m}_x$ to the local leaves of $\mathcal{B}$. It is also clear from the arguments that such a family is essentially unique. We call it the disintegration of m and refer to the $\mathfrak{m}_x$ as conditional classes of m along the leaves of $\mathcal{F}$

3.6. *Foliations whose leaves are fixed under a measure-preserving homeomorphism.* Now suppose the foliation $\mathcal{F}$ is invariant under a homeomorphism $f : M \to M$, meaning that $f(\mathcal{F}_x) = \mathcal{F}_{f(x)}$ for every $x \in M$. Take the measure m to be invariant under f. Then, by essential uniqueness of the disintegration, $f_*(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ for almost every x. We are especially interested in the case when f fixes every leaf, that is, when $f(x) \in \mathcal{F}_x$ for all $x \in M$. Then $f_*(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ for almost every x, which means that every representative m_x of the conditional class $\mathfrak{m}_x$ is f-invariant up to rescaling: $f_*(m_x) = cm_x$ for some $c > 0$. Actually, the scaling factor c is one.

PROPOSITION 3.11. [6, Proposition 3.3] *Suppose that m is invariant under a homeomorphism $f : M \to M$ that fixes all the leaves of $\mathcal{F}$. Then, for almost all $x \in M$, any representative m_x of the conditional class $\mathfrak{m}_x$ is an f-invariant measure.*

3.7. *Absolute continuity.* As previously, let M be a Riemannian manifold. Let λ_Σ denote the volume measure induced by the Riemann metric on a C^1 submanifold Σ of M.

The classical definition of absolute continuity ([2, 3]) goes as follows. A foliation $\mathcal{F}$ on M is *absolutely continuous* if every holonomy map $h_{\Sigma,\Sigma'}$ between a pair of smooth cross-sections Σ and Σ' is absolutely continuous, meaning that, the push-forward $(h_{\Sigma,\Sigma'})_*\lambda_\Sigma$ is absolutely continuous with respect to $\lambda_{\Sigma'}$. Reversing the roles of the cross-sections, one sees that $(h_{\Sigma,\Sigma'})_*\lambda_\Sigma$ is actually equivalent to $\lambda_{\Sigma'}$.

Here it is convenient to introduce the following weaker notion. We say that *$\mathcal{F}$ is leafwise absolutely continuous* (or *volume has Lebesgue disintegration along $\mathcal{F}$-leaves*) if, for any measurable set $Y \subset M$, we have $m(Y) = 0$ if and only if for m-almost every $z \in M$ the leaf L through z meets Y in a zero λ_L-measure set. In other words, for almost every leaf L, the conditional measure m_L of m along the leaf is equivalent to the Riemann measure λ_L on the leaf.

LEMMA 3.12. [6, Lemma 3.4] *If $\mathcal{F}$ is absolutely continuous, then $\mathcal{F}$ is leafwise absolutely continuous.*

The converse is false: one can destroy absolute continuity of holonomy at a single transversal while keeping the Lebesgue disintegration of volume (this is an exercise in Brin and Stuck [11]).

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LEMMA 3.13. *Let $f : M \to M$ be C^2 and partially hyperbolic. The foliations $\mathcal{W}^s(f)$ and $\mathcal{W}^u(f)$ are absolutely continuous and, hence, volume has Lebesgue disintegration along $\mathcal{W}^s(f)$ and $\mathcal{W}^u(f)$-leaves.*

Proof. This is a classical fact going back to Brin and Pesin [10]. □

We say that a foliation $\mathcal{F}$ is *upper leafwise absolutely continuous* if for m-almost every x, we have $m_L \ll \lambda_L$, where L is the leaf of $\mathcal{F}$ through x. Similarly $\mathcal{F}$ is *lower leafwise absolutely continuous* if $\lambda_L \ll m_L$ for almost every L. Note that leafwise absolute continuity = upper leafwise absolute continuity + lower leafwise absolute continuity. In the invariant ergodic case, lower leafwise absolute continuity is actually equivalent to leafwise absolute continuity.

LEMMA 3.14. *If $\mathcal{F}$ is leafwise absolutely continuous and invariant under an ergodic diffeomorphism f, then m_L and λ_L are equivalent for almost every leaf L.*

Proof. Suppose some set Y meets almost every leaf L on a zero λ_L-measure set. We may suppose that Y is invariant because the restriction of f to leaves preserves the class of zero measure sets, because f is smooth. If Y has full measure, then its complement is a zero m-measure that intersects leaves L in full λ_L-measure subsets, a contradiction. □

Remark 3.15. For partially hyperbolic diffeomorphisms whose center leaves are circles with bounded length, the center foliation cannot be *upper* leafwise absolutely continuous unless the center Lyapunov exponent vanishes at almost every point. This follows from the observation in [36] that if the center Lyapunov exponent is non-zero on some set A, then A meets m-almost every leaf $L = \mathcal{W}^c_x$ in a set of λ_L-measure zero.

In the remainder of this section we focus on invariant foliations of C^2 partially hyperbolic diffeomorphisms. Recall that $\mathcal{W}^u$ and $\mathcal{W}^s$ are always absolutely continuous, by [10].

LEMMA 3.16. *Suppose f is C^2, partially hyperbolic, and dynamically coherent. If $\mathcal{W}^c$ is leafwise absolutely continuous, then so are $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$.*

Proof. Suppose that $\mathcal{W}^c$ is leafwise absolutely continuous. Let A be a zero measure set. As $\mathcal{W}^s$ is absolutely continuous, there is a set B of full measure so that $\mathcal{W}^s_x$ meets A in a zero measure set, for every $x \in B$. As $\mathcal{W}^c$ is absolutely continuous, there is a set C of full measure so that $\mathcal{W}^c_y$ meets B in a set of full leaf measure, for every $y \in C$. Let y be a point in C. We claim that $\mathcal{W}^{cs}_y$ meets A in a zero measure set. The reason is that the restriction of $\mathcal{W}^s$ to $\mathcal{W}^{cs}_y$ is an absolutely continuous foliation in the leaf Riemannian metric on $\mathcal{W}^{cs}_y$. Hence, if we unravel the definition of C and apply Fubini's theorem, we get that the leaf measure of A in $\mathcal{W}^{cs}_y$ is zero. □

Remark 3.17. In this argument, $\mathcal{W}^c$ and $\mathcal{W}^s$ do not play symmetric roles. The reason the restriction of $\mathcal{W}^s$ to $\mathcal{W}^{cs}$ leaves is absolutely continuous is dynamical, and does not follow *a priori* from the fact that $\mathcal{W}^s$ is leafwise absolutely continuous.

Problem 3.18. It is observed in [30] that the center foliation is absolutely continuous if the center stable and the center unstable foliations are.

(1) Is the converse true, that is, does absolute continuity of the center imply absolute continuity of the center stable and the center unstable?
(2) Is the converse to Lemma 3.16 true, that is, does leafwise absolute continuity of the center follow from leafwise absolute continuity of the center stable and the center unstable?

LEMMA 3.19. *Let M be a compact Riemannian manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}(M)$. Let m_p be a measure on a local leaf $\mathcal{W}_p^{c,\text{loc}}$, and let $\mathcal{B}$ be the neighborhood of q ($\mathcal{W}^c$ foliation box) obtained by first applying local s-holonomy to $\mathcal{W}_p^c$, and then applying local u-holonomy. Let $\{m_q\}_{q\in\mathcal{B}}$ be the family of measures supported on local $\mathcal{W}^c$ leaves given by pushing forward m_p, first by local s holonomy and then by local u holonomy.*

Suppose that $\{m_q\}_{q\in\mathcal{B}}$ is a disintegration of Lebesgue measure in $\mathcal{B}$. Then $\mathcal{W}^c$ has Lebesgue disintegration in $\mathcal{B}$: for every $q \in \mathcal{B}$, the conditional measure m_q is equivalent to the Riemann measure λ_q^c on $\mathcal{W}_q^{c,\text{loc}}$, and the densities

$$\frac{dm_q}{d\lambda_q^c}$$

are positive, continuous on $\mathcal{W}_q^c \cap \mathcal{B}$, and vary continuously with q.

Proof. Fix a continuous Riemannian metric inducing the Lebesgue measure on M, such that the stable, unstable, and center bundles are orthogonal. For $q \in \mathcal{B}$, denote by $D^*(q)$ the intersection $\mathcal{W}_q^{*,\text{loc}} \cap \mathcal{B}$.

We show that

$$\rho(q) = \lim_{r\to 0} \log m_q(B^c(q,r)) - \log \lambda_q^c(B^c(q,r))$$

exists and is uniformly continuous as a function of $q \in \mathcal{B}$, as this implies that m_q is equivalent to λ_q^c with

$$\frac{dm_q}{d\lambda_q^c}(q) = e^{\rho(q)}.$$

The open set $U(q,r)$ in $\mathcal{B}$ formed by applying stable followed by unstable holonomy in $\mathcal{B}$ to the center ball $B^c(q,r)$ has volume proportional to $m_q(B^c(q,r))$ by a constant that is bounded independent of q, r. On the other hand, it is also given by the formula

$$\int_{B^c(q,r)} \int_{D^s(x)} J_{x,y}^{s,c} \int_{D^u(y)} J_{y,z}^{u,cs}\, d\lambda_y^u(z)\, d\lambda_x^s(y)\, d\lambda_z^c(x),$$

where $J_{x,y}^{s,c}$ denotes the Jacobian of the stable holonomy $\mathcal{W}_x^{c,\text{loc}} \to \mathcal{W}_y^{c,\text{loc}}$ and $J_{y,z}^{u,cs}$ denotes the Jacobian of the unstable holonomy $\mathcal{W}_y^{cs,\text{loc}} \to \mathcal{W}_z^{cs,\text{loc}}$, calculated with respect to the fixed Riemannian structure. As the Jacobians are uniformly continuous, this gives that ρ is the uniformly continuous function:

$$q \mapsto \log \int_{D^s(q)} J_{q,y}^{s,c} \int_{D^u(y)} J_{y,z}^{u,cs}\, d\lambda_y^u(z)\, d\lambda_q^s(y),$$

up to an additive constant. □

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3.8. *Smoothness of foliations.* A foliation is C^r if there is a C^r foliation atlas. Note that the leaves of a C^r foliation are uniformly C^r, but a foliation with C^r leaves is not necessarily a C^r foliation.

A useful criterion for checking whether a foliation with C^r leaves is C^r is given by the examining the holonomy maps. Here we describe a C^∞ version of the criterion. The same arguments yield a C^r version of the criterion, with some modifications. The main tool is the following.

THEOREM 3.20. (Journé [20]) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be transverse foliations of a manifold M whose leaves are uniformly C^∞. Let $\psi : M \to \mathbb{R}$ be any continuous function such that the restriction of ψ to the leaves of $\mathcal{F}_1$ is uniformly C^∞ and the restriction of ψ to the leaves of $\mathcal{F}_2$ is uniformly C^∞. Then ψ is C^∞.*

This has the following corollary.

COROLLARY 3.21. (See [28]) *A local foliation with uniformly C^∞ leaves and uniformly C^∞ holonomies (with respect to a fixed C^∞ transverse local foliation) is a C^∞ local foliation.*

Proof. Let $\mathcal{F}$ be a local foliation with uniformly C^∞ leaves, and let $\mathcal{T}$ be a C^∞ transverse local foliation to $\mathcal{F}$. By a C^∞ change of coordinates, we may assume that $\mathcal{T}$ is the foliation by vertical coordinate planes in $\mathbb{R}^n$. Now, the standard rectification of $\mathcal{F}$ in $\mathbb{R}^n$ (via holonomy between $\mathcal{T}$-leaves) sends $\mathcal{F}$-leaves to horizontal vertical planes. The assumption that the leaves of $\mathcal{F}$ are uniformly C^∞ implies that the rectification is C^∞ along leaves of $\mathcal{F}$. The assumption that the holonomy maps between $\mathcal{T}$-leaves are uniformly C^∞ implies that the rectification is uniformly C^∞ along vertical planes. Journé's theorem implies that the rectification is C^∞, so that $\mathcal{F}$ is a C^∞ foliation. This proves the corollary. □

A simple application of Corollary 3.21 gives the following criterion for smoothness, which will be applied to local center-stable and center-unstable foliations of a partially hyperbolic diffeomorphism.

PROPOSITION 3.22. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be local foliations whose leaves are C^∞ and intersect transversely in a local foliation $\mathcal{F}$. Suppose there exist local foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ with the following properties:*

(1) *$\mathcal{F}_1$ is transverse to $\mathcal{G}_2$ and $\mathcal{F}_2$ is transverse to $\mathcal{G}_1$;*
(2) *$\mathcal{F}_1$ C^∞ subfoliates the leaves of $\mathcal{G}_1$, and $\mathcal{F}_2$ C^∞ subfoliates the leaves of $\mathcal{G}_2$;*
(3) *$\mathcal{F}$-holonomy between $\mathcal{F}_1$-leaves is uniformly C^∞, and $\mathcal{F}$-holonomy between $\mathcal{F}_2$-leaves is uniformly C^∞.*

Then $\mathcal{F}$ is a C^∞ foliation, as are the restrictions of $\mathcal{F}$ to $\mathcal{G}_1$ and $\mathcal{G}_2$.

Proof. As the leaves of $\mathcal{F}$ are uniformly C^∞, to prove the proposition, by Corollary 3.21, it suffices to show that the $\mathcal{F}$ holonomy maps are uniformly C^∞. To this end, fix a C^∞ local foliation $\mathcal{T}$ transverse to $\mathcal{F}$. Fix one leaf $\mathcal{T}_p$ and for $q \in \mathcal{F}_p$, consider the associated family of $\mathcal{F}$ holonomy maps $\psi_{p,q} : \mathcal{T}_p \to \mathcal{T}_q$. We use Theorem 3.20 to prove that $\psi_{p,q}$ is C^∞, uniformly in q.

To do this, we first show that the restriction of $\mathcal{F}$ to the leaves of $\mathcal{G}_1$ is uniformly C^∞, and the restriction of $\mathcal{F}$ to the leaves of $\mathcal{G}_2$ is uniformly C^∞. To see this, observe that by assumption $\mathcal{F}_1$ is (uniformly) a C^∞ subfoliation of $\mathcal{G}_1$, and the $\mathcal{F}$-holonomy maps between $\mathcal{F}_1$ leaves are uniformly C^∞. The leaves $\mathcal{F}$ are uniformly C^∞, because the leaves of $\mathcal{G}_1$ and $\mathcal{G}_2$ are. Corollary 3.21 then implies that the restriction of $\mathcal{F}$ to the leaves of $\mathcal{G}_1$ is uniformly C^∞. Similarly, the restriction of $\mathcal{F}$ to the leaves of $\mathcal{G}_2$ is uniformly C^∞.

Intersecting the leaves of $\mathcal{T}$ with the leaves of $\mathcal{G}_1$, we obtain a foliation $\mathcal{T}_1$ with uniformly C^∞ leaves that subfoliates both $\mathcal{T}$ and $\mathcal{G}_1$. Restricting our attention to the leaves of $\mathcal{G}_1$, because $\mathcal{F}$ is a C^∞ subfoliation of $\mathcal{G}_1$, we obtain that the $\mathcal{F}$-holonomy maps between $\mathcal{T}_1$ transversals are uniformly C^∞. Similarly, intersecting the leaves of $\mathcal{T}$ with the leaves of $\mathcal{G}_2$, we obtain foliation $\mathcal{T}_2$ with uniformly C^∞ leaves that subfoliates both $\mathcal{T}$ and $\mathcal{G}_2$; the $\mathcal{F}$-holonomy maps between $\mathcal{T}_2$ transversals are uniformly C^∞.

The foliations $\mathcal{T}_1$ and $\mathcal{T}_2$ transversely subfoliate the leaves of $\mathcal{T}$ and have uniformly C^∞ leaves. For a fixed $q \in M$, we have just shown that the holonomy map $\psi_{p,q}$ defined previously is uniformly C^∞ along $\mathcal{T}_1$-leaves and uniformly C^∞ along $\mathcal{T}_2$-leaves. Now Theorem 3.20 implies that $\psi_{p,q}$ is C^∞, uniformly in q, completing the proof of Proposition 3.22. □

3.9. *Bunching and smoothness of stable and unstable holonomies.* Our final set of preliminaries concerns the regularity of stable and unstable holonomy maps and the related spectral property of r-bunching. Let f be a partially hyperbolic diffeomorphism. For $r > 0$, we say that f is *r-bunched* if there exists an integer $k \geq 1$ such that for every $p \in M$:

$$\|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{E^c})^{-1}\|^r < 1, \quad \|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{E^c}\|^r < 1,$$

$$\|D_p f^k|_{E^s}\| \cdot \|(D_p f^k|_{E^c})^{-1}\| \cdot \|D_p f^k|_{E^c}\|^r < 1$$

and

$$\|(D_p f^k|_{E^u})^{-1}\| \cdot \|D_p f^k|_{E^c}\| \cdot \|(D_p f^k|_{E^c})^{-1}\|^r < 1.$$

Note that every partially hyperbolic diffeomorphism is r-bunched, for some $r > 0$. The condition of 0-bunching is merely a restatement of partial hyperbolicity, and 1-bunching is center bunching. The first pair of inequalities in this definition are r-normal hyperbolicity conditions; when f is C^r and dynamically coherent, these inequalities ensure that the leaves of $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$, and $\mathcal{W}^c$ are C^r. Combined with the first group of inequalities, the second group of inequalities imply that E^u and E^s are 'C^r in the direction of E^c'. More precisely, in the case that f is C^{r+1} and dynamically coherent, the r-bunching inequalities imply that the restriction of E^u to $\mathcal{W}^{cu}$ leaves is a C^r bundle, and the restriction of E^s to $\mathcal{W}^{cs}$ leaves is a C^r bundle. Hence, if such a system is r-bunched, then the local stable and unstable holonomies $h^*_{x,y}$ are C^r local diffeomorphisms. See Pugh *et al* [**28**, **37**].

LEMMA 3.23. *Suppose $f \in \mathcal{P}(M)$ is such that $Df|_{E^c}$ is an isometry for some choice of Riemannian metric.*

Then the leaves of $\mathcal{W}^c$, $\mathcal{W}^{cs}$, and $\mathcal{W}^{cu}$ are uniformly C^∞ and the stable and unstable holonomy maps between $\mathcal{W}^c$-leaves are C^∞.

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Proof. The assumption implies that f is r-bunched, for any $r \geq 1$. Now, as discussed previously, r-bunching contains r-normal hyperbolicity, which implies that the leaves of $\mathcal{W}^c$, $\mathcal{W}^{cu}$, and $\mathcal{W}^{cs}$ are C^r. See [19]. Moreover, r-bunching implies that $\mathcal{W}^s$ C^r-subfoliates $\mathcal{W}^{cs}$ and $\mathcal{W}^u$ C^r-subfoliates $\mathcal{W}^{cu}$. See [28]. This gives the lemma. □

3.10. *Lyapunov exponents and an invariance principle.* In this subsection, we describe the main results we use concerning Lyapunov exponents and invariant measures of diffeomorphism cocycles.

Let $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ be a continuous diffeomorphism cocycle over f, in the sense of [4, 5]. This means that $\pi : \mathcal{E} \to M$ is a continuous fiber bundle with fibers modeled on some Riemannian manifold and $\mathfrak{F}$ is a continuous fiber bundle morphism over a Borel measurable map $f : M \to M$ acting on the fibers by diffeomorphisms with uniformly bounded derivative. Let $\hat{\mu}$ be an $\mathfrak{F}$-invariant probability measure on $\mathcal{E}$ that projects to an f-invariant measure μ. We denote by $\mathcal{E}_x$ the fiber $\pi^{-1}(x)$ and by $\mathfrak{F}_x : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ the induced diffeomorphism on fibers.

We say that a real number χ is a *fiberwise exponent of* $\mathfrak{F}$ *at* $\xi \in \mathcal{E}$ if there exists a non-zero vector $v \in T_\xi \mathcal{E}_{\pi(\xi)}$ in the tangent space to the fiber at ξ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_\xi \mathfrak{F}^n(v)\| = \chi.$$

By Oseledec's theorem, this limit $\chi(\xi, v)$ exists for $\hat{\mu}$-almost every $\xi \in \mathcal{E}$ and every non-zero $v \in T_\xi \mathcal{E}_{\pi(\xi)}$, and it takes finitely many values at each such ξ. Let

$$\bar{\chi}(\xi) = \sup_{\|v\|=1} \chi(\xi, v) \quad \text{and} \quad \underline{\chi}(\xi) = \inf_{\|v\|=1} \chi(\xi, v).$$

The following result follows almost immediately from Theorem II in [34] and uses no assumptions on the base dynamics $f : M \to M$ other than invertibility. The hypothesis on the fibers can be weakened, but the following statement is sufficient for our purposes.

THEOREM 3.24. [34] *Let $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ be a diffeomorphism cocycle over f. Assume that the fibers of $\mathcal{E}$ are compact. Assume that $\mathfrak{F}$ preserves an ergodic probability measure $\hat{\mu}$ that projects to a (f-invariant, ergodic) probability μ on M and that f is invertible on a full μ-measure set in M. Let $\mathcal{X}_-$ be the set of $\xi \in \mathcal{E}$ such that $\bar{\chi}(\xi) < 0$ and $\mathcal{X}_+$ be the set of $\xi \in \mathcal{E}$ such that $\underline{\chi}(\xi) > 0$.*

Then both $\mathcal{X}_-$ and $\mathcal{X}_+$ coincide up to zero $\hat{\mu}$-measure subsets with measurable sets that intersect each fiber of $\mathcal{E}$ in finitely many points.

The next result, from [4, 5], treats the possibility that all fiberwise exponents vanish. It admits more general formulations, but we state it in the context in which we will use it, namely, when f is a partially hyperbolic diffeomorphism.

We say that $\mathfrak{F}$ *admits a $*$-holonomy* for $* \in \{s, u\}$ if, for every pair of points x, y lying in the same $\mathcal{W}^*$-leaf, there exists a Hölder continuous homeomorphism $H^*_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ with uniform Hölder exponent, satisfying:

(i) $H^*_{x,x} = \mathrm{id}$;

(ii) $H^*_{x,z} = H^*_{y,z} \circ H^*_{x,y}$;

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(iii) $\mathfrak{F}_y \circ H^*_{x,y} = H^*_{f(x),f(y)} \circ \mathfrak{F}_x$; and
(iv) $(x, y) \mapsto H^*_{x,y}(\xi)$ is continuous on the space of pairs of points (x, y) in the same local $\mathcal{W}^*$-leaf, uniformly on ξ.

The existence of a $*$-holonomy is equivalent to the existence of an $\mathfrak{F}$-invariant foliation (with potentially non-smooth leaves) of $\mathcal{E}$ whose leaves project homeomorphically (in the intrinsic leaf topology) to $\mathcal{W}^*$-leaves in M.

A disintegration $\{\hat{\mu}_x : x \in M\}$ is $*$-*invariant* over a set $X \subset M$, $* \in \{s, u\}$ if the homeomorphism $H^*_{x,y}$ pushes $\hat{\mu}_x$ forward to $\hat{\mu}_y$ for every $x, y \in X$ with $y \in \mathcal{W}^*_x$. We call a set $X \subset M$ $*$-*saturated*, $* \in \{s, cs, c, cu, u\}$ if it consists of entire leaves of $\mathcal{W}^*$. Observe that f is accessible if and only if the only non-empty set in M that is both s-saturated and u-saturated is M itself.

THEOREM 3.25. [4, Theorem C] *Let $\mathfrak{F}$ be a diffeomorphism cocycle on $\pi : \mathcal{E} \to M$ over the C^2, volume-preserving, center-bunched, partially hyperbolic diffeomorphism $f : M \to M$. Assume that f is accessible and that $\mathfrak{F}$ preserves a probability measure $\hat{m}$ that projects to the volume m. Suppose that $\bar{\chi}(\xi) = \underline{\chi}(\xi) = 0$ for $\hat{m}$-almost every $\xi \in \mathcal{E}$.*

Then there exists a continuous disintegration $\{\hat{m}^{su}_x : x \in M\}$ of $\hat{m}$ that is invariant under both s-holonomy and u-holonomy.

A slight modification of the proof in [4] gives us the following result.

THEOREM 3.26. *Let $\mathfrak{F}$ be a diffeomorphism cocycle on $\pi : \mathcal{E} \to M$ over the C^2, volume-preserving, center-bunched, partially hyperbolic diffeomorphism $f : M \to M$. Assume that f has an open accessibility class $U \neq \emptyset$, and let $\mu = m(\cdot : U)$ be the conditional volume on U:*

$$\mu(A) := m(A : U) = \frac{m(A \cap U)}{m(U)}.$$

Suppose that $\mathfrak{F}$ preserves a probability measure $\hat{\mu}$ on $\mathcal{E}$ that projects to μ and that $\bar{\chi}(\xi) = \underline{\chi}(\xi) = 0$ for $\hat{\mu}$-almost every $\xi \in \mathcal{E}$.

Then there exists a continuous disintegration $\{\hat{\mu}^{su}_x : x \in U\}$ of $\hat{\mu}$ that is invariant under both s-holonomy and u-holonomy.

Proof. One observes that the proof of part (a) of [4, Theorem D], which is stated for μ in the same measure class as volume, extends to μ absolutely continuous with respect to volume, provided that supp (μ) is bisaturated. This is the case here, because $\mu := m(\cdot : U)$ is supported on the closure of the accessibility class U, which is bisaturated. The conclusion of (b) of [4, Theorem D] then holds if f is accessible on the support of μ.

To see this, the main thing to note is that [4, Theorem 6.1] makes no assumption on whether f preserves volume. In the application of [4, Theorem 6.1] to prove [4, Theorem D], the function Ψ is defined by $\Psi(x) = m_x$, where m_x is the disintegration of $\hat{m}$ along the fibers of $\mathcal{E}$. In the case where μ is supported on an open accessibility class U, we fix a disintegration of $\hat{\mu}$ along the fibers of $\mathcal{E}$, and set

$$\Psi(x) = \begin{cases} \hat{\mu}_x & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

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Similarly, [4, Theorem 4.1] makes no assumptions on volume-preservation of f. Thus, Theorem 3.26 can be deduced from Theorems D and 4.1 in [4] in the same way that [4, Theorem C] is deduced from Theorems D and 4.1 in [4], replacing the function Ψ there with Ψ defined by (2). □

4. *A generalized invariance principle*

In this section, we prove an abstract criterion for holonomy invariance of probability measures preserved by diffeomorphism cocycles with vanishing Lyapunov exponents. A main novelty with respect to previous related results by Avila *et al* [4, 5] is that we also deal with invariance under *center holonomy*, not only stable and unstable holonomies. Implications of this refined theory will be exploited in the forthcoming sections.

4.1. *c-holonomies.* Let $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ be a continuous diffeomorphism cocycle over f. Recall from §3.10 that $\mathfrak{F}$ admits $*$ holonomy, for $* \in \{s, u\}$ if the foliation $\mathcal{W}^*$ in M lifts to an $\mathfrak{F}$-invariant foliation $\widehat{\mathcal{W}}^*$ of $\mathcal{E}$ whose leaves are homeomorphic to the leaves of $\mathcal{W}^*$. If $\mathfrak{F}$ admits a $*$-holonomy, then for any two points x, y in the same $\mathcal{W}^*$ leaf, there is a well-defined holonomy map $H^*_{x,y}$ between the fibers $\mathcal{E}^x$ and $\mathcal{E}^y$ satisfying the conditions (i)–(iv) described in §3.10, which gives an equivalent definition.

There is an analogous way to define c-holonomy, but a little more care must be taken because the leaves of $\mathcal{W}^c$, unlike those of $\mathcal{W}^s$ and $\mathcal{W}^u$, are not necessarily simply connected. The notion of c-holonomy will be used to formulate a new version of Theorem 3.25 for cocycles admitting s, u and c holonomies.

We say that $\mathfrak{F}$ admits a *c-holonomy* if, for every path $\gamma : [0, 1] \to \mathcal{W}^c(\gamma(0))$ lying in a $\mathcal{W}^c$ leaf, there exists a Hölder continuous homeomorphism $H^c_\gamma : \mathcal{E}_{\gamma(0)} \to \mathcal{E}_{\gamma(1)}$ with uniform Hölder exponent, satisfying:

(i) $H^c_\epsilon = \mathrm{id}$, where ϵ is any constant path;

(ii) $H^c_{\gamma_1 \cdot \gamma_2} = H^c_{\gamma_2} \circ H^c_{\gamma_1}$, where $\gamma_1 \cdot \gamma_2$ denotes the concatenated path;

(iii) $H^c_{\gamma_1} = H^c_{\gamma_2}$ whenever γ_1 and γ_2 are homotopic via an endpoint-fixing homotopy in $\mathcal{W}^c(\gamma_1(0)) \quad (= \mathcal{W}^c(\gamma_2(0)))$,

(iv) $\mathfrak{F}_{\gamma(1)} \circ H^c_\gamma = H^c_{f\circ\gamma} \circ \mathfrak{F}_{\gamma(0)}$; and

(v) $\gamma \mapsto H^c_\gamma(\xi)$ is continuous on the space of paths γ whose image lies in a fixed local $\mathcal{W}^c$-leaf, uniformly on ξ.

We say that the c-holonomy is *product type* if H^*_γ depends only on the endpoints of γ; when this is the case, we denote H^*_γ by $H^*_{\gamma(0),\gamma(1)}$. In particular, if the leaves of $\mathcal{W}^c$ are simply connected, then any c-holonomy is product type. Note that H^c holonomy is always product type when restricted to paths in the local $\mathcal{W}^c$-foliation of any $\mathcal{W}^c$-foliation box $\mathcal{B}$. We denote by $H^{c,\mathcal{B}}_{x,y}$ the c-holonomy in $\mathcal{B}$ determined by a path from x to y lying in the local leaf of $\mathcal{W}^c_x$ in $\mathcal{B}$. For short, we refer to 'local c-holonomy' and use the notation $H^c_{x,y}$, when x and y lie in the same local $\mathcal{W}^c$-leaf. Properties (i)–(iii) of c-holonomy imply that c-holonomy is determined by local c-holonomy. The existence of c-holonomy is equivalent to the existence of an $\mathfrak{F}$-invariant foliation (with potentially non-smooth leaves) of $\mathcal{E}$ whose leaves project to $\mathcal{W}^c$ leaves in M; if the holonomy is product type, the c-leaves for $\mathfrak{F}$ project homeomorphically to c-leaves for f; more generally, the projection is a covering map.

We now state our general invariance criterion. Let $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ be a continuous diffeomorphism cocycle on a fiber bundle $\mathcal{E} \to M$. For $* \in \{s, u, c\}$, we say that $\mathfrak{F}$ *admits* $*$ *holonomies over* $X \subset M$ if it admits local $*$-holonomies $H^*_{x,y}$ for every pair $x, y \in X$. Recall that a set $X \subset M$ is $*$-*saturated*, $* \in \{s, cs, c, cu, u\}$ if it consists of entire leaves of $\mathcal{W}^*$ and *essentially* $*$-*saturated* if X coincides with some $*$-saturated up to zero volume sets. Fix $* \in \{s, u\}$ and suppose X^{c*} is a $c*$-saturated set over which $\mathfrak{F}$ admits both $*$ and c holonomies. We say that *c-holonomy commutes with* $*$-*holonomy over* X^{c*} if for any $\mathcal{W}^{c*}$ foliation box $\mathcal{B}$, and two points $x, x' \in \mathcal{B} \cap X^{c*}$ lying in the same local $\mathcal{W}^{c*}$-leaf, we have

$$H^*_{y,x'} \circ H^{c,\mathcal{B}}_{x,y} = H^{c,\mathcal{B}}_{y',x'} \circ H^*_{x,y'} \tag{3}$$

where y is the point in $\mathcal{B} \cap X^{c*}$ where the local $\mathcal{W}^c$-leaf of x intersects the local $\mathcal{W}^*$-leaf of x', and y' is the point where the local $\mathcal{W}^*$-leaf of x intersects the local $\mathcal{W}^c$-leaf of x'.

Let m denote the normalized volume measure on M, and let $\hat{m}$ be any probability measure on $\mathcal{E}$ that projects down to m. A disintegration $\{\hat{m}_x : x \in M\}$ of m along $\mathcal{E}$ fibers is *c-invariant* over a c-saturated subset $X \subset M$ if the homeomorphism H^c_γ pushes $m_{\gamma(0)}$ forward to $m_{\gamma(1)}$ for every path $\gamma : [0, 1] \to \mathcal{W}^c(\gamma(0))$. When X has full m-measure we call the disintegration *essentially c-invariant*. Properties (i)–(iii) imply that c-invariance is equivalent to invariance under local c-holonomy.

THEOREM 4.1. *Fix a diffeomorphism $f \in \mathcal{P}(M)$. Let $\mathcal{E}$ be a fiber bundle defined over a full measure, c-saturated subset $O \subset M$, and let $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ be a continuous diffeomorphism cocycle over $f|_O$. Assume that there exist $c*$-saturated, full measure subsets $O^{c*} \subset O$, for $* \in \{s, u\}$ such that $\mathfrak{F}$ admits commuting c and $*$ holonomies in O^{c*}.*

Let $\hat{m}$ be an $\mathfrak{F}$-invariant measure projecting down to normalized Lebesgue measure. Assume that the center foliation of f is leafwise absolutely continuous and that the fiberwise Lyapunov exponents of $\mathfrak{F}$ vanish $\hat{m}$-almost everywhere. Suppose that $\hat{m}$ admits a disintegration that is c-invariant over $O^c = O^{cs} \cap O^{cu}$.

Then $\hat{m}$ admits a disintegration that is continuous and $*$-invariant over O^c for all $* \in \{s, c, u\}$.

The conclusion means that $(H^*_{x,x'})_*\hat{m}_x = \hat{m}_{x'}$ for every $x \in O^c$ and $x' \in \mathcal{W}^*(x) \cap O^c$.

4.2. *Proof of the invariance theorem.* Let us prove Theorem 4.1.

Proof. Let $\{\hat{m}^c_x : x \in O^c\}$ be a c-invariant disintegration of $\hat{m}$ over the c-saturated set O^c. Consider any $* \in \{s, u\}$. Clearly, $\hat{m}$ may be viewed as an $\mathfrak{F}^{c*}$-invariant probability measure on $\mathcal{E}^{c*}$, with $\mathcal{E}^c$ as a full measure subset. The hypothesis implies that the Lyapunov exponents of $\mathfrak{F}^{c*}$ vanish $\hat{m}$-almost everywhere. Theorem 3.25 implies that $\hat{m}$ admits a disintegration $\{\hat{m}^*_x : x \in O^{c*}\}$ that is $*$-invariant over a full m-measure subset $O^* \subset O^{c*}$. As disintegrations are essentially unique, the set

$$Z = \{x \in O^c : \hat{m}^c_x = \hat{m}^s_x = \hat{m}^u_x\}$$

has full m-measure. We combine this fact with the leafwise absolute continuity assumption, to obtain the conclusion of the theorem.

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Let λ_z^c, λ_z^{cs}, and λ_z^{cu} denote the Riemannian measures on the leaves of $\mathcal{W}^c$, $\mathcal{W}^{cs}$, and $\mathcal{W}^{cu}$ through any point $z \in M$, respectively. All three foliations are leafwise absolutely continuous, by our assumption and Lemma 3.16. Leafwise absolute continuity of $\mathcal{W}^c$ and $\mathcal{W}^{cs}$ implies that Z meets $\mathcal{W}_p^c$ in a set of full λ_p^c-measure and meets $\mathcal{W}_p^{cs}$ in a set of full λ_p^{cs}-measure, for almost every $p \in O^c$. Starting with the c-invariant family of measures $\hat{m}_x^c$ on $\mathcal{W}_p^{c,\mathrm{loc}}$, we define a family of measures ν_x^u on $\mathcal{W}_p^{cs,\mathrm{loc}}$ by pushing $\hat{m}_x^c$ around by (local) s-holonomy. This family is s-invariant, of course, and the assumption that the H^c commutes with H^s ensures that it is also c-invariant. As $\hat{m}_x^c = \hat{m}_x^s$ for λ_p^c-almost every $x \in \mathcal{W}_p^{c,\mathrm{loc}}$ and $\hat{m}_x^s$ is s-invariant and the restriction of $\mathcal{W}^s$ to $\mathcal{W}_p^{cs}$ is absolutely continuous, we also have $\nu_x^u = \hat{m}_x^s$ for λ_p^{cs}-almost everywhere $x \in \mathcal{W}_p^{cs,\mathrm{loc}}$. Then $\nu_x^u = \hat{m}_x^u$ for λ_p^{cs}-almost every $x \in \mathcal{W}_p^{cs,\mathrm{loc}}$ because Z intersects the center-stable leaf on a full measure subset.

The intersection of O^{cu} with the center-stable leaf also has full λ_p^{cs}-measure, because O^{cu} is a u-saturated full m-measure subset of M and $\mathcal{W}^u$ is absolutely continuous. Restricting ν_x^u to this intersection and then pushing it around by u-holonomy we extend ν_x^u to a u-invariant family on a whole neighborhood V_p^u of the point p inside O^{cu}. The fact that H^c commutes with H^u ensures that this extension remains c-invariant. Moreover, ν_x^u is continuous, because of the continuity property (v) in the definition of holonomies. Finally, because $\nu_x^u = \hat{m}_x^u$ for λ_p^{cs}-almost every $x \in \mathcal{W}_p^{cs}$ and $\hat{m}^u$ is u-invariant and $\mathcal{W}^u$ is absolutely continuous, we have $\nu_x^u = \hat{m}_x^u$ for m-almost every $x \in V^u$. This also shows that ν_x^u defines a disintegration of $\hat{m}$ restricted to V_p^u.

In just the same way, we construct a continuous, c-invariant, and s-invariant disintegration ν_x^s of the measure $\hat{m}$ restricted to a neighborhood V_p^s of p inside O^{cs}. As disintegrations are essentially unique, these two continuous disintegrations ν_x^u and ν_x^s must coincide at *every* point in the intersection V_p of the domains. Thus,

$$\hat{m}_x = \nu_x^u = \nu_x^s$$

defines a disintegration of $\hat{m}$ as in the conclusion of Theorem 4.1 locally, on a neighborhood V_p of p inside O^c. The global definition is obtained by covering O^c with such neighborhoods. Continuity ensures that local definitions agree on the intersection of their domains. The proof of the theorem is complete. □

4.3. *An invariance theorem on open accessibility classes.* There is an analog of Theorem 4.1 for su-saturated sets, that is, accessibility classes, in place of c-saturated sets.

THEOREM 4.2. *Fix a diffeomorphism $f \in \mathcal{P}(M)$, and suppose that f has an open accessibility class $U \neq \emptyset$. Let $\mu = m(\cdot : U)$, and fix $\ell \geq 1$ such that $f^\ell(U) = U$.*

Let $\pi : \mathcal{E} \to M$ be a fiber bundle and let $\mathfrak{F} : \mathcal{E} \to \mathcal{E}$ be a continuous diffeomorphism cocycle over f admitting commuting c and $*$ holonomies.

Let $\hat{\mu}$ be an $\mathfrak{F}^\ell$-invariant measure projecting down to μ. Assume that:

(1) *μ has Lebesgue disintegration with respect to the partition*

$$\mathcal{W}^c \cap U := \{\mathcal{W}_x^c \cap U : x \in U\};$$

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(2) *the fiberwise Lyapunov exponents of $\mathfrak{F}$ vanish $\hat{\mu}$-almost everywhere; and*
(3) *$\hat{\mu}$ admits a disintegration $\{\hat{\mu}_x : x \in U\}$ that is c-invariant over U, meaning that for all $x \in U$ and $x' \in \mathcal{W}^c_{x,\mathrm{loc}}$,*

$$(H^c_{x,x'})_*\hat{\mu}_x = \hat{\mu}_{x'}.$$

Then $\hat{\mu}$ admits a disintegration that is continuous and $*$-invariant over U for all $* \in \{s, c, u\}$.

Proof. The proof is the same, except we are in the simplified situation where $O = M$, and we use Theorem 3.26 in place of Theorem 3.25. □

4.4. *Center leaf fiber bundles.* We describe a construction that will be used at some key places in this paper. Let B be a topological space and let N be a manifold. A *continuous fiber bundle with fiber N and base B* is a continuous surjective map $\pi : E \to B$ together with a family of homeomorphisms $g_\alpha : U_\alpha \times N \to \pi^{-1}(U_\alpha)$ (called a π-adapted atlas), where $\{U_\alpha\}$ is some open cover of B and every $\pi \circ g_\alpha$ coincides with the canonical projection to the first coordinate.

PROPOSITION 4.3. *Suppose that $f \in \mathcal{P}(M)$ admits global su-holonomy. Then there exists a continuous fiber bundle $\pi : \mathcal{E}^c \to M$ and a second projection map $p : \mathcal{E}^c \to M$ with the following properties:*
(1) *p sends each $\mathcal{E}^c_x = \pi^{-1}(x)$, $x \in M$ homeomorphically onto $\mathcal{W}^c_x$;*
(2) *the fiber bundle $\mathcal{E}^c$ admits a canonical continuous section sending each x to $p^{-1}(x) \cap \mathcal{E}^c_x$;*
(3) *there is a canonical continuous map $\mathfrak{F} : \mathcal{E}^c \to \mathcal{E}^c$ satisfying $\pi \circ \mathfrak{F} = f \circ \pi$ and $p \circ \mathfrak{F} = f \circ p$;*
(4) *the fiber bundle admits $\mathfrak{F}$-invariant stable, unstable, and center foliations $\mathcal{F}^*$, $* \in \{s, u, c\}$ projecting under π to the corresponding foliations $\mathcal{W}^*$ in M; the u and s holonomies are C^1 and commute with c holonomy.*

Proof. Let $\mathcal{E}^c = \{(x, y) \subset M \times M : y \in \mathcal{W}^c_x\}$ and take π and p to be the first and second coordinate projections. The topology on $\mathcal{E}^c$ is induced by the π-adapted atlas defined as follows. Given any $x \in M$ and w in a small neighborhood U of x in M, define y to be the point in $\mathcal{W}^{s,\mathrm{loc}}_x \cap \mathcal{W}^{cu,\mathrm{loc}}_w$ and z to be the point in $\mathcal{W}^{u,\mathrm{loc}}_y \cap \mathcal{W}^{c,\mathrm{loc}}_w$. Note that y and z depend continuously on w. Then $h_{x,w} = h^u_{y,z} \circ h^s_{x,y}$ is a homeomorphism from $\mathcal{W}^c_x$ to $\mathcal{W}^c_w$ that depends continuously on w. It follows that

$$g_{x,U} : U \times \mathcal{W}^c_x \to \pi^{-1}(U), \quad (w, w') \mapsto (w, h_{x,w}(w'))$$

is a homeomorphism mapping each vertical $\{w\} \times \mathcal{W}^c_x$ to $\pi^{-1}(w)$. This proves that $\mathcal{E}^c$ is a continuous fiber bundle. It is clear that every fiber $\pi^{-1}(x) = \{x\} \times \mathcal{W}^c_x$ is mapped homeomorphically to $\mathcal{W}^c_x$ by the second projection p, as claimed in (1). The diagonal embedding $M \to \mathcal{E}$ defines a section as in (2), and the map $\mathfrak{F} := (f, f) : \mathcal{E}^c \to \mathcal{E}^c$ is a lift of f as in (3). For each fixed $x \in M$ and $y \in \mathcal{W}^c_x$, the set

$$\mathcal{F}^s_{(x,y)} = \{(x', y') \mid x' \in \mathcal{W}^s_x,\ y' \in \mathcal{W}^s_y \cap \mathcal{W}^c_{x'}\},$$

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is a continuous submanifold of $\mathcal{E}^c$, and these submanifolds form an $\mathfrak{F}$-invariant stable foliation that projects down to the stable foliation of f. Analogously, one obtains an $\mathfrak{F}$-invariant unstable foliation $\mathcal{F}^u$.

To obtain a center foliation we set, for $(x, y) \in \mathcal{E}^c$:

$$\mathcal{F}^c_{(x,y)} = \{(x', y) \mid x' \in \mathcal{W}^c_x\}.$$

Clearly the foliation $\mathcal{F}^c$ is $\mathfrak{F}$-invariant and the leaves of $\mathcal{F}^c$ project to the leaves of $\mathcal{W}^c$.

The stable and unstable foliations of $\mathfrak{F}$ define $*$-holonomy, of product type, for the diffeomorphism cocycle:

$$H^*_{x,y} : \mathcal{E}^c_x \to \mathcal{E}^c_y, \quad x \text{ and } y \text{ in the same leaf of } \mathcal{W}^*$$

for either $* \in \{s, u\}$. Furthermore, for every x and y in the same local center leaf, let $H^c_{x,y} : \mathcal{E}^c_x \to \mathcal{E}^c_y$ be the map defined by $p \circ H^c_{x,y} = p$, where p is the second projection associated with $\mathfrak{F}$. It is clear that this c-holonomy is $\mathfrak{F}$-invariant and commutes with both s-holonomy and u-holonomy. □

Lemma 3.5 implies that any $f \in \mathcal{P}_{\text{fib}}(M)$ admits global su-holonomy. In this context, we obtain the following.

THEOREM 4.4. *Let M be a closed Riemannian manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}_{\text{fib}}(M)$. Let $\pi : \mathcal{E}^c \to M$ and projection $p : \mathcal{E}^c \to M$ be given by Theorem 4.3.*

Then for every subset $U \subseteq M$ of positive measure, there exists a probability measure $\hat{m}_U$ on $\mathcal{E}^c$ with the property that for every measurable $A \subset M$:

$$\pi_* \hat{m}_U(A) = m(A : U) = \frac{m(U \cap A)}{m(U)},$$

and for m-almost every $x \in U$,

$$p_* \hat{m}_x = m^c_x(\cdot : U),$$

where $\{m^c_x : x \in M\}$ is any disintegration of m along $\mathcal{W}^c$ leaves, and $\{(\hat{m}^c_U)_x : x \in M\}$ is any disintegration of $\hat{m}_U$ along $\mathcal{E}^c$ fibers.

If U is f-invariant, then $\hat{m}_U$ is $\mathfrak{F}$-invariant, and the Lyapunov exponents of the diffeomorphism cocycle $\mathfrak{F}$ with respect to $\hat{m}_U$ coincide almost everywhere with the center Lyapunov exponents of $f|U$ with respect to m.

Proof. Let $\{m^c_x : x \in M\}$ be a disintegration of m along center leaves, and let $\hat{m}$ be the measure defined on $\mathcal{E}^c$ by re-integration (recall $p(\mathcal{E}^c_x) = \mathcal{W}^c_x$):

$$\hat{m}_U(E) = \int_X m^c_x(p(E) : U)\, dm(x : U) \quad \text{for every measurable set } E \subset \mathcal{E}^c. \tag{4}$$

In other words, $\hat{m}$ projects down to $m(\cdot : U)$ under π and admits $\{m^c_x(\cdot : U) : x \in M\}$ as a disintegration along the fibers of $\mathcal{E}^c$.

It is also clear that $\hat{m}_U$ is $\mathfrak{F}$-invariant if U is f-invariant and that the Lyapunov exponents of the diffeomorphism cocycle $\mathfrak{F}$ with respect to $\hat{m}_U$ then coincide with the center Lyapunov exponents of f. □

THEOREM 4.5. *Let M be a closed manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}_{\text{fib}}(M)$. Suppose that $\mathcal{W}^c$ is leafwise absolutely continuous and the center Lyapunov exponents of f vanish m-almost everywhere. Then m admits some disintegration along center leaves that is continuous and invariant under the holonomy maps of both the stable foliation and the unstable foliation of f.*

Proof. Let $\{m_x^c : x \in M\}$ be a disintegration of m along center leaves, and let $\hat{m}$ be the measure given by Theorem 4.4 with $U = M$. The Lyapunov exponents of the diffeomorphism cocycle $\mathfrak{F}$ coincide with the center Lyapunov exponents of f and so, by assumption, they vanish almost $\hat{m}$-everywhere. Hence, we may use Theorem 4.1 (with $O = M$) to conclude that $\hat{m}$ admits some disintegration $\{\hat{m}_x : x \in M\}$ along the fibers that is continuous and invariant under all three holonomies. By essential uniqueness, $p_*\hat{m}_x = m_x^c$ at m-almost every point. Each $\hat{m}_x$ is a probability on $\mathcal{E}_x^c$ and the property of c-invariance just means that $x \mapsto \hat{m}_x$ is constant on each center leaf. It follows that $\{m_x := p_*\hat{m}_x : x \in M\}$ defines a continuous disintegration of m along center leaves. Finally, s-invariance and u-invariance of $\{\hat{m}_x : x \in M\}$ translate to invariance of $\{m_x : x \in M\}$ under stable and unstable holonomy maps. The proof of the theorem is complete. □

Remark 4.6. The leafwise absolute continuity hypothesis is actually necessary in Theorem 4.5.

As for one-dimensional center, absolute continuity implies zero central exponents, the following statement is contained in Theorem 4.5.

COROLLARY 4.7. *Let $f : M \to M$ be any element of $\mathcal{P}^1_{\text{fib}}(M)$ whose center foliation is absolutely continuous. Then m admits a disintegration along center leaves that is continuous and invariant under the holonomy maps of both the stable foliation and the unstable foliation.*

The next result addresses the case where $f \in \mathcal{P}_{\text{fib}}(M)$ has a non-trivial open accessibility class, in particular when f is accessible.

THEOREM 4.8. *Let M be a closed Riemannian manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}_{\text{fib}}(M)$. Suppose that there exists an open accessibility class $U \neq \emptyset$ of f and that the center Lyapunov exponents of f on U vanish, m-almost everywhere (equivalently, on a positive measure subset of U). Fix $\ell \geq 1$ such that $f^\ell(U) = U$. Let $\pi, p \colon \mathcal{E}^c \to M$, and $\mu := \hat{m}_U$ be given by Theorem 4.4.*

Then $\hat{\mu}$ admits a $\mathfrak{F}^\ell$-invariant disintegration $\{\hat{\mu}_x^{su} : x \in U\}$ along the fibers of $\mathcal{E}^c$ that is invariant under s- and u-holonomies and continuous in $x \in U$.

Proof. Let $\mu = m(\cdot : U) = \pi_*\hat{\mu}$, and note that f^ℓ is ergodic with respect to μ. Let $\{\mu_x^c : x \in M\}$ be a disintegration of μ along center leaves, and note that the disintegration of $\hat{\mu}$ along $\mathcal{E}^c$-fibers satisfies $p_*\hat{\mu}_x = \mu_x^c$, for μ-almost every $x \in U$.

Note that the Lyapunov exponents of the diffeomorphism cocycle $\mathfrak{F}^\ell$ with respect to $\hat{\mu}$ coincide with the center Lyapunov exponents of $f^\ell|_U$, which by ergodicity are constant μ-almost everywhere.

If the central exponent of f is zero on U, then the exponents of $\mathfrak{F}^\ell$ vanish $\hat{\mu}$-almost everywhere. Theorem 3.26 then gives a continuous disintegration $\{\hat{\mu}^{su}_x : x \in U\}$ of $\hat{\mu}$ over U that is invariant under both s-holonomy and u-holonomy. □

We deduce a complete converse to Theorem 4.5, when the center leaves are compact and have dimension one (the statement does not extend to higher-dimensional center foliations).

COROLLARY 4.9. *Let M be a compact Riemannian manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}^1_{\text{fib}}(M)$. Then the following are equivalent:*

(1) *$\mathcal{W}^c$ is leafwise absolutely continuous and the center Lyapunov exponent vanishes m-almost everywhere;*
(2) *there exists a disintegration $\{m^c_x : x \in M\}$ along center leaves satisfying the conclusions of Theorem 4.5;*
(3) *for any disintegration $\{m_x : x \in M\}$ of m along center leaves, the measures m_x and λ^c_x are equivalent for m-almost every x.*

Proof. Theorem 4.5 states that (1) implies (2). Lemma 3.19 gives that (2) implies (3). To prove the remaining claim, suppose that (3) holds. Let $\{m^c_x : x \in M\}$ be a disintegration of m along center leaves. The hypothesis that λ^c_x is equivalent to m^c_x for μ-almost every x contains the conclusion that $\mathcal{W}^c$ is lower leafwise absolutely continuous. It also contains upper leafwise absolute continuity and, as observed in Remark 3.15, this implies the conclusion that the center Lyapunov exponent vanishes almost everywhere. □

COROLLARY 4.10. *Let M be a compact Riemannian manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}^1_{\text{fib}}(M)$. Then one of the following alternatives holds:*

(1) *$\mathcal{W}^c$ is upper leafwise absolutely continuous and the center Lyapunov exponent vanishes m-almost everywhere;*
(2) *the center Lyapunov exponent vanishes m-almost everywhere, and there exist $A, Z \subset M$ with $m(A) > 0$ and $m(Z) = 0$, such that, for every $x \in A$, the leaf $\mathcal{W}^c_x$ meets Z in a set of positive λ^c_x-measure;*
(3) *the center Lyapunov exponent does not vanish m-almost everywhere, and there is $B \subset M$ with $m(B) > 0$ that meets every leaf $\mathcal{W}^c_x$ in a set of λ^c_x-measure zero.*

When f is ergodic the sets A in (2) and B in (3) can be taken to have full measure.

Proof. The case when the center exponent vanishes almost everywhere and the center foliation is upper leafwise absolutely continuous is alternative (1), of course. Suppose the center exponent vanishes almost everywhere, but the center foliation is not upper leafwise absolutely continuous. By definition, the latter means that there exists a zero volume measure set Z that intersects $\mathcal{W}^c_x$ on a positive Lebesgue measure subset, for all x in some positive volume measure set A. This gives (2). Next, let B be the set of points where the center Lyapunov exponent is different from zero and suppose B has positive volume. As observed in Remark 3.15, B must intersect every center leaf on a zero Lebesgue measure subset. This gives alternative (3). Finally, up to replacing Z by the union of its iterates, we may assume right from the start that Z is invariant under f. Then the set A of points

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whose center leaves intersect Z on a positive Lebesgue measure subset is also invariant. It is clear from the definition that B is also invariant under f. This implies the statements for the ergodic case. The proof of the corollary is complete. □

5. *Homogeneity: a tool for establishing smoothness*

Let P be a manifold without boundary. We say that a subset $N \subset P$ is C^r *homogeneous in* P if for any two points $p, q \in N$, there is a C^r local diffeomorphism of P sending p to q and preserving N. The C^1-homogeneous subsets of a manifold have a remarkable property as follows.

THEOREM 5.1. ([31], see also [37]) *Any locally compact subset N of a C^1 manifold P that is C^1 homogeneous in P is a C^1 submanifold of P.*

For any integer $k \geq 2$, any C^k homogeneous, C^1 submanifold of a C^k manifold is a C^k submanifold.

The following proposition is an easy corollary of Theorem 5.1.

PROPOSITION 5.2. *Let P be a manifold without boundary, and let $\mathcal{F}$ be a foliation of P. Suppose that for some $k \geq 2$ and every $p, q \in P$ there exists a C^k diffeomorphism sending p to q and preserving the leaves of $\mathcal{F}$. Then $\mathcal{F}$ is a C^{k-1} foliation with uniformly C^k leaves.*

Proof. Suppose that the leaves of f are m-dimensional. The hypotheses imply that the tangent bundle $T\mathcal{F}$, viewed as a section of the Grassmann bundle of m-planes over P, is C^{k-1} homogeneous. Theorem 5.1 implies that $T\mathcal{F}$ is C^{k-1}, which gives the conclusion. □

We state and prove our first application of Theorem 5.1 to fibered systems.

PROPOSITION 5.3. *Let M be a closed Riemannian manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}_{\text{fib}}(M)$. Suppose that there exists an open accessibility class $U \neq \emptyset$ of f and that the center Lyapunov exponents of f on U vanish on a positive measure subset of U. Let $\mu = m(\cdot : U)$, and fix $\ell \geq 1$ such that $f^\ell(U) = U$.*

Let $\hat{\mu} := \hat{m}_U$ be given by Theorem 4.4, and let $\{\hat{\mu}^{su}_x : x \in U\}$ be the $\mathfrak{F}^\ell$-invariant, su-holonomy invariant, disintegration of $\hat{\mu}$ given by Theorem 4.8.

Then for any $x \in U$, the set $\operatorname{supp} \hat{\mu}^{su}_x \cap p^{-1}(U) \subset \mathcal{E}^c_x \cap p^{-1}(U)$ is C^1 homogeneous. In particular, for any $\xi, \xi' \in p^{-1}(U) \cap \mathcal{E}^c_x$, there is an orientation-preserving, C^1 diffeomorphism $H_{\xi,\xi'} : \mathcal{E}^c_x \to \mathcal{E}^c_x$ (a composition of s, u and c holonomies in $\mathcal{E}^c$) with the following properties:

(1) $H_{\xi,\xi'}(\xi) = \xi'$;
(2) $(H_{\xi,\xi'})_* \hat{\mu}^{su}_x = \hat{\mu}^{su}_x$;
(3) *if $\xi, \xi' \in \operatorname{supp} \hat{\mu}^{su}_x$, then $H_{\xi,\xi'}(\operatorname{supp} \hat{\mu}^{su}_x) = \operatorname{supp} \hat{\mu}^{su}_x$;*
(4) *if f is r-bunched, then $H_{\xi,\xi'}$ is a C^r diffeomorphism.*

Proof of Proposition 5.3. Note that $p_* \hat{\mu}^{su}_y = \mu^c_x$ for every $y \in \operatorname{supp} \mu^c_x$ and μ-almost every x, because $p_* \hat{\mu}^{su}_x = \mu^c_x$ almost everywhere, $\hat{\mu}^{su}_x$ is continuous in x, and $\mu^c_x = m^c_x(\cdot : U)$ is constant on every center leaf.

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Fix $x \in U$, and let $z, z' \in U$ be the p-projections of $\xi, \xi' \in p^{-1}(U)$. As U is an open accessibility class, there is an su-path γ in U connecting z to z'. As π maps leaves of $\mathcal{F}^*$ homeomorphically to leaves of $\mathcal{W}^*(f)$, for $* \in \{s, u\}$, we can lift γ to an su-path in $\mathcal{E}^c$ connecting $\eta = (z, z)$ to $\eta' = (z', z')$. Let $H : \mathcal{E}^c_x \to \mathcal{E}^c_{x'}$ be the su-holonomy map along this su-path. Then H sends η to η' and because the disintegration $\{\hat{\mu}^{su}_x : x \in M\}$ is invariant under su-holonomy, it maps $\hat{\mu}_x$ to $\hat{\mu}_{x'}$.

Suppose first that $x \in \operatorname{supp} \mu^c_x$ (this holds μ-almost everywhere). Then the condition $\xi \in \operatorname{supp} \hat{\mu}_x \cap p^{-1}(U)$ means that $z \in \operatorname{supp} \mu^c_x \cap U$, which implies $p_*\hat{\mu}_z = \mu^c_x = p_*\hat{\mu}_x$. Analogously, $z' \in \operatorname{supp} \mu^c_x \cap U$ and $p_*\hat{\mu}_{z'} = \mu^c_x = p_*\hat{\mu}_x$. Identifying the fibers $\mathcal{E}^c_z, \mathcal{E}^c_{z'}$ to $\mathcal{E}^c_x$ through c-holonomy in $\mathcal{E}^c$, we obtain a homeomorphism $H_{\xi,\xi'} : \mathcal{E}_x \to \mathcal{E}_x$ satisfying properties (1)–(3).

The assumption on x is readily removed, as follows. Given any $x \in U$ let x_0 be any point such that $x_0 \in \operatorname{supp} \mu^c_{x_0} \cap U$, and let γ be an su-path in U connecting x to x_0. The su-holonomy $H_0 : \mathcal{E}_x \to \mathcal{E}_{x_0}$ along the π-lift of γ maps $\operatorname{supp} \hat{\mu}_x$ to $\operatorname{supp} \hat{\mu}_{x_0}$. Let ξ_0, ξ'_0 be the images of ξ, ξ' under H_0. Conjugating H_{ξ_0,ξ'_0} by H_0 we obtain a homeomorphism $H_{\xi,\xi'}$ satisfying conclusions (1)–(3).

As f is partially hyperbolic with one-dimensional center it is center bunched, and so the (globally defined) su-holonomy maps between $\mathcal{W}^c(f)$ leaves are C^1. This implies that $H_{\xi,\xi'}$ is a C^1 diffeomorphism. Moreover, if f is r-bunched, then so is f, and the leaves of $\mathcal{W}^c(f)$ and all holonomies are C^r; in this case $H_{\xi,\xi'}$ is a C^r diffeomorphism, verifying property (4). □

COROLLARY 5.4. *For $f \in \mathcal{P}_{\text{fib}}(M)$, U, and $\{\hat{\mu}^{su}_x : x \in U\}$ as in Proposition 5.3, the set $X_x := \operatorname{supp} \hat{\mu}^{su}_x \cap p^{-1}(U)$ is a C^1 submanifold (possibly zero-dimensional) of $\mathcal{E}^c_x \cap p^{-1}(U)$. The connected components of X_x are diffeomorphic to each other, and for all $x, y \in U$, X_x is diffeomorphic to X_y.*

Proof. Proposition 5.3 shows that for $x \in U$, the support of $\hat{\mu}^{su}_x$ is C^1 homogeneous in U, and so Theorem 5.1 implies that it is a C^1 submanifold. As any two points in U are connected by an su-path, for $x, x' \in U$, the support of $\hat{\mu}^{su}_x$ is C^1 diffeomorphic to the support of $\hat{\mu}^{su}_{x'}$. □

We specialize to the one-dimensional fiber case.

THEOREM 5.5. *For $f \in \mathcal{P}^1_{\text{fib}}(M)$, U, and $\{\hat{\mu}^{su}_x : x \in U\}$ as in Proposition 5.3, either the disintegration of μ is atomic, or $\hat{\mu}^{su}_x$ projects to a measure on M with continuous density Δ on $\mathcal{W}^c \cap U$.*

Proof. (See [6, §7.1].) The support of $\hat{\mu}^{su}_x$ is either finite for all $x \in U$ or equal to $p^{-1}(\overline{U}) \cap \mathcal{E}^c_x$. Suppose that $\operatorname{supp}(\hat{\mu}^{su}_x) = p^{-1}(\overline{U}) \cap \mathcal{E}_x$, for all $x \in U$.

For $x \in M$, denote by λ_x the Riemannian measure on the fiber $\mathcal{E}_x$ and denote by $B(\xi, r)$ the ball in $\mathcal{E}^c_x$ centered at ξ of radius r, with respect to the p-pullback metric of the Riemann structure on $\mathcal{W}^c(f)_x$.

LEMMA 5.6. *For each* $x \in U$, *the measure* $\hat{\mu}_x^{su}$ *is equivalent to the restriction* $\lambda_x | p^{-1}(U) \cap \mathcal{W}_x^c$. *The limit*

$$\Delta_x(\xi) = \lim_{r \to 0} \frac{\hat{\mu}_x(B(\xi, r))}{\lambda_x(B(\xi, r))}$$

exists for every $x \in U$ *and* $\xi \in p^{-1}(U) \cap \mathcal{E}_x^c$, *is continuous in both* x *and* ξ, *and takes values in* $(0, \infty)$.

Proof. For $x \in U$ and $\xi \in \mathcal{E}_x^c \cap p^{-1}(U)$ let

$$\overline{\Delta}_x(\xi) = \limsup_{r \to 0} \frac{\hat{\mu}_x(B(\xi, r))}{\lambda_x(B(\xi, r))}, \quad \underline{\Delta}_x(\xi) = \liminf_{r \to 0} \frac{\hat{\mu}_x(B(\xi, r))}{\lambda_x(B(\xi, r))}.$$

For $\hat{\mu}_x$-almost every $\xi \in \mathcal{E}_x^c$, we have

$$\overline{\Delta}_x(\xi) = \underline{\Delta}_x(\xi) \in (0, \infty].$$

As supp $(\hat{\mu}_x^{su}) = \overline{p^{-1}(U)} \cap \mathcal{E}_x^c$, Proposition 5.3 implies that for any two points $\xi, \xi' \in U \cap \mathcal{E}_x^c$, there is a diffeomorphism $H_{\xi,\xi'} \colon \mathcal{E}_x^c \to \mathcal{E}_x^c$ preserving $\hat{\mu}_x^{su}$ and sending ξ to ξ'. As C^1 diffeomorphisms have continuous and positive Jacobians, it follows that for any $\xi, \xi' \in \overline{p^{-1}(U)} \cap \mathcal{E}_x^c$:

$$\underline{\Delta}_x(\xi) = \overline{\Delta}_x(\xi) \quad \Longleftrightarrow \quad \underline{\Delta}_x(\xi') = \overline{\Delta}_x(\xi').$$

Thus, $\underline{\Delta}_x = \overline{\Delta}_x$ everywhere on $\mathcal{E}_x^c \cap p^{-1}(U)$; denote this function by Δ_x.

Then $\hat{\mu}_x^{su}$ has a singular part with respect to λ_x if and only if there is a positive $\hat{\mu}_x^{su}$-measure set $X \subset p^{-1}(U) \cap \mathcal{E}_x^c$ such that, for $\xi \in X$, $\Delta_x(\xi) = \infty$. On the other hand, again using the diffeomorphisms $H_{\xi,\xi'}$ we see that for every $\xi, \xi' \in p^{-1}(U) \cap \mathcal{E}_x^c$:

$$\Delta_x(\xi) = \infty \quad \Longleftrightarrow \quad \Delta_x(\xi') = \infty.$$

Hence, if $\hat{\mu}_x^{su}$ had a singular part with respect to λ_x, this would imply that $\Delta_x \equiv \infty$ on $\mathcal{E}_x^c$, contradicting the local finiteness of $\hat{\mu}_x^{su}$. Therefore, $\hat{\mu}_x^{su}$ is absolutely continuous with respect to λ_x. Similarly, we see that λ_x is absolutely continuous with respect to $\hat{\mu}_x^{su}$, and so the two measures are equivalent.

For $x \in p^{-1}(U)$, the function $\Delta \colon \mathcal{E}_x^c \cap p^{-1}(U) \to (0, \infty)$ is a pointwise limit of the continuous functions

$$\xi \mapsto \frac{\hat{\mu}_x^{su}(B(\xi, r))}{\lambda_x(B(\xi, r))}$$

and, hence, is a Baire class 1 function; it follows that Δ has a point of continuity [26, Theorem 7.3]. Again using Proposition 5.3, we see that every point in $p^{-1}(U)$ is a point of continuity of Δ, and so Δ is continuous on U. □

Recall that for almost every $x \in M$, we have $p_* \hat{\mu}_x^{su} = \mu_x$, where μ_x is a representative of the disintegration of $\mu = m(\cdot : U)$ on $\mathcal{W}^c(f)_x$. The previous lemma thus implies that $p_* \hat{\mu}_x^{su}$ is equivalent to Lebesgue measure on $U \cap \mathcal{W}^c(f)_x$, for almost every x. □

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6. *Circle bundles: proofs of Theorems 1.1, 1.2, and 2.2*

6.1. *Proof of Theorem 2.2.* Let M be a closed manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}^1_{\text{fib}}(M)$. We first prove part (1), which has no accessibility assumptions.

Proof of part (1) of Theorem 2.2 (Compare with [6, §7.2].) As $\mathcal{W}^c$ is absolutely continuous and one dimensional, the center Lyapunov exponents for f vanish m-almost everywhere [34]. Theorem 4.5 then gives a continuous disintegration $\{m^c_x : x \in M\}$ that is invariant under $\mathcal{W}^s$ and $\mathcal{W}^u$ holonomy in M.

Let ψ_t be the continuous flow on M tangent to the leaves of $\mathcal{W}^c$ and uniquely defined by the condition

$$m^c_x([y, \psi_t(y))^c) = t \mod 1,$$

for all $x \in M$, $y \in \mathcal{W}^c_x$, and $t \in \mathbb{R}$, where $[p, q)^c$ denotes the oriented arc between p and q on $\mathcal{W}^c_p$. Note that $\psi_{t+1} = \psi_t$, so ψ in fact defines an action of the circle $\mathbb{R}/\mathbb{Z}$ on M.

The invariance properties of m^c_x translate into invariance properties of the flow:

- ψ_t commutes with f; and
- ψ_t commutes with u, s, and c holonomy.

LEMMA 6.1. *The flow ψ preserves the volume m.*

Proof. Fix $t \in \mathbb{R}$, and write $dm = dm^c_x\, d\bar{m}(x)$, where $\bar{m}$ is the projection of M to the leaf space $B = M/\mathcal{W}^c$ As ψ is tangent to the leaves of $\mathcal{W}^c$, we have that $(\psi_t)_*\bar{m} = \bar{m}$. For any $p, q \in \mathcal{W}^c_x$ sufficiently close, we have

$$m^c_x([p, \psi_t(p)]^c) + m^c_x([\psi_t(p), \psi_t(q)]^c) = m^c_x([p, q]^c) + m^c_x([q, \psi_t(q)]^c);$$

from the definition of ψ_t, it follows that

$$m^c_x([\psi_t(p), \psi_t(q)]^c) = m^c_x([p, q]^c),$$

so that $(\psi_t)_* m^c_x = m^c_x$. As $dm = dm^c_x\, d\bar{m}(x)$, we obtain that ψ_t preserves m.

Fix $t \in \mathbb{R}$. As $\mathcal{W}^c(f)$ is leafwise absolutely continuous, and ψ_t is C^1 along the leaves of $\mathcal{W}^c(f)$, the map ψ_t preserves the measure class of m. Hence, ψ_t has a Jacobian with respect to volume:

$$\text{Jac}(\psi_t) = \frac{d((\psi_t)^* m)}{dm}.$$

As $\psi_t \circ f = f \circ \psi_t$, it follows that $\text{Jac}\psi_t(f(t)) = \text{Jac}(\psi_t)$. This immediately implies that $(\psi_t)_* m = m$. □

Lemma 3.19 implies that the densities

$$\Delta(x) = dm^c_x / d\lambda|_{\mathcal{W}^c_x}$$

vary continuously in x. Thus, we have a continuous vector field X on M given by

$$X(x) = \frac{X_0(x)}{\Delta(x)},$$

where X_0 is the positively oriented unit speed vector field tangent to the $\mathcal{W}^c$-fibers of M. The vector field X generates the flow ψ_t, and so ψ_t is C^1 along the fibers of $\mathcal{W}^c$.

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The analogous properties holds for the vector field X, in particular:

- X is preserved by Df_*;
- X is preserved by the derivative of u, s holonomy.

To show C^∞ smoothness along the leaves of $\mathcal{W}^c$ one first must establish that the leaves of $\mathcal{W}^c$ are C^∞. *A priori*, these leaves have only finite smoothness determined by the C^1 distance from f to φ_1. However, in the case under consideration, in which volume has Lebesgue disintegration along $\mathcal{W}^c$ leaves, we have more information about the action of f on center leaves.

In particular, because Df preserves a non-vanishing vector field X, it also preserves a continuous Riemannian metric along the leaves of $\mathcal{W}^c$. Lemma 3.23 implies that the leaves of $\mathcal{W}^{cs}$, $\mathcal{W}^{cu}$, and $\mathcal{W}^c$ are C^∞, and the $\mathcal{W}^s$-holonomies and $\mathcal{W}^u$-holonomies between $\mathcal{W}^c$-leaves are also C^∞.

LEMMA 6.2. *Assume that f is accessible. Then the function Δ given by Lemma 5.6 is C^∞ along leaves of $\mathcal{W}^c$, with derivatives varying continuously from leaf to leaf. Consequently X is C^∞ along the leaves of $\mathcal{W}^c$, as is the flow ψ_t.*

Proof. Fix $x \in M$. For any $y \in \mathcal{W}^c_x$ and any diffeomorphism h of $\mathcal{W}^c_x$ preserving m^c_x, we have

$$\Delta_x(h(y)) = \frac{\Delta(y)}{\mathrm{Jac}(h)(y)}. \tag{5}$$

If h is C^∞, then so is the Jacobian $\mathrm{Jac}(h)$. Consider the graph of Δ_x:

$$\mathrm{graph}(\Delta_x) = \{(y, \Delta(y)) : y \in \mathcal{W}^c_x\} \subset \mathcal{W}^c_x \times \mathbb{R}.$$

As the function Δ is continuous, $\mathrm{graph}(\Delta_v)$ is locally compact. If h is an m^c_x-preserving C^∞ diffeomorphism, then (5) implies that the C^∞ diffeomorphism

$$(y, t) \mapsto \left(h(y), \frac{t}{\mathrm{Jac}(h)(y)}\right)$$

preserves $\mathrm{graph}(\Delta_x)$.

Combining this observation with accessibility of f and the fact that f admits global su-holonomy, we obtain that for any pair of points $q = (y, \Delta_x(y))$ and $q' = (y', \Delta_x(y'))$ in $\mathrm{graph}(\Delta_x)$, there is a C^∞ diffeomorphism of $\mathcal{W}^c_x \times \mathbb{R}$ sending q to q' and preserving $\mathrm{graph}(\Delta_x)$. That is, the locally compact set $\mathrm{graph}(\Delta_x)$ is C^∞ homogeneous. Theorem 5.1 implies that $\mathrm{graph}(\Delta_x)$ is a C^∞ submanifold of $\mathcal{W}^c_x \times \mathbb{R}$. Thus, Δ_x is C^∞ off of its singularities (by 'singularities' we mean points where the projection of $\mathrm{graph}(\Delta_x)$ onto $\mathcal{W}^c_x$ fails to be a submersion). However, if Δ_x has any singularities, then it is easy to see that *every point* in $\mathcal{W}^c_x$ must be a singularity, which violates Sard's theorem. Hence, Δ_x has no singularities and, therefore, is C^∞.

To see that the derivatives of Δ_x vary continuously as a function of x, note that one can move from the leaf $\mathcal{W}^c_x$ to any neighboring leaf by a composition of local $\mathcal{W}^u$ and $\mathcal{W}^s$ holonomies. The derivatives of these holonomy maps very continuously with the fiber. Equation (5) implies that the fiberwise derivatives vary continuously. □

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Proof of part (2) of Theorem 2.2

Proof. Let $\pi, p\colon \mathcal{E}^c \to M$, and $\mu := \hat{m}_U$ be given by Theorem 4.4. Denote by χ^c the central exponent of $\mathfrak{F}^\ell$ with respect to $\hat{\mu}$.

The case of non-vanishing exponents. Suppose that $\chi^c \neq 0$. Let

$$X = \{x \in U : \chi^c(x) = \chi^c\},$$

which is a full measure subset of U. Let $\mathcal{X} = p^{-1}(X)$, which is the set of $\xi \in p^{-1}(U)$ where the fiberwise exponent of $\mathfrak{F}$ is equal to χ^c.

Then [6, Theorem 4.1] implies that $\mathcal{X}$ coincides, up to zero $\hat{\mu}$-measure, with a measurable set $\mathcal{Y} \subset \mathcal{E}^c$ meeting almost every fiber $\mathcal{E}^c_x$, $x \in U$ in finitely many points. Setting $Y = p(\mathcal{Y}) \subset U$, we obtain a full measure subset of U that meets $\mathcal{W}^c_x$, for almost every $x \in U$, in finitely many points. Hence, case 2a holds in Theorem 2.2.

The case of vanishing exponents. If the central exponent of f is zero on U, then the exponents of $\mathfrak{F}^k$ vanish $\hat{\mu}$-almost everywhere. Theorem 3.26 then gives a continuous disintegration $\{\hat{\mu}^{su}_x : x \in U\}$ of $\hat{\mu}$ over U that is invariant under both s-holonomy and u-holonomy.

Theorem 5.5 implies that either the disintegration of μ is atomic or $\{\hat{\mu}^{su}_x : x \in U\}$ projects to a continuous disintegration $\{\mu^c_x := p_*(\hat{\mu}^{su}_x) : x \in U\}$ of μ with continuous density Δ on U. If the disintegration of μ is atomic, then conclusion (a) holds.

In the latter case, arguing exactly as in the proof of part (1) of Theorem 2.2, we define a flow ψ_t on M such that:

- ψ_t is supported in U and is tangent to the leaves of $\mathcal{W}^c$;
- ψ_t is generated by a nonsingular vector field X;
- $(\psi_t)_*\mu^c_x = \mu^c_x$, for all $x \in U$.

Now consider the action of ψ_t on a single leaf $\mathcal{W}^c_x$. If $U \cap \mathcal{W}^c_x \neq \mathcal{W}^c_x$, then restricting ψ_t to a connected component of $U \cap \mathcal{W}^c_x \neq \mathcal{W}^c_x$, we obtain an open interval $I \subset \mathcal{W}^c_x$ with $\mu^c_x(I) < 1$ with a μ^c_x-preserving non-singular flow. This is impossible and, hence, $U \cap \mathcal{W}^c_x = \mathcal{W}^c_x$, for all $x \in U$. As f admits global su-holonomy, the accessibility class U must meet every leaf $\mathcal{W}^c_x$. We thus conclude that f is accessible, and so conclusion (b) holds. □

6.2. *Proof of Theorem 1.1*. Let M be a 3-manifold, and suppose that $f \in \mathrm{Diff}(M)$ is partially hyperbolic and preserves a foliation by C^1 circles. Bohnet proved ([7]; see also [18, 14]) that there is a finite cover (at most four-fold) $\widehat{M}$ of M such that the lifts of E^u, E^c to $\widehat{M}$ are orientable, a lift $\hat{f} \in \mathrm{Diff}(\widehat{M})$ of f, and a fibration $\pi : \widehat{M} \to \mathbb{T}^2$ such that $\pi \circ \hat{f} = A \circ \pi$, where $A \in \mathrm{SL}(2, \mathbb{Z})$ is hyperbolic. The fibers of π are the leaves of the foliation $\widehat{\mathcal{W}}^c$, which is the lift of $\mathcal{W}^c$. In particular, the lift $\hat{f}$ on $\widehat{M}$ is a fibered partially hyperbolic system. Henceforth, we assume that $M = \widehat{M}$, $\hat{f} = f$, and $\mathcal{W}^c = \widehat{\mathcal{W}}^c$.

Assume the center is absolutely continuous. Then part (1) of Theorem 2.2, gives a flow ψ_t that is C^1 along the leaves of $\mathcal{W}^c$. We show ψ_t is smooth. Note that we are *not* assuming accessibility here, but we will use in an essential way the assumption that M is three-dimensional.

The first step is to establish the smoothness of the foliation $\mathcal{W}^c$.

PROPOSITION 6.3. *Let M be a 3-manifold, and let $f \in \mathcal{P}_{\text{fib}}(M)$. If $\mathcal{W}^c$ is leafwise absolutely continuous and the center Lyapunov exponents of f vanish m-almost everywhere, then $\mathcal{W}^c$ is C^∞.*

Proof. As noted previously, because Df preserves the non-vanishing vector field X, Lemma 3.23 implies that the leaves of $\mathcal{W}^{cs}(f)$, $\mathcal{W}^{cu}(f)$, and $\mathcal{W}^c(f)$ are C^∞, and the $\mathcal{W}^s(f)$-holonomies and $\mathcal{W}^u(f)$-holonomies between $\mathcal{W}^c(f)$-leaves are also C^∞.

We next verify that the restriction of $\mathcal{W}^c$ to $\mathcal{W}^{cs}$-leaves and to $\mathcal{W}^{cu}$-leaves is C^∞. Both items will follow from the fact that $\mathcal{W}^c$-holonomy preserves the disintegration of volume along $\mathcal{W}^u$ and $\mathcal{W}^s$ leaves.

The following lemma is well-known (see formula (11.4) in [8]).

LEMMA 6.4. *For any foliation box $\mathcal{B} \subset M$ for $\mathcal{W}^s$, there is a continuous disintegration of $m|_{\mathcal{B}}$ along leaves of $\mathcal{W}^s$ (defined at every point $p \in \mathcal{B}$). These disintegrations are equivalent to Riemannian measure in the $\mathcal{W}^s$ leaves. The densities of the disintegrations are C^∞ along leaves and transversely continuous. The same is true for $\mathcal{W}^u$.* □

LEMMA 6.5. *For any foliation box $\mathcal{B}$, any $t \in \mathbb{R}$, and any $p \in \mathcal{B}$, the time-t map ψ_t sends the disintegration m^s_p of $m \mid \mathcal{B}$ along $\mathcal{W}^s$ leaves at p to the disintegration $m^s_{\psi_t(p)}$ of $m|_{\psi_t(\mathcal{B})}$ along $\mathcal{W}^s$ leaves at $\psi_t(p)$.*

Proof. Denote by $\{m^s_p : p \in \mathcal{B}\}$ the disintegration of m along $\mathcal{W}^s(g)$ leaves inside the box $\mathcal{B}$. By Lemma 6.4, the map $p \mapsto m^s_p$ is continuous.

Fix $t \in \mathbb{R}$. Restricted to a $\mathcal{W}^s$ leaf, ψ_t is the $\mathcal{W}^c$-holonomy map between that leaf and its image. As ψ_t preserves both m and the leaves of $\mathcal{W}^s$, we obtain that

$$\psi_{t*} m^s_p = m^s_{\psi_t(p)}, \tag{6}$$

for m-almost every $p \in M$, where the disintegration on the right-hand side takes place in the box $\psi_t(\mathcal{B})$. As $p \mapsto m^s_p$ is continuous (on both sides of the equation) and ψ_t is a homeomorphism, equation (6) holds everywhere.

As t was arbitrary, this shows that between any two $\mathcal{W}^s$-leaves, the $\mathcal{W}^c$-holonomy map preserves conditional densities. □

LEMMA 6.6. *For every $t \in \mathbb{R}$, the map ψ_t is uniformly C^∞ along $\mathcal{W}^s$ leaves and uniformly C^∞ along $\mathcal{W}^u$ leaves.*

Proof. Lemma 6.5 implies that ψ_t satisfies an ordinary differential equation along $\mathcal{W}^s$ leaves with C^∞ coefficients, and so the solutions are C^∞ and vary continuously with the leaf. □

Returning to the proof of Proposition 6.3, we have just shown that the $\mathcal{W}^c$-holonomy maps between $\mathcal{W}^s$-leaves and between $\mathcal{W}^u$-leaves are uniformly C^∞. Applying Proposition 3.22 completes the proof of Proposition 6.3.

Remark 6.7. For a general $f \in \mathcal{P}_{\text{fib}}(M)$, it is possible to show by similar methods that if $\mathcal{W}^c$ is leafwise absolutely continuous and the center Lyapunov exponents of f vanish, then

$\mathcal{W}^c$ satisfies the stronger property of being *absolutely continuous with bounded Jacobians*: the center holonomy maps between any two smooth transversals have Jacobian with respect to volume that is bounded above and below.

6.2.1. *The conjugacy is as smooth as the foliation.* Finally, we prove the following result.

PROPOSITION 6.8. *Let* $f \in \mathcal{P}^1_{\mathrm{fib}}(M)$, *where* $\dim M = 3$. *If* $\mathcal{W}^c$ *is a* C^∞ *foliation, then* f *is* C^∞ *conjugate to a circle extension of a volume-preserving Anosov diffeomorphism.*

Proof. The assumption that $\mathcal{W}^c$ is C^∞ implies that the bundle projection $M \to B = M/\mathcal{W}^c$ is C^∞. Using the C^∞ flow ψ_t, we endow this bundle with a $\mathbb{T}$-structure on the fibers in which f acts as a translation on the fibers.

To this end, let $\{U_\alpha\}$ be an open cover of B, and let $h_\alpha : U_\alpha \times S^1 \to \pi^{-1}(U_\alpha)$ be C^∞ foliation charts for $M \to B$. Define new charts $\hat{h}_\alpha : U_\alpha \times \mathbb{T} \to \pi^{-1}(U_\alpha)$ by

$$\hat{h}_\alpha(b, t) = \psi_t(h_\alpha(b, 0)).$$

Note that if $U_\alpha \cap U_\beta$ is non-empty, then

$$\hat{h}_\beta \circ \hat{h}_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{T} \to (U_\alpha \cap U_\beta) \times \mathbb{T}$$

is of the form $h_\beta \circ h_\alpha^{-1}(b, t) = (b, t + \theta_{\alpha,\beta})$, which gives B the structure of a $\mathbb{T}$ bundle over B. As f commutes with ψ_t these charts, f acts by a translation on the $\mathbb{T}$-fibers, projecting to a diffeomorphism $\bar{f} : B \to B$.

Write $dm = m^c_x \, d\bar{m}(x)$, where $\{m^c_x : x \in M/\mathcal{W}^c\}$ is the smooth disintegration of m along $\mathcal{W}^c$ leaves, and $\bar{m}$ is pushforward of m under $M \to B$. Clearly the map $\bar{f}$ is a C^∞ Anosov diffeomorphism, preserving $\bar{m}$, which is smooth measure on B. □

This completes the proof of Theorem 2.2. We end this section with some remarks about the case where center exponents vanish and atomic disintegration holds.

6.2.2. *Remarks about atomic disintegration.* Suppose $f \in \mathcal{P}^1_{\mathrm{fib}}(M)$ is accessible and the center Lyapunov exponents vanish. In the case of atomic disintegration, one can show (using similar methods to the Lebesgue disintegration case) that there is a C^1 volume-preserving homeomorphism $\psi : M \to M$ commuting with f such that $\psi^k = id$, where k is the number of atoms in the disintegration. Thus, typically one should expect that the number of atoms to be one (one can make this notion of typicality precise using codimension arguments). One might ask where this atom lies in $\mathcal{W}^c_x$.

LEMMA 6.9. *If* $k = 1$, *then* $m_y = \delta_y$ *for every* $y \in M$.

Proof. Let $m_x = \delta_{\phi(x)}$. By definition, $m^c_x = m_x$ for m-almost every x, that is, for μ-almost every leaf and m^c_x-almost every point in the leaf. In particular, for μ-almost every leaf we have $m^c_x = m_x = \delta_{\phi(x)}$ for some point x in the leaf. As m^c_x is a disintegration, the point $\bar{x} = \phi(x)$ depends only on the leaf. Then $E = \phi(M)$ is a full m-measure set, because it has full m^c_x-measure on almost every leaf, restricted to which $m_y = \delta_y$. In particular, E is dense, and so, by continuity, $m_y = \delta_y$ for every y. □

More generally, we have the following result.

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LEMMA 6.10. *There exists $y \mapsto (y_1 = y, y_2, \ldots, y_k)$ continuous with $y_i \neq y_j$ for all $i \neq j$ such that $m_y = (1/k) \sum_{i=1}^{k} \delta_{y_i}$.*

We can then prove the assertion above, which is the following proposition.

PROPOSITION 6.11. *Let M be a closed manifold of dimension at least three, and let $f \in \mathcal{P}^1_{\mathrm{fib}}(M)$. Suppose that f is accessible and that the center Lyapunov exponents vanish almost everywhere. If the disintegration of volume is atomic along $\mathcal{W}^c$, with $k \geq 1$ atoms per $\mathcal{W}^c$ leaf, then there exists a homeomorphism $\psi : M \to M$, commuting with f and fixing the leaves of $\mathcal{W}^c$, such that $\psi^k = id$.*

If $\dim M = 3$, *then ψ is a $C^{1+\alpha}$ diffeomorphism, for some $\alpha \in (0, 1)$.*

Proof. Let $y \mapsto (y_1 = y, y_2, \ldots, y_k)$ be given by the previous lemma, and let $\psi : M \to M$ be defined by $\psi(y_1) = y_2$ (cyclically). Then ψ is a homeomorphism, it commutes with f, it fixes the center leaves, it preserves volume. Moreover, it is $C^{1+\alpha}$ on center leaves, because center bunching implies that the $\mathcal{W}^u$-holonomy is $C^{1+\alpha}$ on $\mathcal{W}^{cu}$ leaves, and the $\mathcal{W}^s$-holonomy is $C^{1+\alpha}$ on $\mathcal{W}^{cs}$ leaves. The graph of ψ is $C^{1+\alpha}$-homogeneous and is therefore $C^{1+\alpha}$. □

Finally, we remark that if $f \in \mathcal{P}^1_{\mathrm{fib}}(M)$ has an open accessibility class $U \notin \{\emptyset, M\}$, and the exponents of f vanish on U, then the atomic disintegration $\{m^c_x : x \in U\}$ of $m|_U$ has the property that for every $x \in M$, every interval in $U \cap \mathcal{W}^c_x$ contains exactly one atom. This implies in particular that $U \cap \mathcal{W}^c_x$ consists of finitely many intervals (the number does not depend on x), each with the same mass in the disintegration of volume. The details of the argument are left to the reader.

6.3. *Proof of Theorem 1.2.* Let M be a 3-manifold, and suppose that $f \in \mathrm{Diff}(M)$ is partially hyperbolic and preserves a foliation by C^1 circles. As in the proof of Theorem 1.1, after taking a finite cover, we may assume that $f \in \mathcal{P}_{\mathrm{fib}}(M)$. Then Theorem 1.2 follows immediately from case (ii) of Theorem 2.2.

7. *Higher center dimension: proof of Theorem 2.3*

Here we prove Theorem 2.3.

Let $f \in \mathcal{P}_{\mathrm{fib}}(M)$, and assume that f is accessible and that the center exponents of f vanish. Let $\{m^c_x\}$ be a disintegration of volume along center leaves and $\pi : \mathcal{E}^c \to M$, p be given by Proposition 4.3. Let $\hat{m}$ be a measure on $\mathcal{E}^c$ with $\pi_* \hat{m} = m$ and $p_*(\hat{m}_x) = m^c_x$ for m-almost every $x \in M$ and any disintegration $\{\hat{m}_x : x \in M\}$ along fibers of $\mathcal{E}^c$.

Let $\{\hat{m}^{su}_x : x \in M\}$ be the continuous, holonomy-invariant disintegration of $\hat{m}$ given by Theorem 4.8, and for $x \in M$, let $X_x \subset \mathcal{E}^c_x$ be the support of $\hat{m}^{su}_x$. As f is assumed to be accessible, Theorem 5.4 implies that X_x is a $C^{1+\alpha}$ submanifold of $\mathcal{E}^c_x$, for every $x \in M$ and the connected components of X_x are diffeomorphic to each other.

LEMMA 7.1. *For almost every $x \in M$ and every $y \in p(X_x)$, we have $p_* \hat{m}^{su}_y = m^c_x$. That is, $p_* \hat{m}^{su}_y$ is constant on $p(X_x)$.*

Proof. We start by noting that because $\hat{m}_x^{su}$ is a disintegration of $\hat{m}$, we have that $p_*\hat{m}_x^{su} = m_x^c$ and $p(\text{supp}\,(\hat{m}_x^{su})) = \text{supp}\,(m_x^c)$ at m-almost every point $x \in M$. That is, for m-almost every x, and m_x^c-almost every $y \in \mathcal{W}_x^c$, we have $p_*\hat{m}_y^{su} = m_y^c = m_x^c$ and $p(X_y) = \text{supp}\,(m_y^c) = \text{supp}\,(m_x^c)$. Hence, for m-almost every x and a dense set of $y \in \text{supp}\,(m_x^c)$, we have $p_*\hat{m}_y^{su} = m_x^c$ and $p(X_y) = \text{supp}\,(m_x^c)$. The left-hand side of the latter equation depends continuously on $y \in M$, and the right-hand side is constant on $\mathcal{W}_x^c$. Thus, for m-almost every x and every $y \in \text{supp}\,(m_x^c) = P(X_x)$, we have $p_*\hat{m}_y^{su} = m_x^c$ (and $p(X_y) = \text{supp}\,(m_x^c)$). The lemma is proved. □

LEMMA 7.2. *For $x, x' \in M$, $p(X_x)$ and $p(X_{x'})$ are either disjoint or coincide.*

Proof. Suppose that for some $x, x' \in M$, we have $p(X_x) \cap p(X_{x'}) \neq \emptyset$. Using accessibility and applying su-holonomy to X_x and $X_{x'}$, we may assume that x is m-typical, and by Lemma 7.1, in particular that for $y \in p(X_x)$, we have $p_*\hat{m}_y^{su} = m_x^c$. Thus, for $y \in X_x \cap X_{x'}$, $p(X_y) = p(X_x)$. Reversing the roles of x, x', we obtain that $p(X_y) = p(X_{x'})$, and so $p(X_x) = p(Xx')$. □

The collection $\mathcal{W}^{cc} := \{p(X_x) : x \in M\}$ is a continuous family of compact, $C^{1+\alpha}$ submanifolds on M, tangent to the leaves of $\mathcal{W}^c$, and preserved by both s and u holonomies. It is thus a foliation of M that subfoliates $\mathcal{W}^c$.

LEMMA 7.3. *The foliation $\mathcal{W}^{cc}$ is leafwise absolutely continuous.*

Proof. The proof is similar to the proof of Theorem 5.5.

For $x \in M$, denote by λ_x the Riemannian measure on X_x and denote by $B(\xi, r)$ the ball in $\mathcal{E}_x^c$ centered at ξ of radius r, with respect to the p-pullback metric of the Riemann structure on $\mathcal{W}^c(f)_x$.

LEMMA 7.4. *For each $x \in M$, the measure $\hat{m}_x^{su}$ is equivalent to the restriction λ_x. The limit*

$$\Delta_x(\xi) = \lim_{r\to 0} \frac{\hat{m}_x(B(\xi, r))}{\lambda_x(B(\xi, r))}$$

exists for every $x \in M$ and $\xi \in X_x$, is continuous in both x and $\xi \in X_x$, and takes values in $(0, \infty)$.

Proof. For $x \in M$ and $\xi \in X_x$ let

$$\overline{\Delta}_x(\xi) = \limsup_{r\to 0} \frac{\hat{m}_x(B(\xi, r))}{\lambda_x(B(\xi, r))}, \qquad \underline{\Delta}_x(\xi) = \liminf_{r\to 0} \frac{\hat{m}_x(B(\xi, r))}{\lambda_x(B(\xi, r))}.$$

For $\hat{m}_x$-almost every $\xi \in X_x$, we have

$$\overline{\Delta}_x(\xi) = \underline{\Delta}_x(\xi) \in (0, \infty].$$

As supp $(\hat{m}_x^{su}) = X_x$, Proposition 5.3 implies that for any two points $\xi, \xi' \in X_x$, there is a diffeomorphism $H_{\xi,\xi'} \colon X_x \to X_x$ preserving $\hat{m}_x^{su}$ and sending ξ to ξ'. As C^1 diffeomorphisms have continuous and positive Jacobians, it follows that for any

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$\xi, \xi' \in X_x$:

$$\underline{\Delta}_x(\xi) = \overline{\Delta}_x(\xi) \quad \Longleftrightarrow \quad \underline{\Delta}_x(\xi') = \overline{\Delta}_x(\xi').$$

Thus, $\underline{\Delta}_x = \overline{\Delta}_x$ everywhere on X_x; denote this function by Δ_x.

Then $\hat{m}_x^{su}$ has a singular part with respect to λ_x if and only if there is a positive $\hat{m}_x^{su}$-measure set $B \subset X_x$ such that, for $\xi \in B$, $\Delta_x(\xi) = \infty$. On the other hand, again using the diffeomorphisms $H_{\xi,\xi'}$ we see that for every $\xi, \xi' \in X_x$:

$$\Delta_x(\xi) = \infty \quad \Longleftrightarrow \quad \Delta_x(\xi') = \infty.$$

Hence, if $\hat{m}_x^{su}$ had a singular part with respect to λ_x, this would imply that $\Delta_x \equiv \infty$ on X_x, contradicting the local finiteness of $\hat{m}_x^{su}$. Therefore, $\hat{m}_x^{su}$ is absolutely continuous with respect to λ_x. Similarly, we see that λ_x is absolutely continuous with respect to $\hat{m}_x^{su}$, and so the two measures are equivalent.

For $x \in M$, the function $\Delta_x : X_x \to (0, \infty)$ is a pointwise limit of the continuous functions

$$\xi \mapsto \frac{\hat{m}_x^{su}(B(\xi, r))}{\lambda_x(B(\xi, r))}$$

and, hence, is a Baire class 1 function; it follows that Δ_x has a point of continuity [**26**, Theorem 7.3]. Again using Proposition 5.3, we see that every point in M is a point of continuity of $x \mapsto \Delta_x$, and so Δ is continuous on M. □

Now for almost every $x \in M$, we have $p_* \hat{m}_x^{su} = m_x^c$, and so Lemma 7.4 implies that $p_* \hat{m}_x^{su}$ is equivalent to Lebesgue measure on $p(X_x) = \mathcal{W}_x^{cc}$, for almost every x. Thus, $\mathcal{W}^{cc}$ is leafwise absolutely continuous. □

LEMMA 7.5. *If f is k-bunched, for some $k \geq 2$, then the restriction of $\mathcal{W}^{cc}$ to $\mathcal{W}^c$-leaves is uniformly C^{k-1}.*

Proof. Fix $x \in M$ and consider the leaf $\mathcal{W}_x^c$. The restriction of $\mathcal{W}^{cc}$ to $\mathcal{W}_x^c$ is a subfoliation invariant under su-holonomy in M. As f is k-bunched and accessible, the holonomy acts C^k and transitively on $\mathcal{W}_x^c$. Proposition 5.2 implies that $\mathcal{W}^{cc}$ is a C^{k-1} subfoliation of $\mathcal{W}_x^c$. □

8. *Systems with mostly compact leaves: proof of Theorem 2.5*

Let f be a C^2 volume-preserving, partially hyperbolic, dynamically coherent diffeomorphism of a closed manifold M.

THEOREM 8.1. *Assume the center foliation of f is leafwise absolutely continuous, the center leaves are compact for all points in a dense G_δ, and the center Lyapunov exponents vanish m-almost everywhere. Then all leaves are compact and have bounded Riemannian volume.*

Before proving Theorem 8.1 we discuss some preliminary facts about the leafwise properties of foliations.

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8.1. *Foliations with the generic leaf compact.* Recall that if $\mathcal{F}$ is a foliation of a manifold M, then we say that *the generic leaf of $\mathcal{F}$ is compact* if there exists a dense G_δ subset $C \subset M$ such that for every $x \in C$, the leaf $\mathcal{F}_x$ is compact.

LEMMA 8.2. *Let $\mathcal{F}$ be a foliation of M with C^1 leaves. If the generic leaf of $\mathcal{F}$ is compact, then there exists an open and dense, $\mathcal{F}$-saturated set $O \subset M$ restricted to which $\mathcal{F}$ is a fiber bundle.*

Proof. Consider the function $\phi : x \mapsto \mathrm{vol}(\mathcal{F}_x)$ assigning to each point the volume (possibly infinite) of the leaf through it. As the leaves are a locally continuous family of submanifolds, the function ϕ is lower semi-continuous:

$$\liminf \mathrm{vol}(\mathcal{F}_{x_n}) \geq \mathrm{vol}(\mathcal{F}_x)$$

for any sequence $(x_n)_n$ converging to some point $x \in M$. Hence, there exists a residual subset $\mathcal{R}$ of M such that every $x \in \mathcal{R}$ is a continuity point for ϕ. Note that ϕ is constant on $\mathcal{F}$-leaves and the set of continuity points is $\mathcal{F}$-saturated. Thus, we may take $\mathcal{R}$ to be $\mathcal{F}$-saturated. Intersecting with the dense G_δ in the statement, we may have assume that every leaf through $\mathcal{R}$ is compact. Then $\mathcal{F}$ is a fiber bundle on an (open) neighborhood of every leaf through $\mathcal{R}$. The union of such neighborhoods is a set O as in the statement. □

PROPOSITION 8.3. *If f is partially hyperbolic, volume-preserving, and dynamically coherent, and the generic leaf of $\mathcal{W}^c$ is compact, then the set O in Lemma 8.2 is f-invariant and has full volume. Moreover, for almost every $x \in M$ the stable and unstable leaves of x are contained in O.*

Proof. Invariance follows replacing O by its f-orbit, if necessary. Now let μ be an ergodic component of the volume measure. The conditional probabilities along (local) unstable leaves of the measure μ and of the volume measure itself coincide μ-almost everywhere. This is because the σ-algebra of measurable invariant sets is contained in the σ-algebra of measurable sets consisting of entire unstable leaves (cf. also [**1**, Lemma 6.2]). As the unstable foliation is absolutely continuous, it follows that for almost every ergodic component μ, its conditional probabilities along unstable leaves are equivalent to the Riemannian measure on the leaf. In particular, the support of almost every ergodic component is u-saturated and, by a dual argument, s-saturated. It follows that the ω-limit set of Lebesgue almost every $x \in M$ contains some su-saturated set. Then the c-saturate of $\omega(x)$ has non-empty interior, and so it intersects the dense set O. As O is c-saturated, open, and invariant, it follows that $x \in O$. This proves O that has full volume. Finally, because O is open and invariant, we have that $\mathcal{W}^s_x$ and $\mathcal{W}^u_x$ are contained in O whenever $x \in O$ is recurrent. This completes the proof of the proposition. □

8.2. *Foliations whose leaves have bounded volume.* Let $\mathcal{F}$ be a foliation on some manifold M and L be some compact leaf. Let Σ be a cross-section to the foliation at some point $p \in L$. The *holonomy group* of L is the group of germs at p of the projections along $\mathcal{F}$-leaves from Σ back to itself. The choice of p and Σ is irrelevant because different choices give rise to groups that are isomorphic. The following result is contained in Theorem 4.2 of Epstein [**17**].

THEOREM 8.4. *Let $\mathcal{F}$ be a foliation of a manifold M whose leaves are all compact, with bounded volume. Then every center leaf has finite holonomy group.*

We use this to show the following result.

THEOREM 8.5. *Let f be a partially hyperbolic, dynamically coherent diffeomorphism with* $\dim E^s = \dim E^u = 1$ *and center leaves of which are compact with uniformly bounded volume.*

Then there exists a covering map $\pi : \widetilde{M} \to M$ (at most four-to-one) such that the lift of the center foliation to $\widetilde{M}$ is a fiber bundle, and f lifts to a fibered diffeomorphism on $\widetilde{M}$.

Proof. By Theorem 8.4, the assumption implies that the holonomy group of every leaf is finite. Let $\pi : \widetilde{M} \to M$ be the covering map that orients both the stable foliation and the unstable foliation: each point of $\widetilde{M}$ is a triple $(x, \epsilon_s, \epsilon_u)$ with $x \in M$ and ϵ_s and ϵ_u are orientations of the stable and unstable directions, and π is just the projection to the first coordinate. Endow $\widetilde{M}$ with the smooth structure obtained from M by pull-back under π. Then the natural lift $\tilde{f} : \widetilde{M} \to \widetilde{M}$ of f is a diffeomorphism. The covering space $\widetilde{M}$ needs not be connected, if either the stable foliation or the unstable foliation are orientable. However, the connected components are canonically identified through diffeomorphisms

$$(x, \epsilon_s, \epsilon_u) \sim (x, \pm\epsilon_s, \pm\epsilon_u). \tag{7}$$

Thus, it is no restriction to suppose $\widetilde{M}$ is connected: just replace it by any connected component and replace $\tilde{f}$ by its composition with an appropriate identification map as in (7). It is clear that the invariant foliations of f lift to $\tilde{f}$-invariant foliations $\widetilde{\mathcal{W}}^c$, $\widetilde{\mathcal{W}}^s$, $\widetilde{\mathcal{W}}^u$, $\widetilde{\mathcal{W}}^{cs}$, and $\widetilde{\mathcal{W}}^{cu}$ on the covering space. Moreover, the leaves of $\widetilde{\mathcal{W}}^c$ are compact and the leaves of $\widetilde{\mathcal{W}}^s$ and $\widetilde{\mathcal{W}}^u$ have dimension one. Consider any leaf $\widetilde{L}$ and let $\tilde{p} \in \widetilde{L}$. By dynamical coherence, each element of the holonomy group defines a germ of orientation-preserving homeomorphisms on the stable leaf $\widetilde{\mathcal{W}}^s_{\tilde{p}}$. As the holonomy group is finite, this germ must have finite order. In dimension one this implies that the germ is the identity. The same argument proves that every element of the holonomy group is the identity along the unstable leaf $\widetilde{\mathcal{W}}^u_{\tilde{p}}$. Hence, by product structure, the holonomy group is trivial, for every leaf $\widetilde{L}$ of $\widetilde{\mathcal{W}}^c$. Equivalently, the center foliation $\widetilde{\mathcal{W}}^c$ is a fiber bundle, as we wanted to prove. □

8.3. *Proof of Theorem 8.1.* Having made these preliminary observations, we now return to the proof of Theorem 8.1.

Proof. Recall from §8.1 that there is an open and dense subset $O \subset M$ so that the restriction of $\mathcal{W}^c$ to O is a fiber bundle.

The invariance principle (Theorem 4.1) with $O^c = O^{cs} = O^{cu} = O$ implies that there exists a continuous disintegration $\{m_x : x \in O\}$ of m into probabilities measures supported in $\mathcal{W}^c_x$ with $x \in O$. Moreover, for each $x \in O$ and $y \in \mathcal{W}^*(x)$, with $* \in \{s, u\}$, we have that m_x is pushed forward by $h^*_{x,y}$ to m_y. □

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LEMMA 8.6. *For any $x, y \in O$, if x is connected to y by an su-path in M, then m_x pushes forward to m_y under the corresponding composition of holonomies.*

Proof. The conclusion obviously holds if the corners of the su-path lie in O. However, because O is open and dense, any su-path can be approximated arbitrarily well by a path with corners in O. Continuity of the disintegration then implies the result for arbitrary su-paths. □

COROLLARY 8.7. *There exists a disintegration $\{m_x : x \in M\}$ of volume into measures m_x in M such that:*

(1) *m_x is constant on every center leaf and is absolutely continuous with respect to the Riemannian measure along $\mathcal{W}^c_x$;*
(2) *for any $C > 0$, there exists $\epsilon_0 > 0$ such that for any su-path of length at most C from x to y, the corresponding holonomy h sends the restriction $m_x \mid B^c(x, \epsilon_0)$ to $m_y|_{h(B^c(x,\epsilon_0))}$;*
(3) *m_x depends continuously on x in the local sense that for every ε sufficiently small, the function $x \mapsto m_x(B_\varepsilon(x))$ is continuous (B_ε denotes the Riemannian ball of radius ε);*
(4) *for every $\epsilon > 0$ there exists $\delta > 0$ such that for every center ball $B_c \subset \mathcal{W}^c_x$ of radius ϵ, we have $m_x(B_\epsilon) > \delta$;*
(5) *there exists $\delta > 0$ such that $\delta \leq m_x(\mathcal{W}^c_x) \leq 1$ for every x.*

Proof. As O is open, dense, and c-saturated, every point in M may be connected to a point in O by a two-leg su path of arbitrarily small length. For any $y \in \mathcal{W}^c_x$ we define m_x on a small ball $B(y)$ around y by connecting y to some $z \in O$ by such a path and then pulling m_z back under stable and unstable holonomies. Lemma 8.6 ensures that this is consistent. By construction, m_x is constant on the center leaf $\mathcal{W}^c_x$. Moreover, it is absolutely continuous, since m_z is absolutely continuous and the stable and unstable holonomies are absolutely continuous (indeed, C^1 in the fiber bunched case at hand).

Claim (2) also follows from the construction.

Now we prove claim (3). Continuity in the center direction follows, simply, from the fact that the boundary of $B_\varepsilon(x)$ has zero measure (because m_x is absolutely continuous). Then transverse continuity follows from the holonomy invariance in claim (2), using once more that boundaries have zero measure.

Claim (4) follows from compactness, the continuity property in (3), and the fact that the measure of balls never vanishes: otherwise, by holonomy invariance, it would vanish on a whole open set, contradicting the fact that the m_x are a disintegration of Lebesgue measure.

Concerning claim (5), note first that $m_x(\mathcal{W}^c_x) \leq 1$ for every $x \in M$: if there existed $L \subset \mathcal{W}^c_x$ with $m_x(L)$, then by considering a short two-leg su-path we could map this to some L' inside a leaf $\mathcal{W}^c_z \subset O$, getting a contradiction. □

Parts (3) and (5) of Corollary 8.7 imply that the center leaves have bounded volume. This completes the proof of Theorem 8.1.

Finally, we prove Theorem 2.5.

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Proof of Theorem 2.5. Suppose that f satisfies the hypotheses of Theorem 2.5. By Theorem 8.5, all center leaves are compact and they have bounded volume. Then, by Theorem 8.4, every center leaf has finite holonomy. Moreover, if $\dim \mathcal{W}^s = \dim \mathcal{W}^u = 1$, we can use Theorem 8.5: there exists $\pi : \widetilde{M} \to M$ such that the lift of the center foliation to $\widetilde{M}$ is a fiber bundle, and f lifts to a diffeomorphism on $\widetilde{M}$. This completes the proof of Theorem 2.5. □

9. *Center fixing maps: proof of Theorem 2.7*

The proof is similar in structure to the proof of [6, Theorem A], where the same result is shown for perturbations of the time-one map of the geodesic flow on a negatively curved surface. The difficulty is constructing a fiber bundle in which one can carry out the arguments. We indicate where the appropriate modifications occur.

9.1. *Setting up a fiber bundle.*

As before, let $f \in \mathcal{P}(M)$ with leafwise absolutely continuous center foliation, and let m denote the volume measure. Here we provide a setup for the application of the invariance criterion in Theorem 4.1 under the assumption that $f \in \mathcal{P}^1_{\mathrm{fix}}(M)$. Recall this means that the center is one-dimensional and all center leaves are fixed by the diffeomorphism:

$$\dim E^c_x = 1 \quad \text{and} \quad f(\mathcal{W}^c_x) = \mathcal{W}^c_{f(x)} \quad \text{for every } x \in M.$$

Each center leaf $\mathcal{W}^c_x$ is either a circle or an injectively immersed copy of the real line. In the latter case, we denote by $[y, z]$ the closed leaf segment determined by any two points $y, z \in \mathcal{W}^c_x$ and similarly define the half-open segment $[y, z)$ (here we do not assume that $\mathcal{W}^c$ is orientable, but in the course of the proof, we will show this).

We now construct a circle bundle $\mathcal{E}^c$ over M admitting s, u, and c holonomies, a diffeomorphism cocycle $\mathfrak{F}^c : \mathcal{E}^c \to \mathcal{E}^c$ covering f and an $\mathfrak{F}$- and c-invariant probability measure $\hat{m}$, covering m. Roughly, the fiber of $\mathcal{E}^c$ over x will correspond to $\mathcal{W}^c_x/f$, and the conditional measures m_x will be the probability measures whose $\mathcal{W}^c$-lifts are representatives of the disintegration of Lebesgue. In practice, there are issues, such as closed $\mathcal{W}^c$-leaves and potential fixed points for f, that complicate the construction, which we now address.

As the leaves of $\mathcal{W}^c$ can be both circles and lines, there is no global su-holonomy, and the construction in Proposition 4.3 no longer produces a fiber bundle in the present setting. We remedy this problem by working instead in the continuous line bundle E^c.

Regarding the fiber E^c_x over x as the universal cover $\widetilde{\mathcal{W}^c_x}$, we construct a lift $\widetilde{\mathfrak{F}} : E^c \to E^c$ of f and compatible holonomies on E^c. We also construct a special bundle map $\mathfrak{G}$ on E^c, covering the identity, and commuting with $\widetilde{\mathfrak{F}}$ and the holonomies. The bundle $\mathcal{E}^c$ will be constructed as the quotient $E^c/\mathfrak{G}$.

For each $x \in M$, the manifold $\mathcal{W}^c_x$ carries an induced Riemannian structure and, hence, has a 'center exponential map' $\exp^c_x : E^c_x \to \mathcal{W}^c_x$ which is a covering map, sending 0 to x and the point $t \in E^c_x \cong \mathbb{R}$ to the point a signed distance t from x on $\mathcal{W}^c_x$ (in the induced metric). We define $\widetilde{\mathfrak{F}}$ to be the lift of the action of f by $\exp^c$; it is the unique continuous map fixing the 0 section of E^c and satisfying $(\exp^c_{f(x)})^{-1} \circ \widetilde{\mathfrak{F}}_x = f \circ \exp^c_x$ at every $x \in M$ (note that this map is well-defined for points with compact center leaves, because every

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circle homeomorphism has a unique lift to the universal cover once the image of a single point is specified: in this case, the image of $0 \in E^c_x$.)

We now address the issue of defining holonomies for $\widetilde{\mathfrak{F}}$ and a special map $\mathfrak{G}$ on E^c. Let $O \subset M$ be the set of points in M with open (that is, non-compact) center leaves. The next lemma implies that O is a dense, full-volume subset of M.

LEMMA 9.1. *For $f \in \mathcal{P}^1_{\mathrm{fix}}(M)$, the set of compact center leaves is countable, and so is the set of center leaves containing fixed points.*

Proof of Lemma 9.1. Let L_0 be any compact center leaf, and suppose it is accumulated by compact center leaves L_n with bounded length. As L_0 is normally hyperbolic, there exists $\delta > 0$ such that every L_n has some forward or backward iterate at distance $\geq \delta$ from L_0. That cannot be, because every L_n is fixed under f. This contradiction proves that the set of compact leaves with length bounded by any large constant is discrete and, hence, finite. Thus, the set of compact center leaves is countable, as stated in the first part of the lemma.

The claim in the second part uses the same kind of argument. Let p_0 be a fixed point contained in some center leaf L_0 and suppose it is accumulated by fixed points p_n in center leaves L_n distinct from L_0. By normal hyperbolicity, each p_n has some iterate at distance $\geq \delta$ from p_0, but that cannot be, because $f(p_n) = p_n$. This contradiction proves that any fixed point close to p_0 must be contained in the same local center leaf. It follows that at most countably many leaves contain fixed pints, as claimed. □

The center exponential map gives a natural means to define c-holonomies on E^c. Let γ be a path lying in the leaf $\mathcal{W}^c(\gamma(0))$, and let $\widetilde{\gamma}$ be the unique lift of γ to $E^c_{\gamma(1)}$ under $(\exp^c_{\gamma(1)})^{-1}$ with $\widetilde{\gamma}(0) = 0$. Setting $H^c_\gamma(0) = \widetilde{\gamma}(1)$ determines a unique continuous map $H^c_\gamma : E^c_{\gamma(0)} \to E^c_{\gamma(1)}$ satisfying $\exp^c_{\gamma(1)} : H^c_\gamma = \exp^c_{\gamma(0)}$. It is clear that this construction depends only on the leafwise homotopy type of the path γ and that it is continuous. The restriction of c-holonomy to O is of product type: for $x \in O$ and $y \in \mathcal{W}^c_x$, the map $H^c_{x,y}$ is just the diffeomorphism $(\exp^c_y)^{-1} \circ \exp^c_x$. Clearly c-holonomy is $\widetilde{\mathfrak{F}}$-invariant on O; because O is dense, this invariance extends to all of M.

We next define $\mathfrak{G}$. As f is center fixing, for each $x \in O$, the restriction of f to $\mathcal{W}^c_x$ lifts to a unique diffeomorphism $\mathfrak{G}_x = (\exp^c_x)^{-1} \circ f \circ \exp^c_x : E^c_x \to E^c_x$. Note that, by construction, $H^c_{x,x'} \circ \mathfrak{G}_x = \mathfrak{G}_x \circ H^c_{x,x'}$ for every $x \in O$. This defines a bundle map $\mathfrak{G}$, covering the identity, over O. We use the next lemma to extend $\mathfrak{G}$ to continuous bundle over all of M. For $x \in M$, let $\ell(x)$ denote the length of the central segment $[x, f(x)]$, which vanishes precisely when x is fixed by f. Note that ℓ is a continuous function on O. We have the following result.

LEMMA 9.2. *If $f \in \mathcal{P}^1_{\mathrm{fix}}(M)$, then there exists $\delta_0 > 0$ such that $\ell(x) \geq \delta_0$, for every $x \in O$.*

Proof of Lemma 9.2. Recall that because E^c is one-dimensional, the local stable and unstable holonomy maps between center manifolds are uniformly C^1. Hence, there exists a constant $c_0 \geq 1$ such that for $* \in \{s, u\}$, and for any x, x' with $x' \in \mathcal{W}^{*,\mathrm{loc}}_x$, the derivative of $h^*_{x,x'}$ lies in $[c_0^{-1}, c_0]$.

There exist positive constants $k \in \mathbb{N}$ and $R \in \mathbb{R}$ such that for every $x, y \in M$, there is a sequence of points $x_0, x_1, \ldots x_k$ with $x_{i+1} \in \mathcal{W}^{a_i,\mathrm{loc}}_{x_i}$, for $a_i \in \{s, u\}$, and $x_k \in \mathcal{W}^{c,R}_y$, where $\mathcal{W}^{c,R}_y$ denotes the ball of radius R in $\mathcal{W}^c_y$. Fix such k and R.

As ℓ is not identically zero, there exists a point $x_0 \in M$ with $\mathcal{W}^c_{x_0}$ open, such that $\ell(x_0) > 0$; let $\ell_0 = \ell(x_0)$. Let y be any point whose center leaf is open, and fix a sequence $\{x_0, \ldots, x_k = y\}$ as previously. Consider the arc $[x_0, f(x_0)]$ of $\mathcal{W}^c_{x_0}$ connecting x_0 to $f(x_0)$, parametrized as a unit-speed path γ_0. The image of γ_0 under $\mathcal{W}^{a_1}$-holonomy $h^{a_i}_{x_0,x_1}$ is a non-singular path γ_1 in $\mathcal{W}^c_{x_1}$ from x_1 to $h^{a_1}_{x,x_1}(f(x)) = f(x_1)$.

Inductively, we set $\gamma_i = h^{a_i}_{x_{i-1},x_i} \circ \gamma_{i-1}$. Then γ_k is a non-singular path from x_k to $f(x_k)$; because the center leaf of $y = x_k$ is open, it follows that $\ell(y)$ is equal to the length of γ_k. It follows that if ℓ_0 is sufficiently small (for example, $\ell_0 = O(c_0^{-k})$), then the length of γ_k is less than or equal to $c_0^k \ell_0$. As y was arbitrary, this implies that $\sup_y \inf_{\mathcal{W}^{c,R}_y} \ell(y) \leq c_0^k \ell_0$. Hence, ℓ_0 cannot be arbitrarily small, for then every open center leaf in M would have a fixed point for f, contradicting Lemma 9.1. By the same token, if ℓ vanishes on some open center leaf, then every open center leaf in M has a fixed point for f. It follows that ℓ is bounded below on open center leaves. □

As $\ell(x) \geq \delta_0 > 0$ for all points with open center leaf, there is an orientation on the open $\mathcal{W}^c$ leaves so that $[x, f(x))$ is positively oriented. This orientation is preserved by f and by su-holonomy and so extends continuously to compact leaves and, thus, to the bundle E^c. It follows that $\mathfrak{G}_x(v) - v \geq \delta_0 > 0$, for all $v \in E^c_x$ and $x \in O$. Note also that $\mathfrak{G}$ is continuous over the set of points with open center leaves (though *a priori* not uniformly continuous, as we have not shown that ℓ is bounded above). To extend $\mathfrak{G}$ to M we use the stable holonomy maps.

Let y be a point with compact center leaf. To define $\mathfrak{G}_y$, we note that for any such y and any $x \in \mathcal{W}^{s,\mathrm{loc}}_y$ different from y, the leaf $\mathcal{W}^c_x$ is open (because normal hyperbolicity forbids one compact center leaf from lying in the local stable manifold of another compact leaf). Fix such an x; because $\mathcal{W}^c_x$ is f-invariant, and x lies in the stable manifold of $\mathcal{W}^c_y$, we may assume that the positive arc $[x, \infty)$ of $\mathcal{W}^c_x$ lies in the local stable manifold of $\mathcal{W}^c_y$; then the stable holonomy $h^s_{x,y}$ onto $\mathcal{W}^c_y$ is defined on $[x, \infty)$ and is a local homeomorphism. The image of the interval $[0, \mathfrak{G}_x(0))$ under the covering map $\exp^c_x$ is the path $[x, f(x))$ in $\mathcal{W}^c_x$. The image of this path under $h^s_{x,y}$ is a path in $\mathcal{W}^c_y$ from y to $f(y)$. We lift this path by $(\exp^c_y)^{-1}$ to a path from 0 to $t' \in E^c_y$, and we set $\mathfrak{G}_y(0) = t'$. This choice of $\mathfrak{G}_y(0)$ determines a continuous map $\mathfrak{G}_y$ on all of E^c_y satisfying $\exp^c_y \mathfrak{G}_y = f \circ \exp^c_y$, via the usual lifting procedure. Observe that, because $\mathfrak{G}$ is continuous over the set of points with open center leaves, this definition of $\mathfrak{G}_y$ does not depend on the choice of $x \in \mathcal{W}^{s,\mathrm{loc}}_y$ and is continuous at y along $\mathcal{W}^s_y$.

As $\mathfrak{G}$ is continuous over the set of points with open center leaves, and $\mathfrak{G}$ is continuous along $\mathcal{W}^s$-leaves, this defines a continuous bundle map $\mathfrak{G} : E^c \to E^c$ covering the identity on M; it has the two key properties that $\exp^c_x \circ \mathfrak{G}_x = f \circ \exp^c_x$ and $\mathfrak{G}_x(v) - v \geq \delta_0 > 0$, for all $x \in M$ and $v \in E^c(x)$. In particular, it follows that

$$E^c_x = \bigsqcup_{k \in \mathbb{Z}} [\mathfrak{G}^k_y(0), \mathfrak{G}^{k+1}_y(0)),$$

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for each $x \in M$. As $\mathfrak{G}$ is continuous and commutes with c-holonomy on the dense set O, it commutes with c-holonomy everywhere on M.

We next describe how to define s- and u-holonomy maps on E^c, commuting with $\widetilde{\mathfrak{F}}$ and $\mathfrak{G}$ and compatible with c-holonomy. Suppose $x \in M$ and $y \in \mathcal{W}_x^{*,\mathrm{loc}}$, for $* \in \{s, u\}$. We define a map $H^*_{x,y} : E^c_x \to E^c_y$ as follows. We first define $H^*_{x,y}$ on the interval $[0, \mathfrak{G}_x(0))$ in E^c_x. For $t \in [0, \mathfrak{G}_x(0))$, the image of $[0, t)$ under the covering map $\exp^c_x$ is a path in $\mathcal{W}^c_{x_0}$ from x to $\exp^c_x(t)$. The image of this path under holonomy $h^*_{x,y}$ to $\mathcal{W}^c_y$ is a path from y to $h^*_{x,y}(\exp^c_x(t))$. We lift this path by $(\exp^c_y)^{-1}$ to a path from 0 to $t' \in E^c_y$, and we set $H^s_{x,y}(t) = t'$. As f commutes with $\mathcal{W}^*$ holonomy, which is a local homeomorphism, the interval $[0, \mathfrak{G}_x(0))$ is mapped by $H^*_{x,y}$ homeomorphically onto the interval $[0, \mathfrak{G}_y(0))$.

We extend the definition of $H^*_{x,y}$ to all of $E^c_x = \bigsqcup_k [\mathfrak{G}^k_x(0), \mathfrak{G}^{k+1}_x(0))$ by setting $H^*_{x,y} = \mathfrak{G}^k_y \circ H^*_{x,y} \circ \mathfrak{G}^{-k}_x$ on $[\mathfrak{G}^k_x(0), \mathfrak{G}^{k+1}_x(0))$. Then $H^*_{x,y}$ is a homeomorphism onto

$$\bigsqcup_k [\mathfrak{G}^k_y(0), \mathfrak{G}^{k+1}_y(0)) = E^c_y.$$

This defines $H^*_{x,y}$, for $* \in \{s, u\}$; by construction, $H^*_{x,y}$ commutes with $\widetilde{\mathfrak{F}}$ and $\mathfrak{G}$ and is compatible with c-holonomy.

Now let $\mathcal{E}^c = E^c/\mathfrak{G}$ be the quotient of E^c under the action of $\mathfrak{G}$. As $\mathfrak{G}$ fixes the fibers of E^c and has no fixed points, $\mathcal{E}^{cs}$ is still a fiber bundle over M, whose leaves are all circles. We also get that $\widetilde{\mathfrak{F}}$ projects down to a diffeomorphism cocycle $\mathfrak{F} : \mathcal{E}^c \to \mathcal{E}^c$ and the holonomies of $\widetilde{\mathfrak{F}}$ project down to compatible holonomies of $\mathfrak{F}$.

The next step is to construct a σ-finite measure m^c on E^c whose restriction to a $\mathfrak{G}$ fundamental domain is a probability measure that projects down to m. The measure m^c is both $\widetilde{\mathfrak{F}}$-invariant and c-invariant. Let $\{\mathfrak{m}_x\}$ be a disintegration of m along center leaves, which is defined on a full-volume c-saturated set which we denote by M^c. For each $x \in M^c \cap O$, choose a representative m_x of the conditional class $\mathfrak{m}_x$ normalized by

$$m_x([x, f(x))) = 1. \tag{8}$$

This choice of normalization immediately implies that

$$f_* m_x = m_{f(x)}. \tag{9}$$

By Proposition 3.11, equation (8) implies that

$$m_x([y, f(y))) = 1 \quad \text{for every } y \in \mathcal{W}^c_x, \tag{10}$$

so that we have

$$m_y = m_x \quad \text{for every } y \in \mathcal{W}^c_x. \tag{11}$$

Pushing m_x forward by $\exp^{c\,-1}_x$ gives a measure m^c_x on E^c_x, and letting $m^c = m^c_x \, dm(x)$ we obtain an invariant (by (9)) and c-invariant (by (11)) measure for $\widetilde{\mathfrak{F}}$.

By the choice of normalization in (8), m^c is the lift of a probability measure $\hat{m}$ on $\mathcal{E}^c$ which is c- and $\mathfrak{F}$-invariant.

The induced Riemannian metric on $\mathcal{W}^c$ leaves pulls back via $\exp^c_x$ to a Riemannian metric on E^c_x, with respect to which the Lyapunov exponent of any $z \in E^c_x$ under $\widetilde{\mathfrak{F}}$ coincides with that of $\exp^c_x(z)$ under f.

9.1.1. *Application of the invariance principle.*

LEMMA 9.3. *If the center Lyapunov exponent vanishes m-almost everywhere, then there is a continuous disintegration* $\{\hat{m}^c_x : x \in M\}$ *of* m^c *along* E^c *fibers, which is* cF*-invariant, and* s, u *and* c*-holonomy invariant.*

Proof. By assumption the center Lyapunov exponents of $\widetilde{\mathfrak{F}}$ are zero for m^c-almost every z (by construction of m^c).

Let $S \subset \mathcal{E}^c$ be the 'half-closed' set bounded by the zero-section of E^c and its image under $\mathfrak{G}$, including the former and excluding the latter. As S is precompact, the quotient map $E^c \to \mathcal{E}^c$ has bounded derivative at S, hence the Lyapunov exponents of $\hat{m}$ (which is the push-forward of $m^c|S$ by the quotient map) are also zero. Applying Theorem 4.1 now yields that there is a holonomy-invariant disintegration $\hat{m}_x$ of $\hat{m}$ along the fibers of $\mathcal{E}^c$ over M. □

9.2. *Proof of part (1) of Theorem 2.7.* We assume that $\mathcal{W}^c$ is leafwise absolutely continuous, which implies as in [6] that the center Lyapunov exponents vanish m-almost everywhere. Then Lemma 9.3 gives a holonomy-invariant disintegration $\hat{m}_x$ of $\hat{m}$ along the fibers of $\mathcal{E}^c$ over M. This lifts to a family of Radon measures $\hat{m}^c_x$ on E^c that is invariant under $\mathcal{F}$, $\mathcal{G}$, and s, u, and c holonomies.

Continuity of the foliation $\mathcal{W}^c$ implies that a small enough interval $(-\epsilon, \epsilon)$ in E^c projects under $\exp^c$ to a local center manifold in M. The c-invariance of the measures $\hat{m}^c_x$ implies that there is a coherent projection to a continuous family of Radon measures m^c_x on the leaves of $\mathcal{W}^c$ invariant under f and su-holonomy. In any local foliation chart, these measures restrict to a disintegration of m, and for any open leaf of $\mathcal{W}^c$, we have that $m^c_x[x, f(x)) = 1$.

As in §6.1 we define a local flow ψ_t on M via the relation

$$m_x([y, \psi_t(y))^c) = t,$$

for $t \in (-\epsilon, \epsilon)$. This extends to a global flow in the obvious way, and by construction we have $\psi_1 = f$. The proof now proceeds exactly as the proof of part (1) of Theorem 2.2 in §6.1, where the arguments establishing the properties of ψ_t are entirely local in nature (see also [6], where the same thing is proved assuming accessibility).

9.3. *Proof of Part (2) of Theorem 2.7.* We prove part (2) of Theorem 2.7. Suppose $U \neq \emptyset$ is an open accessibility class for $f \in \mathcal{P}^1_{\text{fix}}(M)$.

9.3.1. *The case of non-vanishing exponents.* Suppose that $\chi^c \neq 0$. Let

$$X = \{x \in U : \chi^c(x) = \chi^c\},$$

which is a full measure subset of U. Let $\mathcal{X} = (\exp^c)^{-1}(X) \subset E^c$, which is the set of $\xi \in (\exp^c)^{-1}(U)$ where the fiberwise exponent of $\mathfrak{F}$ is equal to χ^c; it is clearly $\mathcal{F}$ and $\mathcal{G}$-invariant. Let $\mathcal{X}'$ be the projection of $\mathcal{X}$ to $\mathcal{E}$. Then [6, Theorem 4.1] implies that $\mathcal{X}'$ coincides, up to zero $\hat{\mu}$-measure, with a measurable set $\mathcal{Y}' \subset \mathcal{E}^c$ meeting almost every fiber

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$\mathcal{E}^c_x$, $x \in U$ in finitely many points. Pulling back to E^c, we obtain an $\mathcal{F}$-invariant measurable subset $\mathcal{Y} \subset E^c$ whose projection to M has full measure in U and that meets each E^c fiber in finitely many $\mathcal{G}$-orbits. Setting $Y = \exp(\mathcal{Y}) \subset U$, we obtain a full measure subset of U that meets $\mathcal{W}_x$, for almost every $x \in U$, in finitely many f-orbits. Hence case 2a holds in Theorem 2.7.

9.3.2. *The case of vanishing exponents.* As in the proof of part 2 of Theorem 2.2, we deduce that either the restriction of m to U is atomic, or there is a flow ψ_t supported in U and non-singular in U, tangent to the leaves of $\mathcal{W}^c$ and preserving the restriction $m|_U$. This implies that for every $x \in U$, we have $\mathcal{W}^c_x \cap U = \mathcal{W}^c_x$. Thus, U is c-saturated. However, U is an accessibility class, and so is u- and s-saturated. It follows that $U = M$ and f is accessible.

10. *Examples and questions*

We have seen that there is a dichotomy for some conservative, accessible systems with one-dimensional center: either the center is absolutely continuous or the disintegration of Lebesgue measure along the center foliation is atomic. Although these results are quite general, some interesting questions remain, which we pose here.

10.1. *Zero exponents and atomic disintegrations.* Let us discuss an example of Katok showing that the center foliation may fail to be absolutely continuous and, in fact, the disintegration of Lebesgue measure along center leaves may be atomic even when the center Lyapunov exponents vanish.

Let $\{f_t : \mathbb{T}^2 \to \mathbb{T}^2 : t \in \mathbb{R}/\mathbb{Z}\}$ be a smooth family of area-preserving Anosov diffeomorphisms with the following property: for all $s, t \in \mathbb{R}/\mathbb{Z}$ with $s \neq t$, the diffeomorphisms f_s and f_t are conjugate by a homeomorphism $h_{s,t} : \mathbb{T}^2 \to \mathbb{T}^2$ near the identity, but they are *not* smoothly conjugate. One can obtain such a family by, for example, smoothly perturbing a linear Anosov diffeomorphism in a neighborhood of a fixed point. It follows from [**16**] that $h_{s,t}$ is not absolutely continuous, in fact there is no absolutely continuous conjugacy between f_s and f_t, if $s \neq t$.

Define $f : \mathbb{T}^2 \times \mathbb{R}/\mathbb{Z} \to \mathbb{T}^2 \times \mathbb{R}/\mathbb{Z}$ by $f(x, t) = (f_t(x), t)$. Then f is partially hyperbolic and preserves the Lebesgue measure λ_3 on $\mathbb{T}^2 \times \mathbb{R}/\mathbb{Z}$. The leaf of the center foliation through each $(x, s) \in \mathbb{T}^2 \times \mathbb{R}/\mathbb{Z}$ is the smooth curve

$$\mathcal{W}^c_{(x,s)} = \{(h_{s,t}(x), t) : t \in \mathbb{R}/\mathbb{Z}\}.$$

It is easy to see that the center Lyapunov exponent of f vanishes almost everywhere. Let $\mathcal{Z}$ be the set of points $(x, s) \in \mathbb{T}^2 \times \mathbb{R}/\mathbb{Z}$ such that x is λ_2-regular for the diffeomorphism f_s and the Lebesgue measure λ_2 on $\mathbb{T}^2$. Observe that $\mathcal{Z}$ has full λ_3-measure.

LEMMA 10.1. *The set $\mathcal{Z}$ meets each leaf of $\mathcal{W}^c$ in at most one point. Hence, any disintegration of m along the leaves of $\mathcal{W}^c$ is atomic, supported on a single point in each leaf.*

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Proof. Let $(x, s) \in \mathcal{Z}$, and fix $t \neq s$. The measures λ_2 and $(h_{s,t})_*(\lambda_2)$ are both ergodic for f_t. As $h_{s,t}$ is not absolutely continuous, these measures are therefore mutually singular. As x is regular for f_s and the measure λ_2, it follows that $h_{s,t}(x)$ is regular for f_t and the measure $(h_{s,t})_*(\lambda_2)$. Thus, $h_{s,t}(x)$ cannot be regular for f_t and the measure λ_2. This means that $(h_{s,t}(x), t) \notin \mathcal{Z}$, for all $t \neq s$ or, in other words, $\mathcal{W}^c_{(x,s)} \cap \mathcal{Z} = \{(x, s)\}$. This proves the first statement in the lemma. The second is a direct consequence, because the set $\mathcal{Z}$ has full measure. □

One can also give an explicit description of the disintegrations m^s_x and m^u_x that appeared in the proof of the invariance theorem. Define $\mathcal{Z}^s$ to be the set of (x, s) such that x is a forward-regular point for f_s and λ_2, and define $\mathcal{Z}^u$ to be the set of (x, s) such that x is a backward-regular point for f_s and λ_2. Then $\mathcal{Z}^s = \mathcal{Z}^u = \mathcal{Z}$ mod 0, all three sets are f-invariant, the set $\mathcal{Z}^s$ is $\mathcal{W}^s$-saturated, and the set $\mathcal{Z}^u$ is $\mathcal{W}^{uu}$-saturated. Arguing as in the proof of Lemma 10.1, it is easy to see that $\mathcal{Z}^s$ meets each leaf of $\mathcal{W}^c$ in at most one point, as does $\mathcal{Z}^u$. Hence, for almost every point $x \in M/\mathcal{W}^c$, there exists $p \in M$ such that $\mathcal{Z}^s \cap \mathcal{W}^c_x = \{p\}$; for such x, we set $m^s_x = \delta_p$. The measures m^u_x are defined analogously. Then $x \mapsto m^s_x$ is s-invariant, and $x \mapsto m^u_x$ is u-invariant. Although the two functions coincide almost everywhere, Lemma 3.19 implies that there is no disintegration $x \mapsto m_x$ that is simultaneously s-invariant and u-invariant (at all points, not just almost all). Thus, the conclusion of Theorem 4.5 does not hold in this case.

10.2. *Non-accessible ergodic cases.* The preceding discussion leads us naturally to the following question.

Problem 10.2. Let $f : M \to M$ be an ergodic (but not accessible), C^2 volume-preserving perturbation of an Anosov skew product with circle fiber. Is it possible for the disintegration of Lebesgue along the center foliation to be continuous (that is, non-atomic), but singular with respect to Lebesgue measure on the leaves?

If such an example exists, it must have jointly integrable stable and unstable foliations.

PROPOSITION 10.3. *Let M be a manifold of dimension $d \geq 3$, and let $f \in \mathcal{P}^1_{\text{fib}}(M)$ or $f \in \mathcal{P}^1_{\text{fix}}(M)$. If the disintegration of Lebesgue along the center foliation is continuous but singular with respect to Lebesgue, then the stable and unstable foliations are jointly integrable.*

Proof. The accessibility classes consist of either compact su-leaves or connected open sets bounded by compact su-leaves. Suppose there is a non-empty open accessibility class. Then part (2) of Theorems 2.2 and 2.7 imply that either the disintegration contains atoms, or f is accessible and the center foliation is leafwise absolutely continuous. As both possibilities are excluded by the hypotheses, it follows that the stable and unstable foliations are jointly integrable. □

Hence, there is a natural class of examples in which to consider this question, which are related to the example mentioned in §10.1. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^∞ Anosov

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diffeomorphism. Then there is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}^\infty_m(\mathbb{T}^2)$ such that, for any C^∞ map $\phi : \mathbb{R}/\mathbb{Z} \to \mathcal{U}$:

(1) for each $t \in \mathbb{R}/\mathbb{Z}$, the map $f_{\phi,t} := \phi_t \circ f$ is an area-preserving Anosov diffeomorphism, topologically conjugate to f;

(2) for any $\alpha \in \mathbb{R}/\mathbb{Z}$, the map $g_{\phi,\alpha} : \mathbb{T}^2 \times \mathbb{R}/\mathbb{Z} \to \mathbb{T}^2 \times \mathbb{R}/\mathbb{Z}$ given by

$$g_{\phi,\alpha}(x, t) = (f_{\phi,t}(x), t + \alpha)$$

is partially hyperbolic, dynamically coherent and topologically conjugate modulo $\mathcal{W}^c(g_{\phi,\alpha})$ to f.

For a fixed C^∞ map $\phi : \mathbb{R}/\mathbb{Z} \to \mathcal{U}$, consider the family $\{g_{\phi,\alpha}\}_{\alpha\in\mathbb{R}/\mathbb{Z}}$ defined previously; it is a partially hyperbolic, volume-preserving skew product over a rotation by α (in the second factor). Although singular continuous center decomposition of Lebesgue might occur in this family of examples, it turns out that the generic example has Dirac disintegration.

PROPOSITION 10.4. *There is a residual subset $\mathcal{R} \subset C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U}) \times \mathbb{R}/\mathbb{Z}$ such that for $(\phi, \alpha) \in \mathcal{R}$, the map $g_{\phi,\alpha}$ is ergodic (and non-accessible), and the disintegration of Lebesgue along center leaves of $g_{\phi,\alpha}$ is Dirac.*

Proof. The strategy is to establish first that for the generic ϕ, and any rational p/q, the disintegration of volume along $\mathcal{W}^c(g_{\phi,p/q})$ leaves is Dirac; for generic ϕ, this property then passes to a residual set of irrational α, for which $g_{\phi,\alpha}$ is also ergodic (though non-accessible). □

LEMMA 10.5. *For each $p/q \in \mathbb{Q} \cap [0, 1]$ there is a residual subset $\mathcal{R}_{p/q}$ of the space $C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$ such that for $\phi \in \mathcal{R}_{p/q}$, the disintegration of Lebesgue along center leaves of $g_{\phi,p/q}$ is Dirac.*

Proof. For a fixed $\phi \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$, and $p/q \in \mathbb{Q}$, consider the map $G_{\phi,p/q} = g^q_{\phi,p/q}$: the center foliation for this is the same as the center foliation for $g_{\phi,p/q}$. This map takes the form $G_{\phi,p/q}(x, t) = (F_{\phi,p/q,t}(x), t)$, where

$$F_{\phi,p/q,t} = f_{\phi,t+(q-1)p/q} \circ \cdots \circ f_{\phi,t+p/q} \circ f_{\phi,t}.$$

As $G_{\phi,p/q}(x, t)$ is partially hyperbolic, the maps $F_{\phi,p/q,s}$ are Anosov, for all $s \in \mathbb{R}/\mathbb{Z}$. The leaf of $\mathcal{W}^c(G_{\phi,p/q})$ through $(x, 0)$ is the curve $(H_{\phi,p/q,t}(x), t)_{t\in\mathbb{R}/\mathbb{Z}}$, where $H_{\phi,p/q,t}$ is the conjugacy between $F_{\phi,p/q,t}$ and $F_{\phi,p/q,t}$ given by structural stability (unique the homotopy class of the identity on $\mathbb{T}^2$).

Moreover, the disintegration of Lebesgue measure along $\mathcal{W}^c(G_{\phi,p/q})$ is Dirac if and only if for almost every $t \in \mathbb{R}/\mathbb{Z}$ and every $s \neq t$, the map $H_{\phi,p/q,s,t} := H_{\phi,p/q,t} \circ H^{-1}_{\phi,p/q,s}$ is not C^1 (note that $H_{\phi,p/q,s,t}$ is the conjugacy between $F_{\phi,p/q,s}$ and $F_{\phi,p/q,t}$). □

LEMMA 10.6. *For any $p/q \in \mathbb{Q}$, there is a residual subset $\hat{\mathcal{R}}_{p/q} \subset C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$ such that for every $\phi \in \hat{\mathcal{R}}_{p/q}$, for every $s \in \mathbb{R}/\mathbb{Z}$, if $H_{\phi,p/q,s,t}$ is C^1, then $t = s + kp/q$, for some $0 \leq k < q$.*

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Proof. Fix $p/q \in \mathbb{Q}$. To simplify notation, in the proof we suppress the p/q subscripts in F, G, and H. We first note that if $H_{\phi,s,t}$ is C^1 for some $\phi \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$, then the eigenvalues of the derivatives of the maps $F_{\phi,s}$ and $F_{\phi,t}$ must coincide at all corresponding periodic orbits. Let $\{x_{\phi,k}\}_{k\geq 1}$ be an enumeration of the periodic points for $F_{\phi,0}$ with $\mathrm{per}(x_{\phi,k}) = m_{\phi,k}$, and for $t \in \mathbb{R}/\mathbb{Z}$, let $x_{\phi,k,t} = H_{\phi,t}(x_{\phi,k})$ be the corresponding periodic orbit for $F_{\phi,t}$. Denote by $\lambda_{\phi,k,t}$ the larger eigenvalue of $D_{x_{\phi,k,t}} F_{\phi,t}^{m_{\phi,k}}$. If $H_{\phi,s,t}$ is C^1, then $\lambda_{\phi,k,s} = \lambda_{\phi,k,t}$, for all $k \geq 1$.

Let $\mathcal{K} = \{I_k\}_{k\geq 1}$ be a sequence of compact intervals in $\mathbb{R}/\mathbb{Z}$ with the following properties:

- $\mathrm{diam}(I_k) \to 0$ as $k \to \infty$;
- $\bigcup_{k\geq k_0} I_k = \mathbb{R}/\mathbb{Z}$, for all $k_0 \geq 1$.

For $I, J \in \mathcal{K}$, and $k \geq 1$, let $\mathcal{E}_{I,J,k} \subset C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$ be the set of all ϕ such that, for every $s \in I$, and for every $t \in \bigcup_{j=0}^{q-1}(J + jp/q)$:

$$\lambda_{\phi,k,s} \neq \lambda_{\phi,k,t}.$$

The set $\mathcal{E}_{I,J,k}$ is clearly open in $C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$. Let

$$\mathcal{D}(I) = \left\{ J \in \mathcal{K} : \mathrm{diam}(J) < 1/q \ \text{ and } \ I \cap \bigcup_{j=0}^{q-1}(J + jp/q) = \emptyset \right\}.$$

It is straightforward to check that for $I \in \mathcal{K}$ and $J \in \mathcal{D}(I)$, the set $\mathcal{E}_{I,J} = \bigcup_{k\geq 1} \mathcal{E}_{I,J,k}$ is open and dense in $C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$. Let

$$\hat{\mathcal{R}}_{p/q} = \bigcap_{I\in\mathcal{K}} \ \bigcap_{J\in\mathcal{D}(K)} \mathcal{E}_{I,J}.$$

Then $\hat{\mathcal{R}}_{p/q}$ is residual in $C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$. Suppose that $\phi \in \hat{\mathcal{R}}_{p/q}$. Fix $s \in \mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}/\mathbb{Z} \setminus \bigcup_{j=0}^{q-1}\{s + jp/q\}$. Then there exist intervals $I \in \mathcal{K}$ and $J \in \mathcal{D}(I)$ such that $s \in I$ and $t \in J$. As $\phi \in \hat{\mathcal{R}}_{p/q} \subset \mathcal{E}_{I,J}$, there exists a $k \geq 1$ such that $\lambda_{\phi,k,s} \neq \lambda_{\phi,k,t}$. Then $H_{\phi,s,t}$ is not C^1. □

Remark 10.7. The same type of argument shows that, for the generic ϕ, there is no C^1 conjugacy at all between $F_{\phi,s}$ and $F_{\phi,t}$, if $t \in \mathbb{R}/\mathbb{Z} \setminus \bigcup_{j=0}^{q-1}\{s + jp/q\}$. We next treat the case where $t = s + jp/q$, for some $0 < j \leq q - 1$; here, a C^1 conjugacy between $F_{\phi,s}$ and $F_{\phi,t}$ cannot be avoided: they are always conjugate by the map $f_{s+(q-1)p/q} \circ \cdots \circ f_{\phi,s+jp/q}$. What can be avoided generically is a C^1 conjugacy that is isotopic to the identity map on $\mathbb{T}^2$, as the next lemma shows.

LEMMA 10.8. *For each $p/q \in \mathbb{Q}$, there is a residual subset $\mathcal{R}_{p/q} \subset \hat{\mathcal{R}}_{p/q}$ such that for every $s \in \mathbb{R}/\mathbb{Z}$ and $0 < k \leq q - 1$, there is no C^1 conjugacy between $F_{\phi,s}$ and $F_{\phi,s+kp/q}$ that is isotopic to the identity. In particular, for $\phi \in \mathcal{R}_{p/q}$, the conjugacy $H_{\phi,s,t}$ is not C^1 for $s \neq t$.*

Proof. The set $\mathcal{D}$ of Anosov diffeomorphisms of $\mathbb{T}^2$ with trivial centralizer is C^∞-open and dense; that is, if $F \in \mathcal{D}$ and $FG = GF$, for some C^∞ diffeomorphism G, then $G = F^m$, for some integer $m \in \mathbb{Z}$. See Palis and Yoccoz [27]. From this it follows easily that

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there is an open and dense set $\mathcal{O}_{p/q} \subset C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$ such that for each $\phi \in \mathcal{O}_{p/q}$, and each $t \in \mathbb{R}/\mathbb{Z}$, the map $F_{\phi,t}$ has trivial centralizer.

Fix $\phi \in \mathcal{O}_{p/q}$ and suppose that for some $s \in \mathbb{R}/\mathbb{Z}$ and $t = s + jp/q$, with $0 \leq j \leq q-1$, the map $H_{\phi,s,t}$ is C^1. Then $H_{\phi,s,t}$ is, in fact, C^∞ (see [16]). On the other hand, $F_{\phi,s}$ is conjugate to $F_{\phi,t}$ by the map $H'_{\phi,s,t} = f_{s+(q-1)p/q} \circ \cdots \circ f_{\phi,s+jp/q}$. Hence, the C^∞ map $H'_{\phi,s,t} H^{-1}_{\phi,s,t}$ commutes with the Anosov diffeomorphism $F_{\phi,t}$.

As $\phi \in \mathcal{O}_{p/q}$, the map $F_{\phi,t}$ has trivial centralizer, and so there exists an integer m such that $H'_{\phi,s,t} H^{-1}_{\phi,s,t} = F^m_{\phi,s}$. As $F_{\phi,s}$ is isotopic to f^q and $H'_{\phi,s,t} H^{-1}_{\phi,s,t}$ is isotopic to f^{q-j}, this implies that $j = 0$, and so $s = t$. Hence, $H_{\phi,s,t}$ is not C^1 if $s \neq t$. We conclude the proof by setting $\mathcal{R}_{p/q} = \mathcal{O}_{p/q} \cap \hat{\mathcal{R}}_{p/q}$. □

This completes the proof of Lemma 10.5.

Let $\mathcal{R}_0 = \bigcap_{p/q \in \mathbb{Q}} \mathcal{R}_{p/q}$ and note it is a residual subset of $C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$. For $\phi \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U})$, consider the map $g_{\phi,\alpha}$, for some $\alpha \in \mathbb{R}/\mathbb{Z}$. Condition 2 on $\mathcal{U}$ implies that the quotient space $\mathbb{T}^3/\mathcal{W}^c(g_{\phi,\alpha})$ is the 2-torus $\mathbb{T}^2$. Denote by $\pi_{\phi,\alpha} : \mathbb{T}^3 \to \mathbb{T}^2$ the quotient map. Let $\mu_{\phi,\alpha} = (\pi_{\phi,\alpha})_* m$. The following lemma is easy to check.

LEMMA 10.9. *The disintegration of m along $\mathcal{W}^c(g_{\phi,\alpha})$ leaves is Dirac almost everywhere if and only if $\Delta(\phi, \alpha) = 0$, where*

$$\Delta(\phi, \alpha) = \int_{\mathbb{T}^2} \left(\int_{\pi^{-1}_{\phi,\alpha}(p) \times \pi^{-1}_{\phi,\alpha}(p)} d(x, y)\, dm_{\phi,\alpha,p}(x)\, dm_{\phi,\alpha,p}(y) \right) d\mu_{\phi,\alpha}(p),$$

and $m_{\phi,\alpha,p}$ is the disintegration of m on the leaf $\mathcal{W}^c(g_{\phi,\alpha})$ over p.

Let $\{\mathcal{P}_n\}_{n \geq 0}$ be a nested sequence of finite (mod 0) partitions of $\mathbb{T}^2$ into open sets, generating the Borel σ-algebra. Consider the sequence of functions

$$\{\Delta_n : C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U}) \times \mathbb{R}/\mathbb{Z} \to [0, 1]\}_{n \geq 0}$$

given by

$$\Delta_n(\phi, \alpha) = \sum_{P \in \mathcal{P}_n} \mu_{\phi,\alpha}(P)^{-1} \left(\int_{\pi^{-1}_{\phi,\alpha}(P) \times \pi^{-1}_{\phi,\alpha}(P)} d(x, y)\, dm(x)\, dm(y) \right).$$

We claim that Δ_n is continuous and $\Delta_n \to \Delta$ pointwise. Continuity follows from the fact that the foliation $\mathcal{W}^c(g_{\phi,\alpha})$ depends continuously on (ϕ, α). The pointwise convergence follows from Rokhlin's theorem: for $\mu_{\phi,\alpha}$-almost every $p \in \mathbb{T}^2$, we have

$$m(\cdot \mid \pi^{-1}_{\phi,\alpha}(\mathcal{P}_n(p))) \to m_{\phi,\alpha,p}$$

in the weak* topology, where $\mathcal{P}_n(p)$ denotes the atom of $\mathcal{P}_n$ containing p.

We conclude using the following lemma.

LEMMA 10.10. *Let X be a Baire space and let $\{\Delta_n : X \to [0, 1]\}_{n \geq 0}$ be a sequence of continuous functions such that $\Delta_n \to \Delta$ pointwise. Then $\Delta^{-1}(0)$ is a G_δ. Hence, if $\Delta(x) = 0$ for a dense set of x, then $\Delta^{-1}(0)$ is residual in X.*

Proof. Fix $\varepsilon > 0$ and for $n \geq 0$ let

$$U_\varepsilon^n = \{x : \Delta_m(x) < \varepsilon, \text{ for some } m \geq n\}.$$

Clearly U_ε^n is open for each n, ε. The conclusion follows from the fact that $\Delta^{-1}(0) = \bigcap_{m,n>0} U_{1/m}^n$. □

As Δ vanishes on the dense set $\mathcal{R}_0 \times \mathbb{Q}/\mathbb{Z}$, it follows that $\mathcal{R} = \Delta^{-1}(0)$ is residual in $C^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{U}) \times \mathbb{R}/\mathbb{Z}$. Lemma 10.9 then implies that for $(\phi, \alpha) \in \mathcal{R}$, the disintegration of m along $\mathcal{W}^c(g_{\phi,\alpha})$ leaves is Dirac. This completes the proof of Proposition 10.4.

10.3. *Generic accessible systems.* Another relevant question concerns the number of atoms that can occur in a generic accessible system with atomic disintegration along center fibers.

Problem 10.11. Let f be an accessible, C^2, volume-preserving perturbation of an Anosov skew product with circle fiber. Suppose that the center Lyapunov exponents are non-vanishing (that is, either positive almost everywhere or negative almost everywhere).

Is it possible for such a system to have Dirac disintegration, that is, exactly one atom per (almost every) center leaf? Generically, is the disintegration Dirac?

Is the number of atoms per leaf unbounded in any neighborhood of the skew product?

Note that when the center exponents vanish in such an example, we generically have Dirac disintegration. In addition, it is possible to have more than one atom per leaf and non-vanishing center exponents, at least when the example admits a smooth symmetry (see [**34**] for an example). In dimension three, if the center exponents vanish, then a disintegration with one atom per leaf forces a smooth symmetry in the system (Proposition 6.11). In higher dimensions, there is a continuous, measure-preserving symmetry. More generally, we have the following problem.

Problem 10.12. Let $f : M \to M$ be an accessible, C^2, volume-preserving perturbation of an Anosov skew product with circle fiber. If the disintegration of Lebesgue on the center foliation is atomic with k atoms, then must there exist a (continuous or even smooth) map $\Phi : M \to M$, preserving Lebesgue, such that $\Phi \circ f = f \circ \Phi$ and $\Phi^k = Id$?

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